Inversion of the two dimensional Radon transformation by diagonalisation

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Abstract

Based on an explicit diagonalisation of Abel's integral operator, we give an inversion of the plane Radon transformation

$$R(f)(p,\theta) = \int_{\{x\cos(\theta)+y\sin(\theta)=p\}} f(x,y)\, ds$$

by diagonalising explicitly $\tilde{R}(f)(r,\phi) = R(f(r,\phi)/r)$. We calculate the spectrum and the kernel of $\tilde{R}$.

Keywords

Computerized Tomography, Abel and Radon transformation, Inverse problems.

1 Introduction

The reconstruction of a two-dimensional density function from projections along lines reduces to the solution of the two-dimensional Radon transform. Studied first by Radon in 1917 ([6] Appendix), it is today a basic tool in applications like medical diagnosis, tokamak monitoring in plasma physics or astrophysical applications. The reconstruction is also called *tomography.* See [7],[3] for recent collection of works about the Radon transform. Mathematical tools developed for the solution of this problem lead to the construction of sophisticated scanners (see [8] Ch.1 and Appendix). It is important that the inversion $h = R(f) \mapsto f$ is fast, accurate, robust and requires few data points. It is well known that there are natural obstructions to fulfill these requirements. We will see this explicitly by showing that the Radon operator R has a kernel and arbitrary small singular values.

The explicit calculation of the spectrum and the kernel of the Radon transform might be of further theoretical interest. Whether a serious implemented numerical algorithm can compete with the existing algorithms needs further investigation. The simplicity of the new approach makes it easy to program an inversion.

2 The Radon transform

Given a function $f : \mathbf{R^2} \setminus \{\mathbf{0}\} \to \mathbf{R}$ decaying like $1/r$ at infinity. The Radon transform of f is

$$R(f)(p,\theta) = \int_{L(p,\theta)} f(x,y)\ ds\ ,$$

where the integration is taken along the line $L(p,\theta) = \{(x,y) \mid x\cos(\theta) + y\sin(\theta) = p\}$ for $p > 0$. If we interpret (p,θ) as polar coordinates, the function $g = R(f)$ is again a function $\mathbf{R^2} \setminus \{\mathbf{0}\} \to \mathbf{R}$ decaying at infinity. The problem is to invert R.

The reduction of the Radon transform to modified Abel integrals is due to Cormack [2] (see also [1] p. 380 or [4] p. 119). Using polar coordinates $(r,\phi) = (\sqrt{x^2+y^2}, \arctan(y/x))$ and writing $s = \sqrt{r^2-p^2}, ds = rdr/\sqrt{r^2-p^2}$ with $\frac{p}{r} = \cos(\theta-\phi)$ gives

$$R(f)(p,\theta) = \int_{L(p,\theta)} \frac{rf(r,\phi)\ dr}{\sqrt{r^2-p^2}}\ .$$

We define also $\tilde{R}(f) = R(f/r)$. If f is differentiable, we can make a point wise converging Fourier decomposition

$$f(r,\phi) = \sum_{n \in \mathbf{Z}} f_n(r) e^{in\phi}$$

and get

$$\begin{aligned} R(f)(p,\theta) &= \int_p^\infty \sum_n f_n(r) e^{in\phi}\ ds + \int_p^\infty \sum_n f_n(r) e^{in(2\theta-\phi)}\ ds \\ &= 2\int_p^\infty \sum_n f_n(r) \frac{\cos(n(\theta-\phi))}{\sqrt{r^2-p^2}} e^{in\theta} r\ dr = 2\int_p^\infty \sum_n f_n(r) \frac{T_n(p/r)}{\sqrt{r^2-p^2}} r e^{in\theta}\ dr\ , \end{aligned}$$

where $T_n(x) = \cos(n \arccos(x))$ is the n-th Tschebycheff polynomial of the first kind. The Fourier coefficients g_n of g and f_n are related by the *modified Abel integral equation* $g_n = 2A_n(rf_n)$, where

$$A_n(f)(p) = \int_p^\infty \frac{f(r)T_n(p/r)}{\sqrt{r^2-p^2}}\ dr\ .$$

For $n = 0$, we get the usual Abel operator $A = A_0$.

