Fluctuation bounds for subharmonic functions

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Abstract

We obtain bounds for the angular fluctuations of a subharmonic function $f(z) = \int \log|z - w|\, d\mu(w)$ in terms of the distribution of the angular mean. This can be applied to study the regularity of subharmonic functions.

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1 Introduction

We look at the problem to estimate the fluctuations

$$F_f(r) = \max_{|z-z_0|=r} f(z) - \min_{|z-z_0|=r} f(z)$$

of a potential $f(z) = \int \log|z - w|\, d\mu(w)$ in the complex plane in terms of the defining measure μ. The main result gives a tool to relate $F_f(r)$ with the regularity of the map $r \mapsto \int_{|z-z_0|<r} d\mu(z)$.

While subharmonic functions are upper-semicontinuous, they are not continuous in general. How discontinuous can they become? A classical result of Ransford actually due to Deny tells that if $f(z) > -\infty$, then up to an exceptional polar set of θ's, one has $\lim_{r\to 0} f(z+r\exp(i\theta)) \to f(z)$ ([16] Corollary 5.4.4). An analogue of Lusin's theorem in potential theory assures that for any potential f and any $\epsilon > 0$, there is a set K_ϵ of capacity $< \epsilon$, such that f is continuous on the complement of K_ϵ (see [13, 14, 1]). Since subharmonic functions are the pointwise limit of continuous functions, a theorem of Baire ([8] p.221) assures that the set of discontinuity points of the first Bair category. An other result following an inequality of Milloux-Schmidt [9] states that there exist a sequence $r_n \to 0$ such that for any θ one has $\lim_{n\to\infty} f(z + r_n \exp(i\theta)) = f(z)$. Moreover, on any ray in the direction θ through a point z, there is a set S of one-dimensional Lebesgue density at z such that $\lim_{n\to\infty} f(z+r_n \exp(i\theta)) = f(z)$ if $r_n \in S$. The reason is that the sets $F_k = \{z \mid f(z) < f(0)+1/k\}$ is open and thin at 0 and if $E_k = \{r \in (0,1] : \exists \theta, re^{i\theta} \in F_k\}$ are its circular projections, then $\int_{E_k} 1/x\, dx < \infty$ ([16] Theorem 5.4.2) so that E_k has zero Lebesgue density at 0. Actually also $\bigcup_k E_k$ has zero Lebesgue density at 0 as we will see and more is known: there is a (in $\mathbb{C}$) polar set S on the ray R such that on f is containuous on $R \setminus S$ ([1] Theorem 7.8.4). Polar sets in the plane have zero Lebesgue measure, zero capacity, and zero Hausdorff dimension and so zero α-Hausdorff measure for every $\alpha > 0$ ([1] Theorem 5.9.6).

We obtain regularity results by relating the fluctuations F_f in terms of the regularity of map $r \mapsto \int_{|z|<r} d\mu(z)$. Our approach provides in specific situations explicit estimates of the density of

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the continuity set near 0, estimates which can be useful for applications. It also can be applied to find criteria for a subharmonic function to be continuous at a point which is related to Privalov's theorem on the Hilbert transform. The Hilbert transform defined as the principal value of $H(g) = \frac{1}{\pi}\int_R g(y)/(x-y)\, dy$ is linked to potential theory because $H(g)$ is the derivative of the potential $f(x) = \frac{1}{\pi}\int \log|x-y| g(y)\, dy$, where $g(y)dy$ is interpreted as a measure on $\mathbb{C}$ supported on $\mathbb{R}+i0$. Because we relate the fluctuations of f with the fluctuations of the potential f^* obtained by moving all mass of the Riesz measure to the real line, the Hilbert transform is relevant: a potential with Riesz measure ρ supported on $\mathbf{R}^+$ is the Hilbert transform of $\frac{1}{\pi}\int_0^r \rho(t)\, dt$.

The homogenization idea is to use the sum of the rotated functions $f^{(k)} = \sum_{j=1}^{2^k} f(e^{2\pi i j/2^k})$ with upper fluctuation
$B_{f^{(k)}}(z) \leq \int_0^\infty \exp(-|\log(\frac{t}{|z|})|2^k)\, d\rho_f(t)$ to estimate the lower fluctuation $A_f(z)$ of f by $A_f(z) \leq \sum_{k=0}^\infty B_{f^{(k)}}(z)$. The relation of upper and lower fluctuations is already effective after finitely many homogenization steps: in the case $k=1$, where $f^{(1)}(z) = f(z) + f(-z)$, the lower fluctuation of f is related to the lower fluctuation of $f^{(1)}$ and the upper fluctuation of f: $A_f \leq A_{f^{(1)}} + B_f$. In the case $k=2$, where $f^{(2)}(z) = f(z) + f(iz) + f(-z) + f(-iz)$, the lower fluctuation of f is related to the already smaller lower fluctuation of $f^{(2)}$ and the upper fluctuations of $f^{(1)}$ and f which reads $A_f \leq A_{f^{(2)}} + B_{f^{(1)}} + B_f$. In the limit $k \to \infty$ of the relations $A_f \leq A_{f^{(k)}} + B_{f^{(k-1)}} + \ldots + B_f$, where $\lim_{k\to\infty} A_{f^{(k)}} = 0$, we are left with $A_f \leq B_f + B_{f^{(1)}} + B_{f^{(1)}} + \ldots$. This estimate is true for a general potential f with a Riesz measure of finite mass. While it is far from optimal in many situations, this expression of the lower fluctuations in terms of upper fluctuations can lead to useful estimates.

The applications we have in mind are mentioned in an appendix. Classes of subharmonic functions $f(z)$ are obtained by measuring the exponential growth of a product of matrices along a dynamical process for which the matrices depend analytically on a parameter z. An example is

$$f_n(z) = \frac{1}{n}\log||\begin{bmatrix} a_n z^2+1 & -a_n z \\ a_n z & 0 \end{bmatrix} \cdots \begin{bmatrix} a_2 z^2+1 & -a_2 z \\ a_2 z & 0 \end{bmatrix}\begin{bmatrix} a_1 z^2+1 & -a_1 z \\ a_1 z & 0 \end{bmatrix}||\,,$$

where $a_k = e^{ki\alpha}$ are complex numbers. The subharmonic function $f(z) = \liminf_{n\to\infty} f_n(z)$ is called a Lyapunov exponent. It measures the asymptotic growth rate of the product of the matrices parametrized by the complex number z. Lyapunov exponents are important quantities in classical and quantum mechanics. They are used to express stochastic properties of a classical dynamical system, while in quantum mechanics, where for certain parameterizations the Riesz measure is called "density of states", the potential can be related to the spectral type of certain operators which is relevant for transport properties.

We give here the following application of the fluctuation theorem: if g is a bounded measurable function on a probability space (X, m) such that $\int_X \log|g(x)|\, dm(x) < \infty$ and if T is a measure preserving transformation on (X, m), then the Lyapunov exponent $\mu(A)$ of $A(x) = \begin{bmatrix} E-\lambda g(x) & -1 \\ 1 & 0 \end{bmatrix}$ is positive for 'most' large λ. This result must be compared with other results in that area: for realanalytic g [10], [17], for smooth g see [19], nondeterministic g [6]. Here, no assumptions on T are made and essentially no regularity for g assumed. On the other hand, we don't know whether the Lyapunov exponent is positive for **all** large enough λ, only for most large λ. For a similar result but with E as a parameter, see [18]. An other application is a criterium for the existence of singular continuous spectrum for one-dimensional Jacobi matrices which is based on the Ishii-Pastur-Kotani theorem (see [5]): if the spectrum has positive Lebesgue density at some point E, where the Lyapunov exponent is positive, then there is some singular spectrum.

2 A result on homogenization

Let $z \mapsto f(z)$ be a subharmonic function in the complex plane $\mathbb{C}$ having a harmonic majorant. By the Riesz decomposition theorem, it is the sum of a potential $\int_{\mathbb{C}} \log|z-w|\, d\mu_f(w)$ and a harmonic function

h_f. The maximal function $M_f(z) = \sup_{|w|=|z|} f(w)$ is a radial subharmonic function by Hadamard's three circle theorem (i.e. [2]). Also the mean function $C_f(z) = (2\pi)^{-1} \int_0^{2\pi} f(e^{i\theta} z)\, d\theta$ is a radial subharmonic function. Define $m_f(z) = \inf_{|w|=|z|} f(w)$. Define a measure ρ_f on $\mathbb{R}^+$ by $\rho_f([a,b]) = \int_{a \leq z \leq b} d\mu(z)$. By the Poisson-Jensen's formula (see i.e. [7]), $C_f(r) = f(z) + \int_0^r \log(r/t)\, d\rho_f(t)$. (We use a notation, where $d\rho_f(t) = \rho(t)\, dt$ if ρ is absolutely continuous with respect to Lebesgue measure).

While estimates for the **upper angular fluctuation** $B_f(z) := M_f(z) - C_f(z) \in \mathbb{R}^+$ can be achieved usually quite well due to its boundedness, finding bounds for the **lower angular fluctuation** $A_f(z) := C_f(z) - m_f(z) \in \mathbb{R}^+ \cup \{\infty\}$ can be difficult. In general, there are no finite upper bounds on $A_f(z)$ as for example when the Riesz measure $d\mu = (2\pi)^{-1} \Delta f$ of f has atoms.

The Homogenization Theorem below gives an upper bound of A_f in terms of the radial behavior of the mean function C_f.

We define $\mathcal{K}_l(\rho_f)(r) = 2^{-l} \sum_{k=l}^{\infty} \int_0^{\infty} \exp(-|\log(r/t)| 2^k)\, d\rho_f(t)$, where we understand $\mathcal{K}_l(\rho_f) = \infty$ at points, where the sum does not have a finite limit e.g. when the measure ρ_f has an atom at r. We write shortly $\mathcal{K} = \mathcal{K}_0$. Our main result is:

Theorem 2.1 (Homogenization theorem) *If f is a potential, then $A_f(z) \leq \mathcal{K}(\rho_f)(|z|)$. If additionally $f(z) = f(-z)$ for all $z \in \mathbb{C}$, then $A_f(z) \leq \mathcal{K}_1(\rho_f)(|z|)$.*

Remarks.

1) For a general subharmonic function, we additionally could estimate the fluctuations of the harmonic function $h(z)$ on $|z| = r$ using Harnack's inequality.

2) An application of Theorem (2.1) is to give conditions, under which a subharmonic function is continuous at a point $z = 0$. We will later see how to estimate $\mathcal{K}_1(\rho_f)$ and see that if $||\rho||_p < \infty$ $(1 < p < \infty)$, then $\mathcal{K}_1(\rho_f)(|z|) \to 0$ for $|z| \to 0$ assuring continuity of f at 0. Actually, $K(\rho_f)(|z|) \to 0$ too, because Hilbert transform results show that if ρ_f is in $L^p(\mathbb{R}^+)$ with $1 < p \leq \infty$, then f' is in L^p and so f continuous at z. Such sufficient criteria for the continuity is by no means necessary. There are functions $f(z) = \int_{\mathbb{C}} \log|z-w|\, d\mu(w)$ with rotationally symmetric $\mu = \sum_n \frac{1}{2^n} 1_{|z|=1/n}$ supported on the union of a countable set of concentric circles leading to an atomic ρ but where f is continuous at 0. The radial measure ρ does not determine the continuity property of f at 0: with $\mu = \sum_n 1_{\{1/n\}} \frac{1}{2^n}$, we get a discontinuous f but the same ρ. The corollary applies however in examples like $\mu_f(z) = g(|z|) 1_C(z)/|z|$, where C is a measurable subset of the plane and $g \in L^p(\mathbb{R}^+)$ in which case f is continuous.

3) The measure ρ and the distribution $(f^*)' = f'_\rho$ are related by the Hilbert transform. Therefore, $f^*(x) = \int_{\mathbb{R}} \log|x-t|\, d\rho(t)$ on $\mathbb{R}$ is the Hilbert transform of $g(x) = \frac{1}{\pi} \int_0^x d\rho(t)$. $(f(x) + ig(x)$ extends to a function $f(x+iy) + ig(x+iy)$ which is analytic in the upper half plane.) Privalov's theorem assures that the Hilbert transform preserves α-Lipshitz continuity ($\sup_x |f(x+h) - f(x)| \leq C|h|^\alpha$, $0 < \alpha \leq 1$) It also preserves the L^p property as can be seen by the Fourier transform. If ρ is in L^∞, then f is α-Lipschitz continuous for every $\alpha > 0$. The L^p-invariance property of the Hilbert transform also assures continuity of f at 0 if $\rho \in L^p$.

3 Homogenization

Let $\xi_k = \exp(2\pi i/2^k)$ denote the 2^k'th primitive root of unity.

Definition. For a subharmonic function f, define **the k'th homogenized subharmonic function** $f^{(k)}(z) = \sum_{j=1}^{2^k} f(\xi_k^j z)$.