3 Diagonalisation of the modified Abel operator

Let $\mathbf{C}^* = \mathbf{C} \setminus \{\mathbf{0}\}$. We call a function $g : \mathbf{C}^* \to \mathbf{C}^*$ *entire at* ∞, if $z \mapsto g(z^{-1})$ is an entire function. Denote with $\mathcal{E}$ the linear space of all functions g which are entire at ∞, and satisfy $g(\infty) = 0$. Every function in $\mathcal{E}$ has a Taylor development $g(z) = \sum_{n=1}^\infty g_n z^{-n}$ which converges for all $z \in \mathbf{C}^*$. In [9], we have shown how the Abel operator A can be diagonalised in $\mathcal{E}$. This can be generalized:

Lemma 3.1 *Given $n \in \mathbf{N}$. For all $k \geq 1$, the function $\psi_k(x) = x^{-k} \in \mathcal{E}$ is an eigenfunction of A_n with the eigenvalue*

$$\int_0^{\pi/2} \cos(nx) \cdot \cos^{k-1}(x)\ dx\ .$$

Proof. We use $A(\psi_k) = \lambda_k \psi_k$ with $\lambda_k = \int_0^{\pi/2} \cos^{k-1}(x)\ dx$ of [9] to get

$$\begin{aligned} A_n(\psi_k)(p) &= A(T_n(p/r)r^{-k})(p) = \sum_{i=0}^n a_i A(\frac{p^i}{r^{i+k}})(p) = \sum_{i=0}^n a_i p^i \lambda_{i+k} p^{-i-k} \\ &= (\sum_{i=0}^n a_i \lambda_{i+k})\ \psi_k(p) = \int_0^{\pi/2} \sum_{i=0}^{n-1} a_i \cos^{i+k-1}(x)\ dx\ \psi_k(p) \end{aligned}$$

$$= \int_0^{\pi/2} T_n(\cos(x)) \cdot \cos^{k-1}(x)\, dx\, \psi_k(p)$$
$$= \int_0^{\pi/2} \cos(nx) \cdot \cos^{k-1}(x)\, dx\, \psi_k(p)\ .$$

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4 Diagonalisation of the operator $\tilde{R}$

Let $\mathcal{F}$ be the space of functions $f : \mathbf{R^2} \to \mathbf{R}$ which are in polar coordinates smooth in the angle direction and analytic at ∞ in the radial direction. More precisely, we need and require that the Fourier series $f(r,\phi) = \sum_n f_n(r) e^{in\phi}$ converges uniformly for all $r > 0$ and that $f_n(r) \in \mathcal{E}$ for all $n \in \mathbf{Z}$. A function $f \in \mathcal{F}$ can be written as

$$f(r,\phi) = \sum_{n\in\mathbf{Z}} \sum_{k=1}^{\infty} \psi_{n,k} = \sum_{n\in\mathbf{Z}} \sum_{k=1}^{\infty} f_{n,k} r^{-k} e^{in\phi}\ .$$

Theorem 4.1 *For every $k \in \mathbf{N}$, $n \in \mathbf{Z}$ the function $\psi_{n,k}(r,\phi) = r^{-k} e^{in\phi} \in \mathcal{F}$ is an eigenfunction of $\tilde{R}$ with the eigenvalue*

$$\lambda_{n,k} = 2 \int_0^{\pi/2} \cos(nx) \cos(x)^{(k-1)}\, dx = \frac{\pi}{2^{k-1} \cdot k} \cdot \frac{\Gamma(k+1)}{\Gamma(\frac{k+n+1}{2})\Gamma(\frac{k-n+1}{2})}\ .$$

The operator $\tilde{R}$ is defined on $\mathcal{F}$ and the simultaneous Taylor and Fourier expansion leads to the diagonalization

$$\tilde{R} : \sum_{k=1}^{\infty} f_{k,n} r^{-k} e^{in\phi} \mapsto \sum_{k=1}^{\infty} \lambda_{n,k} f_{k,n} r^{-k} e^{in\phi}\ .$$

The kernel of $\tilde{R}$ (and R) is spanned by $\{\psi_{n,k} \mid (n+k) \text{ odd}\ , |n| > k\}$.