Lemma 3.1 (Homogenization) *For a subharmonic function f one has $A_f(z) \leq \sum_{k=0}^{\infty} B_{f^{(k)}}(z)$. If $f(z) = f(\xi_l z)$ for all $z \in \mathbb{C}$, then $A_f(z) \leq \frac{1}{2^l} \sum_{k=l}^{\infty} B_{f^{(k)}}(z)$.*

Proof. Define $\xi_{k,m} = \xi_k \xi_{k-1} \cdots \xi_m$ for $m \leq k$. We apply recursively the identity $f^{(k)}(z) = f^{(k-1)}(z) + f^{(k-1)}(\xi_k z)$, to get

$$\begin{aligned} f^{(k)}(z) &= f^{(k-1)}(z) + f^{(k-1)}(\xi_k z) \\ &= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + f^{(k-2)}(\xi_{k,k-1} z) \\ &= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + f^{(k-3)}(\xi_{k,k-1} z) + f^{(k-3)}(\xi_{k,k-2} z) \\ &\dots \\ &= f^{(k-1)}(z) + f^{(k-2)}(\xi_k z) + \dots + f^{(1)}(\xi_{k,3} z) + f^{(0)}(\xi_{k,2} z) + f^{(0)}(\xi_{k,1} z) \end{aligned}$$

so that

$$f^{(0)}(\xi_{k,1} z) = f^{(k)}(z) - \sum_{j=0}^{k-1} f^{(j)}(\xi_{k,j+1} z) .$$

Subtracting the angular means on each side gives

$$\begin{aligned} f(\xi_{k,1} z) - C_f(z) &= f^{(k)}(z) - 2^k C_f(z) - \sum_{j=0}^{k-1} (f^{(j)}(\xi_{k,j+1} z) - 2^j C_f(z)) \\ &\geq (f^{(k)}(z) - 2^k C_f(z)) - \sum_{j=0}^{k-1} (M_{f^{(j)}}(z) - C_{f^{(j)}}(z)) . \end{aligned}$$

We apply this to the subharmonic functions $f_n(z) = 2^{n-1} \int_{-2^{-n}}^{2^{-n}} f(z e^{i2\pi x})\, dx$ for which $M_{f_n^{(j)}}(z) \leq M_{f^{(j)}}(z), C_{f_n}^{(j)} = C_f^{(j)}$. and so $B_{f_n^{(j)}} \leq B_{f^{(j)}}$. This gives

$$f_n(\xi_{k,1} z) - C_f(z) \geq (f_n^{(k)}(z) - 2^k C_f(z)) - \sum_{j=0}^{k-1} B_{f^{(j)}}(z) .$$

Taking the limit $k \to \infty$ gives, using $f_n^{(k)}(z) - 2^k C_f(z) = 0$ for $k \geq n$ and $\xi_{k,1} \to 1$ for $k \to \infty$

$$(f_n(z) - C_{f_n}(z)) \geq - \sum_{j=0}^{\infty} B_{f^{(j)}}(z) .$$

Since the right hand side is n-independent, we obtain

$$\inf_n (f_n(z) - C_f(z)) \geq - \sum_{j=0}^{\infty} B_{f^{(j)}}(z) .$$

Because subharmonic functions are upper semicontinuous, we have for given $\alpha > 0$ and $z \in \mathbb{C}$ that $f_n(z) < f(z) + \alpha$ for large n and therefore $\inf_n f_n(z) \leq f(z)$ and obtain

$$(f(z) - C_f(z)) \geq - \sum_{j=0}^{\infty} B_{f^{(j)}}(z)$$

which is equivalent to the claim.

The case with symmetry $f(z) = f(\xi_l z)$ goes similar but starting with

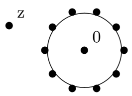
$$2^l f(\xi_{k,1} z) = f^{(k)}(z) - \sum_{j=l}^{k-1} f^{(j)}(\xi_{k,j+1} z) .$$

□

Remark. The case with more symmetry applies when $f(z) = g(z^{2^l})$ for all $z \in \mathbb{C}$. If $l = 1$ that is the case when $f(z) = f(-z)$ is especially relevant in applications.

4 Proof of the homogenization theorem

We estimate first the upper fluctuation bounds $B_{f^{(k)}}(z)$, when f is the potential $f(z) = \log|z-a|$. In this case, $f^{(k)}(z)$ has a Riesz measure representing a circular equidistributed chain of charges with total charge 2^k.



The crucial fact is that for m charges located on the m'th roots of unity of the unit circle, the upper fluctuation $B_f(z)$ of the potential $f(z) = \log|z^m - 1|$ stays bounded above for $m \to \infty$ and decays exponentially $B_f(z) \leq \exp(-|\log(|z|)|m)$ for $|z| \neq 1$.

Lemma 4.1 (Potential of a circular chain of charges) *If $f(z) = \log|z-a|$ with $|a| = r$, there are sharp estimates*

$$B_{f^{(k)}}(z) \leq \begin{cases} \log|1 + (\frac{|z|}{r})^{2^k}| , & \text{for } |z| \leq r , \\ \log|1 + (\frac{r}{|z|})^{2^k}| , & \text{for } |z| \geq r . \end{cases}$$

Proof. Factorisation gives for every $m \in \mathbb{N}$

$$(z^m - 1) = \prod_{j=1}^{m} (z - \xi^j) ,$$

where ξ is the primitive complex m-th root of unity. Therefore, $f^{(k)}(z) = \log|\prod_{j=1}^{2^k}(z-\xi^j)| = \log|z^{2^k} - 1|$ for $f(z) = \log|z-1|$.

Assume now that $f(z) = \log|z-a|$ with $|a| = r$. For $|z| < r = |a|$ and $f^{(k)}(z) = 2^k \log(r) + \log|(z/r)^{2^k} - 1|$, one gets

$$B_{f^{(k)}}(z) \leq \log|1 + (\frac{|z|}{r})^{2^k}| .$$

For $|z| > r = |a|$ and with $f^{(k)}(z) = 2^k \log|z| + \log|1 - (r/z)^{2^k}|$

$$B_{f^{(k)}}(z) \leq \log|1 + (\frac{r}{|z|})^{2^k}| .$$

□

The proof of Theorem (2.1) is obtained along the following steps:

(i) If $f(z) = \log|z-a|$ then $B_{f^{(k)}}(z) \leq \exp(-|\log(r/|z|)|2^k)$. Proof: Because $\log(1+x) \leq x$ for all $x \geq 0$, we obtain $\log|1 + (|z|/r)^{2^k}| \leq (|z|/r)^{2^k} = \exp(-|\log(r/|z|)|2^k)$ for $|z| > r$ and

$$\log|1 + (r/|z|)^{2^k}| \leq (r/|z|)^{2^k} = \exp(-|\log(r/|z|)|2^k)$$

for $|z| < r$. Apply Lemma (4.1).

(ii) "Averaging subharmonic functions does not increase fluctuations". If $\sum_i \alpha_i = 1, \alpha_i > 0$ and $f = \sum_i \alpha_i f_i$, where f_i are subharmonic functions which are potentials, then $M_f(z) \leq \sum_i \alpha_i M_{f_i}(z)$ and $C_f(z) = \sum_i \alpha_i C_{f_i}(z)$ so that $B_f(z) \leq \sum_i \alpha_i B_{f_i}(z)$. More generally, if $x \mapsto f_x(z)$ is a random variable over a probability space $(X, \mathcal{A}, \mu)$ for all z and $z \mapsto f_x(z)$ is subharmonic, then $f(z) = \int_X f_x(z)\, d\mu(x)$ is subharmonic and $B_f(z) \leq \int_X B_{f_x}(z)\, d\mu(x)$.

(iii) Especially, if $f(z) = \int_{\mathbb{C}} f_w\, d\mu(w)$ where $d\mu$ is a finite, positive Borel measure in $\mathbb{C}$ and f_w are subharmonic potentials for which B_{f_w} is bounded above, then f is a subharmonic potential and $B_f(z) \leq \int_{\mathbb{C}} B_{f_w}\, d\mu(w)$.

(iv) If $\mathrm{supp}(d\mu) \subset \{|z| = r\}$, then $B_{f^{(k)}}(z) \leq \exp(-|\log(r/|z|)|2^k)$.
Proof. From (i), the result follows for measures on $\{|z| = r\}$ which are finite point measures. The general case follows from (iii) by integration over the angles.

(v) For a general finite Borel measure ρ on $\mathbb{R}^+$, we have

$$B_{f^{(k)}}(z) \leq \int_0^\infty \exp(-|\log(\frac{t}{|z|})|2^k)\, d\rho_f(t) .$$

Proof. By (ii) and (iv) by integration over the radius.

(vi) The claim of the theorem: from Lemma (3.1) and (v)

$$A_f(z) \leq \sum_{k=0}^\infty B_{f^{(k)}}(z) \leq \sum_{k=0}^\infty \int_0^\infty \exp(-|\log(\frac{t}{|z|})|2^k)\, d\rho(t) = \mathcal{K}(\rho)(z)$$

allowing the right hand side to be $+\infty$.
The estimate of the lower fluctuation of a subharmonic function obtained in the proof (v) is of seperate interest:

Corollary 4.1 (Estimate of upper fluctuations) *For a general potential f, the upper fluctuations of f can be estimated by*

$$B_{f^{(k)}}(z) \leq \int_0^\infty \exp(-|\log(\frac{t}{|z|})|2^k)\, d\rho_f(t) = \int_0^r (t/r)^{2^k}\, d\rho_f(t) + \int_r^\infty (r/t)^{2^k}\, d\rho_f(t) .$$

Especially, for $k = 0$

$$B_f(z) \leq \int_0^\infty \exp(-|\log(\frac{t}{|z|})|)\, d\rho_f(t) \leq \int_0^r (t/r)\, d\rho_f(t) + \int_r^\infty (r/t)\, d\rho_f(t) .$$

This is useful if we know the lower fluctuations of $f^{(1)}$ and the upper fluctuations of f, because $A_f \leq A_{f^{(1)}} + B_f$.

5 The homogenization transformation

The **homogenization transformations**

$$\mathcal{K}_l : \rho \mapsto \frac{1}{2^l} \sum_{k=l}^\infty \int_{\mathbb{R}^+} \exp(-|\log(\frac{r}{t})|2^k)\, d\rho(t)$$

are defined on the linear space of all finite Borel measures ρ on $\mathbb{R}^+ = [0, \infty)$. The image of $\mathcal{K}_l(\rho)$ is a $\mathbb{R}^+ \cup \{+\infty\}$-valued function on $\mathbb{R}^+$.

Proposition 5.1 ($\mathcal{K}_1$ for $\rho \in L^\infty(\mathbb{R}^+)$.) *If $\rho \in L^\infty(\mathbb{R}^+)$ then $\mathcal{K}_1(\rho)(z) \leq 2||\rho||_\infty |z|$. Especially, $\mathcal{K}_1(\rho) \in L^\infty_{loc}(\mathbb{R}^+ \setminus \{0\})$.*

Proof. Using $\int_0^r (t/r)^m\, dt = r/(1+m)$ and $\int_r^\infty (r/t)^m\, dt = r/(m-1)$, we estimate

$$\begin{aligned}
2\mathcal{K}_1(\rho)(r) &= \sum_{k=1}^\infty \int_0^r (\frac{t}{r})^{2^k} \rho(t) dt + \sum_{k=1}^\infty \int_r^\infty (\frac{r}{t})^{2^k} \rho(t) dt \\
&\leq ||\rho||_\infty (\sum_{k=1}^\infty \int_0^r (\frac{t}{r})^{2^k} dt + \sum_{k=1}^\infty \int_r^\infty (\frac{r}{t})^{2^k} dt) \\
&= 2||\rho||_\infty r \sum_{k=1}^\infty \frac{2^k}{2^{2k}-1} \leq 2||\rho||_\infty r \sum_{k=1}^\infty \frac{1}{2^k - 2^{-k}} \leq 4||\rho||_\infty r \; .
\end{aligned}$$

□

The next proposition can be applied if C_f is Lipschitz on some interval $[r/a, ra]$, in which case ρ_f is bounded on that interval.

Proposition 5.2 (**$\mathcal{K}$ for $\rho \in L^\infty_{loc}(\mathbb{R}^+)$**) *Assume $\rho(s) \leq M$ for almost all $s \in [r/a, ar]$, with $a > 1$ and let $c = \rho(\mathbb{C})$ be the total mass of ρ. Then $\mathcal{K}(\rho)(z) \leq 2Mr(1-1/a) + (c/2)/(1-1/a)$. Especially, $\mathcal{K}(\rho) \in L^\infty_{loc}(\mathbb{R}^+)$. Taking $a = 2$, we get $\mathcal{K}_1(\rho)(r) \leq Mr + c$ for $r \in [r/2, 2r]$.*

Proof. From $\int_{r/a}^r (t/r)^m\, dt = (1-1/a^{m+1})r/(1+m) \leq (1-1/a)r/(1+m)$, $\int_r^{ra} (r/t)^m\, dt = (1-1/a^{m-1})r/(m-1) \leq (1-1/a)r/(m-1)$ we get the contribution from the mass in $[r/a, ar]$.
The term $(c/2)/(1-1/a)$ is obtained by assuming the worst case, where the rest of the mass is assumed to be a Dirac measure of maximal mass c at the point r/a. □

A measure ρ on the real line is α-continuous $0 < \alpha \leq 1$, if there exists a constant C such that $\int_a^b \rho(x)\, dx \leq C|b-a|^\alpha$ for all $a < b$. In other words, ρ is α-continuous if its antiderivative $\int_0^r d\rho(t)\, dt$ is Hölder continuous of order α. We call the later also α-Hölder or α-Lipschitz continuous.