Proof. (i) The diagonalisation of $\tilde{R}$.
From Cormack's formula

$$\tilde{R}(f)(p,\theta) = 2 \sum_{n\in\mathbf{Z}} A_n(f_n(r))(p) e^{in\theta}$$

and the diagonalisation of A_n in Lemma 3.1 the claim follows.

(ii) The spectrum of $\tilde{R}$.
Using (i), we get for the eigenvalues $\lambda_{n,k}$ of $\tilde{R}$

$$\lambda_{n,k} = 2 \int_0^{\pi/2} \cos(nx) \cos(x)^{(k-1)}\, dx = \frac{\pi}{2^{k-1} \cdot k} \cdot \frac{\Gamma(k+1)}{\Gamma(\frac{k+n+1}{2})\Gamma(\frac{k-n+1}{2})}\ .$$

For the integral see [5] p.272.

(iii) The kernel $\ker(\tilde{R}) = \ker(R)$.
An eigenvalue $\lambda_{n,k}$ of $\tilde{R}$ is vanishing if and only if $(n+k)$ is odd and $|n| > k$ because Γ has poles for $z = 0, -1, -2, \ldots$.

(iv) Inversion of R.
Because

$$R^{-1}(g)(r) = \tilde{R}^{-1}(g)/r\ ,$$

we have an inversion of R on the complement of the kernel: given $h = R(f) = \sum_{n,k} h_{n,k} \phi_{n,k}$. Then

$$f = \sum_{k,n} \frac{h_{n,k}}{\lambda_{n,k}} \phi_{n,k+1}\ .$$

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5 Discussion, Critics

• The diagonalisation of $\tilde{R}$ corresponds to a singular value decomposition of $R = \tilde{R} \circ M$. The operator $\tilde{R}$ is selfadjoint when considered as a bounded diagonal linear operator on an abstract Hilbert space built by the functions $\phi_{n,k}$. The operator R^{-2} has some similarity with the "square root" of the Laplacian $L = \delta_r^2 + \frac{1}{r}\delta_r + \frac{1}{r^2}\delta_\phi^2$ which has the singular values $(k^2 - n^2)$, because

$$\tilde{L}\psi_{n,k} = M^2 L \psi_{n,k} = (k^2 - n^2)\psi_{n,k} \ .$$

• The fact that the Radon transform has a kernel makes the inversion ambiguous. From numerical interest is, how small eigenvalues of $\tilde{R}$ are needed for the reconstruction. Small eigenvalues correspond to huge errors. A good algorithm should try to fit the measured projection datas as good as possible with eigenfunctions having small eigenvalues $\lambda_{n,k}^{-1}$.

• The eigenfunctions have a singularity at the center of the coordinate system. When using the diagonalisation inversion, one should translate the interesting part of the target away from $(0, 0)$.

• In applications, one is often interested in Radon transforms of functions which have compact support like for example the density function of a human body. Functions with compact support or discontinuous functions can be approximated pointwise by polynomials in $\mathcal{F}$. The discussed diagonalisation in the analytic set-up does not shed light on work done for Radon transformations for such discontinuous functions. For theoretical considerations one should then use the following remark.
• For $\alpha \in \mathbf{R}, \alpha > \mathbf{0}$, $\beta \in \mathbf{Z}$, the functions $\phi_{\beta,\alpha} = r^\alpha e^{i\beta\phi}$ are eigenfunctions of $\tilde{R}$ with eigenvalues

$$\lambda(\beta, \alpha) = \frac{\pi}{2^\beta \cdot \beta} \cdot \frac{\Gamma(\beta + 1)}{\Gamma(\frac{\beta+\alpha+1}{2})\Gamma(\frac{\beta-\alpha+1}{2})} \ .$$

This allows to diagonalise $\tilde{R}$ on a large class of functions which need not to be analytic. Assume the function f has support in a cone and consider $(r, \phi) \mapsto f(r, \phi)$ as a Cartesian function on $\mathbf{R^2}$. Suppose

$$f(r, \phi) = \int \int \mu(\beta, \alpha) r^{-\alpha} e^{i\beta\phi} \, d\alpha d\beta$$

for some distribution μ on $\mathbf{R^2}$. In the coordinates $\mu = \mu(f)$, the operator $\tilde{R}$ is a multiplication operator $\mu(\beta, \alpha) \mapsto \lambda(\beta, \alpha) \cdot \mu(\beta, \alpha)$.

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