Proposition 5.3 (**$\mathcal{K}_1$ for α-continuous ρ.**) *If ρ is α-continuous of compact support, then there exists a constant K such that $\mathcal{K}_1(\rho)(z) \leq K|z|^\alpha$.*

Proof. Divide the interval $[0, r]$ into n small intervals and denote by x_l the maximum of $(x/r)^\beta$ in $[l/n, (l+1)/n]$. Using the α-Hölder continuity, we get

$$\begin{aligned}
\int_0^r x^\beta / r^\beta \; d\rho(x) &\leq \sum_{l=1}^n (x_k^\beta / r^\beta) C r^\alpha / n^\alpha \\
&\leq C r^\alpha \sum_{l=1}^n (r^\beta l_k^\beta / (n^{\alpha+\beta} r^\beta)) \\
&\leq C_1 r^\alpha \; .
\end{aligned}$$

A change of variables $y = r^2/x$, $dy = -r^2 dx/x^2 = -(y^2/r^2) dx$ for the second integral $\int_r^\infty (r^\beta/x^\beta)\; d\rho(x) = \int_0^r (y^{\beta-2}/r^{\beta-2})\; d\rho(r^2/y)$ reduces it to the first case. □

Remark. Proposition 5.3 is strenghend by Privalov's theorem: the Hilbert transform preserves α-Hölder continuity. Define $f(x) = \int \log|x-y|\; d\rho(y)$ and $g(x) = \frac{1}{\pi}\int_0^x d\rho(y)$. Since f' is the Hilbert transform of ρ and because the Hilbert transform commutes with differentiation (the Fourier transform diagonalizes both), the function f is the Hilbert transform of $g(r) = \frac{1}{\pi}\int_0^r \rho(t)\; dt$. If g is α-Hölder continuous, then f is α-Hölder continuous: for all $a < b$,

$$|g(x) - g(y)| \leq C|a-b|^\alpha \Rightarrow |f(a) - f(b)| \leq K|a-b|^\alpha$$

But this is equivalent to the fact that $f'(x) = \int d\rho(y)/(x-y)$ is α-continuous if ρ is α-continuous. Because the Hilbert transform commutes with differentiation, Privalov's result is true also for $\alpha > 1$. Especially, if μ is supported on $\mathbf{R}^+$, has compact support and is in $C^k(\mathbb{R})$, then the potential of μ is in C^{k+1}, a known fact in potential theory (see [1] Corollary 4.5.4).

Proposition 5.4 (**$\mathcal{K}_1$ for $\rho \in L^p(\mathbb{R}^+)$**) *Given $1 < p < \infty$ define q by $1/p + 1/q = 1$. For $\rho \in L^p(\mathbb{R}^+)$,*

$$2\mathcal{K}_1(\rho)(z) \leq ||\rho||_p |z|^{1/q} (2/q)^{1/q} 1/(1 - 2^{-1/q}) \; .$$

Especially, $\mathcal{K}_1(\rho) \in L^q_{loc}(\mathbb{R}^+ \setminus \{0\})$.

Proof. Write $r = |z|$. With Hölder's inequality one has

$$\int_0^r (\frac{t}{r})^m \rho(t) dt \leq ||(\frac{t}{r})^m||_{L^q(0,r)} ||\rho||_p = (\frac{r}{qm+1})^{1/q} ||\rho||_p$$

and

$$\int_r^\infty (\frac{r}{t})^m \rho(t) dt \leq ||(\frac{r}{t})^m||_{L^q(r,\infty)} ||\rho||_p = (\frac{r}{qm-1})^{1/q} ||\rho||_p \; ,$$

so that

$$\int_0^r (\frac{t}{r})^m \rho(t) dt + \int_r^\infty (\frac{r}{t})^m \rho(t) dt \leq r^{1/q} (\frac{2qm}{q^2m^2-1})^{1/q} ||\rho||_p \leq r^{1/q} (2/qm)^{1/q} ||\rho||_p \; .$$

Summing up the right hand side for $m = 2^k$ gives

$$2\mathcal{K}_1(\rho)(z) \leq ||\rho||_p |z|^{1/q} (2/q)^{1/q} \sum_{k=1}^\infty (2^{1/q})^{-k} = ||\rho||_p |z|^{1/q} (2/q)^{1/q} 1/(1 - 2^{-1/q}) \; .$$

□

Examples. For $p = 2$, we get $\mathcal{K}_1(\rho)(z) \leq (2+\sqrt{2})||\rho||_2 \sqrt{|z|}$. In the limit $p \to \infty$ we get the L^∞ case in Proposition (5.1). For $p = 1$, the estimate breaks down. (This break down is related to the fact that the Hilbert transform can not be extended to L^1.)

Definition. Given a subharmonic function f, the potential $f^*(r) = f_{\rho_f}(r) = \int_0^\infty \log|r-s|\; d\rho_f(s)$ is called the **$*$-function** of f. Define

$$C_\rho(r) = \frac{1}{2\pi} \int_0^{2\pi} f^*(r \exp(i\theta))\; d\theta \; .$$

The function f_ρ is obtained from f by sweeping all masses to a positive (or negative) real axes. The $*$ function appears in results on growth estimates for subharmonic functions.

Proposition 5.5 (Bounds by potential of $*$-function)

$$C_\rho(r) - f_\rho(r) \leq \mathcal{K}(\rho)(r) \leq 2(C_\rho(r) - f_\rho(r)) \; .$$

If $f(z) = f(\xi_l z)$ for all $z \in \mathbb{C}$, then $\mathcal{K}_l(\rho)(r) \leq 2^{-(l-1)}(C_\rho(r) - f_\rho(r))$.

Proof. We can assume that ρ has no atom at r because otherwise all three terms are $+\infty$.

(i)

$$\begin{aligned}
\int_0^r \log|r-s|\; d\rho(s) &= \int_0^r \log|1 - \frac{s}{r}|\; d\rho(s) + \int_0^r \log|r|\; d\rho(s) \\
\int_r^\infty \log|r-s|\; d\rho(s) &= \int_r^\infty \log|1 - \frac{r}{s}|\; d\rho(s) + \int_r^\infty \log|s|\; d\rho(s) \; .
\end{aligned}$$

(ii)

$$\int_0^r \log|1-\frac{s}{r}|\,d\rho(s) = -\sum_{k=1}^{\infty}\int_0^r (\frac{s}{r})^k/k\,d\rho(s)$$
$$\int_r^{\infty} \log|1-\frac{r}{s}|\,d\rho(s) = -\sum_{k=1}^{\infty}\int_r^{\infty} (\frac{r}{s})^k/k\,d\rho(s)\,.$$

(iii) For $0 < a < 1$,

$$\sum_{k=0}^{\infty} a^{2^k} \le 2\sum_{k=1}^{\infty}\frac{a^k}{k}\,.$$

Proof. Match each term on the left with a sum on the right hand side: $a \le 2a/1, a^2 \le 2a^2/2, a^4 \le 2(a^3/3+a^4/4), a^8 \le 2(a^5/5+a^6/6+a^7/7+a^8/8)$, etc.

(iv) From (iii), we get

$$\begin{aligned}\mathcal{K}(\rho)(r) &= \sum_{k=0}^{\infty}\int_{\mathbb{R}^+}\exp(-|\log(r/t)|2^k)\,d\rho(t)\\ &\le 2\sum_{k=0}^{\infty}\int_{\mathbb{R}^+}\exp(-|\log(r/t)|k)/k\,d\rho(t)\\ &= -2\left(\int_0^r \log|1-\frac{s}{r}|\,d\rho(s)+\int_r^{\infty}\log|1-\frac{r}{s}|\,d\rho(s)\right)\\ &= -2\left(\int_0^{\infty}\log|r-s|\,d\rho(s)-\int_0^r\log|r|\,d\rho(s)-\int_r^{\infty}\log|s|\,d\rho(s)\right)\end{aligned}$$

so that

$$\mathcal{K}(\rho)(r) \le 2\left(-f_\rho(r)+\int_0^r \log|r|\,d\rho(s)+\int_r^{\infty}\log|s|\,d\rho(s)\right)\,.$$

(v) Subtracting and adding $\int_0^r \log(s)\,d\rho(s)$ in the second and third term respectively gives, using $\rho(\{r\}) = 0$

$$\mathcal{K}(\rho)(r) \le 2\left(-f_\rho(r)+\int_0^r \log(\frac{r}{s})\,d\rho(s)+f_\rho(0)\right)\,.$$

(vi) With the Poisson-Jensen formula [2] $C_\rho(r) = \int_0^r \log(\frac{r}{s})\,d\rho(s)+f_\rho(0)$, the estimate in (v) can be rewritten as

$$\mathcal{K}(\rho)(r) \le 2(C_\rho(r)-f_\rho(r))\,.$$

(vii) The inequality $-(f_\rho(r)-C_\rho(r)) \le \mathcal{K}(\rho)(r)$ is a consequence of the Homogenization theorem applied to the case, when $d\mu$ has support in the positive real line.

(viii) For $f(z) = f(\xi_l z)$, things are similar. □

Remarks.
1) The inequalities can be rewritten as

$$A_{f^*}(r) \le \mathcal{K}(\rho_f)(r) \le 2A_{f^*}(r)\,.$$

If $f(z) = f(-z)$, then $\mathcal{K}_1(\rho_f)(r) \le A_{f^*}(r)$. While the lower bound $A_{f^*}(r) \le \mathcal{K}(\rho_f)(r)$ is sharp as the example shows, when $f = f^*$, we don't expect the upper bound to be optimal. Is a better estimate $\mathcal{K}(\rho_f)(r) \le aA_{f^*}(r)$ with $a < 2$ possible?
2) It follows that $\mathcal{K}(\rho)(r)$ is finite if ρ is absolutely continuous with respect to the $\alpha > 0$-dimensional Hausdorff measure in an open interval containing r.
3) The potential $f(x) = \int_{\mathbb{R}^+}\log|x-w|\,d\rho(w)$ is the Hilbert transform of the function $h(x) = \int_0^x d\rho(y)$.

The Hilbert transform $f \mapsto \frac{1}{x}\star f$ maps α-Lipshitz spaces onto itself. It also is a map on $L^p(\mathbb{R})$ into itself. Especially, if ρ is bounded, then h is Lipschitz and f is α-Hölder for all $\alpha \in (0,1)$. Because C_f is Lipschitz if ρ is bounded, Proposition (5.5) implies that if ρ is bounded, then $\mathcal{K}(\rho)$ is sandwiched between two Hölder continuous maps.

An immediate consequence of these bounds is:

Corollary 5.1 *If f^* is continuous at 0, then f is continuous at 0.*

Proof. If f^* is continuous, then $C_\rho(r)-f_\rho(r) = C_{f^*}(r)-f^*(r)$ goes to zero for $r \to 0$. The inequality (5.5) assures $\mathcal{K}(\rho)(r) \to 0$ and the Homogenization theorem assures $A_f(z) \le \mathcal{K}(\rho)(r) \to 0$ for $|z| \to 0$ assuring continuity of f at 0. □

Remark. The converse is of course not true. There are (i.e. some rotational symmetric) subharmonic functions f which are continuous at 0 but for which f^* is not continuous.

Corollary 5.2 *If ρ is α-continuous in a neighborhood of 0, then f is α-Lipshitz continuous at 0.*

Proof. f ρ is α-continuous, then $g(x) = \int_0^x \rho(t)\,dt$ is α-Lipschitz continous. By Privalov's theorem, its Hilbert transform $f^*(x)$ is α-Lipschitz continuous. The inequality in Proposition 5.5 provides $\mathcal{K}(\rho)(r) \le Cr^\alpha$. The Homogenization theorem assures $A_f(z) \le \mathcal{K}(\rho)(r) \le Cr^\alpha$ so that $|f(r)-f(0)| \le C_1 r^\alpha$. □

6 Radially mollified subharmonic functions

The homogenization theorem is especially useful if ρ_f is bounded. It can then applied to cases, where ρ is smeared out radially in such a way that the mollified ρ appears from a radially mollified subharmonic function.

Definition. Let ϕ be smooth positive mollifier function on $\mathbb{R}^+$ of compact support $K_\phi \subset \mathbb{R}^+$ satisfying $\phi(x) = \phi(1/x)$ and $||\phi||_1 = 1$. If f is a subharmonic function, define a **radially mollified function** as $f_\phi(z) = \int_{\mathbb{R}} f(\beta z)\,d\phi(\beta)$.

Lemma 6.1 *If f is a subharmonic potential, then $f_\phi(z)$ is a subharmonic potential, ρ_{f_ϕ} has the same smoothness as ϕ and $|\rho_{f_\phi}(r)| \le ||\phi||_\infty \rho_f(K_\phi r)$.*

Proof. A positive average of subharmonic functions $f_\beta(z) = f(\beta z)$ is subharmonic.

Taking the Laplacian in the distributional sense of $f_\phi(z) = \int_{\mathbb{R}} f(\beta z)\,d\phi(\beta)$ shows $d\mu_{f_\phi} = \int_0^\infty d\mu(\beta z)\,d\phi(\beta)$ so that f_ϕ is a potential.

The function $\rho_{f_\phi}(r) = \int_0^\infty \rho(\beta r)\,d\phi(\beta) = \int_0^\infty \rho(\beta^{-1}r)\,d\phi(\beta) = \int_{-\infty}^\infty \tilde{\rho}(r'-\beta')\tilde{\phi}(\beta')\,d\beta' = (\tilde{\phi}\star\tilde{\rho})(r')$ is in logarithmic coordinates $r' = \log(r)$ a convolution with $\tilde{\phi}(r') = \phi(\exp(r'))$ and has therefore the same smoothness as ϕ.

The estimate $|\rho_{f_\phi}(r)| \le ||\phi_\infty||\rho(K_\phi r)$ is obtained by assuming ρ to have an atom of mass $\rho(K_\phi r)$ at r in which case we have equality. □

Remark. While we could also smooth out f using the standard smoothing technique in potential theory, radial smoothing will be more convenient for the applications later on.

Notation. Denote by $|Y|$ the Lebesgue measure of a measurable subset Y of $\mathbb{R}$. For $x \in \mathbb{R}$, call $\limsup_{\epsilon\to 0} |Y \cap [x-\epsilon, x+\epsilon]|/(2\epsilon)$ the Lebesgue density of Y at x. We have seen that for any subharmonic function f

$$\limsup_{\lambda\to 0} \inf_\theta f(\lambda \exp(i\theta)) = f(0) .$$

The new proof to this fact given in the next Corollory illustrates to use the homogenisation method.

Corollary 6.1 *Let f be an arbitrary subharmonic function and let $z \in \mathbb{C}$ be given so that $f(z) > -\infty$. For every $\delta > 0$ and every line $z + \lambda e^{i\theta}$ through z, the Lebesgue density of $Y = \{\lambda \in \mathbb{R} \mid f(z+\lambda e^{i\theta}) > f(z) - \delta\}$ at $\lambda = 0$ is equal to 1.*

Proof. Because the claim is obviously true for harmonic functions, it is enough to prove the statement for potentials $f(z) = \int_{\mathbb{C}} \log|z-w| \, d\mu(w)$. After a translation, we can assume $z = 0$. If K is a compact subset of $\mathbb{C}$, then $d\mu(K) < \infty$, otherwise, f would be constant $-\infty$.
We have $B_f(z) = M_f(z) - C_f(z) \to 0$ for $z \to 0$. Let I be an open ball around 0 such that $B_f(z) \leq \delta/3$ for $z \in I$ and therefore $M_f(z) - f(0) \leq \delta/3$ for $z \in I$.
Define $g(z) = (f(z)+f(-z))/2$. We will show that the Lebesgue density of $Y_\delta = \{\lambda \mid g(\lambda e^{i\theta}) \geq g(0) - \delta/3\}$ is 1 near 0. From this we get in $Y \cap I$, $g(\lambda e^{i\theta}) > g(0) - \delta/3 = f(0) - \delta/3$, implies together with $M_f(-\lambda e^{i\theta})) - f(0) \leq \delta/3$ that $f(\lambda e^{i\theta}) = 2g(\lambda e^{i\theta}) - f(-\lambda e^{i\theta}) \geq 2f(0) - 2\delta/3 - (f(0) - \delta/3) = f(0) - \delta$. The claim follows from this because if every $Y_{1/k}$ has Lebesgue density 1, then also the intersection $\bigcap_k Y_{1/k}$ has Lebesgue density 1.

Let ϕ be a radial mollifier function with support in $[1/2, 2]$ and let g_ϕ be the corresponding radially mollified subharmonic potential. Applying Lemma (6.1) and Corollary (5.1) gives

$$A_{g_\phi}(r) \leq \mathcal{K}_1(\rho_\phi)(r) \leq 2||\phi||_\infty r$$

and the right hand side is $\leq \delta/3$ for small enough r. Because $\sup_{\beta\in[1/2,2]} B_g(\beta r) \to 0$ for $r \to 0$ and $A_g(\beta r) \leq \delta/3$ for a set of $\beta \in [1/2, 2]$ which reaches Lebesgue measure 3/2 for $r \to 0$, the normalized Lebesgue measure of parameters $s \in [r/2, 2r]$ with $A_g(s) \leq \delta/3$ approaches 1 for $r \to 0$. □

Remarks.

1) A subharmonic function is by definition always finely continuous, where the fine topology is defined as the coarsest topology on the complex plane for which all subharmonic functions are continuous. Every fine neighborhood of a point z contains rays passing through z in almost all directions. Assuming $z = 0$ and $f(0) = 0$ without loss of generality, one can define two finely closed sets $U^+ = \{z \mid |f| \geq \epsilon\}$ and $U^- = \{z \mid |f| \leq -\epsilon\}$ which are both nonempty. For any ray γ through 0, also $U^\pm \cap \gamma$ are nonempty finely closed sets and which are sets with zero linear density at zero: $U^\pm \cap [0,\epsilon]e^{i\theta}| = o(\epsilon)$ (see for example Corollary 7.2.4 in [1]).
2) A stronger result follows from works by Beurling,Hall and Baernstein: if E is an open set which is thin at 0, then $\Lambda = \{\lambda \in \mathbb{R}^+ \mid \exists\theta, \lambda e^{i\theta} \in E\}$ is thin and has zero density at 0. Especially, $\Lambda_\theta = \{\lambda \in \mathbb{R}^+ \mid \lambda e^{i\theta} \in E\}$ has zero Lebesgue density at 0.
3) Related is the fact (see [16]) that if $f(z) > -\infty$, then for almost all θ, one has $\lim_{r\to 0} f(z + r\exp(i\theta)) \to f(z)$.
4) It follows especially that there is a sequence $r_n \to 0$ such that

$$\limsup_{n\to\infty} \mathcal{K}(\rho)(r_n) = 0 .$$

This can be seen also as a consequence of the Milloux-Schmidt inequality [9] which implies that for any subharmonic function f with $f(0) > -\infty$ one has $\limsup_{r\to 0} \inf_\theta f(r\exp(i\theta)) = f(0)$. Therefore

$\limsup_{r\to 0} f_\rho(r) = f_\rho(0)$. Together with $\lim_{r\to 0} C_\rho(r) = C_\rho(0)$ which follows from the upper-semicontinuity and the submean inequality for subharmonic functions, we get $\limsup_{r\to 0} 2(f_\rho(r) - C_\rho(r)) = 0$. By Proposition (5.5), one has $\mathcal{K}(\rho)(r_n) = 0$ along a subsequence.
5) This result verifies that for "most" points w near z, $f(w)$ is near $f(z)$. A subharmonic function f behaves in a probabilistic sense like a continuous function: if a sequence $z_n \to z$ is chosen at 'random' in the product probability space of a nested sequence of open intervals $I_n \subset \mathbb{R}$ around 0 (all equipped with the normalized Lebesgue measure and $\bigcap_n I_n = \{0\}$), then with probability 1, a point $x = \{x_n\}$ in the product probability space satisfies $f(z + e^{i\theta}x_n) \to f(z)$. For example, if f is the at 0 discontinuous subharmonic function $f = \sum_{n=1}^\infty 2^{-n} \log|z - 2^{-n}|$ and x_n is an independent, identically distributed sequence of random variables with uniform distribution in $[-1, 1]$ then $f(x_n/n) \to f(0)$ with probability 1.
6) The continuity property of a subharmonic function holds on a much larger set near a given point z. One can take away on every ray through z a (in $\mathbb{C}$) polar set such that on the complement in the ray, the function f is continuous.

7 An application in Ergodic theory

Let $(X, \mathcal{A}, m)$ be a probability space and let $T : X \mapsto X$ be a measure preserving invertible transformation. Given $g \in L^\infty(X)$ for which $\int_X \log|g(x)| \, dm(x)$ is finite. Given $E, \lambda \in \mathbb{R}$, the matrix-valued map $A_\lambda(x) = \begin{bmatrix} E - \lambda g(x) & -1 \\ 1 & 0 \end{bmatrix}$ defines a cocycle

$$(n, x) \mapsto A^n_\lambda(x) = A_\lambda(T^{n-1}x) \cdots A_\lambda(x) .$$

Denote by $\mu(A_\lambda) = \lim_{n\to\infty} n^{-1} \int_X \log||A^n_\lambda(x)|| \, dm(x)$ the Lyapunov exponent of A_λ. Define $f(\epsilon) = \mu(B_\epsilon)$, where $\epsilon = 1/\lambda$ and $B_\epsilon(x) := \begin{bmatrix} \epsilon E + g(x) & -\epsilon \\ \epsilon & 0 \end{bmatrix}$. We have $\mu(A_\lambda) = f(\epsilon) - \log|\epsilon| = f(\epsilon) + \log|\lambda|$. The aim is to estimate the subharmonic function $\epsilon \mapsto f(\epsilon)$ from below, using $f(0) = \int_X \log|g(x)| \, dm(x) > -\infty$. The later will be a standing assumption in this section.

Lemma 7.1 *f is a subharmonic function which is a potential: there exists a constant C such that $f(\epsilon) = C + \int_{\mathbb{C}} \log|\epsilon - w| \, d\mu(w)$. The total mass of the Riesz measure $d\mu$ is 1. If there exists a measure preserving involution S on X such that $g(x) = -g(S(x))$ for almost all $x \in X$, then $f(\epsilon) = f(-\epsilon)$.*

Proof. It is an observation of Herman that the Lyapunov exponent of an analytically parameterized cocycle A is subharmonic as the infimum of subharmonic functions

$$f_n(\epsilon) = 2^{-n} \int_X \log||A^{2^n}_\epsilon(x)|| \, dm(x)$$

satisfying $f_n \leq f_m$ for $n \leq m$.

By the Riesz decomposition theorem, the Lyapunov exponent can be written as $f(\epsilon) = \int_{\mathbb{C}} \log|\epsilon - w| \, d\mu(w) + h(\epsilon)$, where $\epsilon \mapsto h(\epsilon)$ is harmonic. With the constant $C = \mu(B_0)$, we have for $\epsilon \to \infty$ $f(\epsilon) = C + \log|\epsilon| + O(1)$. This shows that the order of f (defined as $\limsup_{r\to\infty} \log|M_f(r)|/\log|r|$) is equal to 1 implying that the harmonic function h in the Riesz decomposition theorem must be constant (see [9]).
If $g(x) = -g(S(x))$, the pointwise Lyapunov exponent

$$\mu(\lambda, x) = \lim_{n\to\infty} n^{-1} \log||A^n_\lambda(x)||$$

(which exists according to Kingman's subadditive ergodic theorem m-almost everywhere), satisfies $\mu(-\lambda, S(x)) = \mu(\lambda, x)$ so that the average over X satisfies $\mu(-\lambda) = \mu(\lambda)$ implying $f(\epsilon) = f(-\epsilon)$. □

Theorem 7.1 *For a fixed dynamical system $T : X \to X$ and cocycle A defined by $g \in L^\infty(X)$, there exists a sequence $\lambda_m \to \infty$ such that*

$$\limsup_{m\to\infty} \mu(A_{\lambda_m}) - \log|\lambda_m| - \int_X \log|g(x)|\, dx \geq 0 \ .$$

Proof. This is consequence of Corollary (6.1) and the homogenization theorem. ◻

Example. For $g(x) = 2x + 2\cos(x)$ we obtain $\limsup_{m\to\infty} \mu(A_{\lambda_m}) - \log(|\lambda_m|/2) \geq 0$.

Theorem 7.2 *Let ϕ be a radial mollifier function. Then*

$$\int_0^\infty \mu(A_{\lambda\beta})\, d\phi(\beta) \geq \log|\lambda| + \int_X \log|g(x)|\, dx - 2||\phi||_\infty/\lambda \ .$$

Let ϕ be a radial mollifier function such that $\rho_\phi(s) \leq M_\phi$ for almost all $s \in [r/2, 2r]$. Then

$$\int_0^\infty \mu(A_{\lambda\beta})\, d\phi(\beta) \geq \log(|\lambda|) + \int_X \log|g(x)|\, dx - 2M_\phi/\lambda - 2c \ ,$$

where $c = \rho(\mathbb{C})$.

Proof. The statements follow from the Homogenization theorem, Corollary (6.1) and the estimates in Proposition (5.1) rsp. Proposition (5.2) which can be applied using Lemma (7.1). ◻

Example. For $g(x) = 2\cos(x)$, where $S(x) = x + \pi$ produces the symmetry $f(\epsilon) = f(-\epsilon)$, this gives $\int_{1/2}^2 \mu(A_{\lambda\beta})\, d\phi(\beta) \geq \log(|\lambda|/2) - 2|\phi|_\infty/|\lambda|$.

The set of parameters for which the cocycle has positive Lyapunov exponent reaches full Lebesgue density at $\lambda \to \infty$. This follows from Corollary (6.1). We state this here in a more quantitative way:

Corollary 7.1 *There exists a constant C such that for all λ_0, the Lebesgue measure of parameters $\lambda \in [\lambda_0/2, 2\lambda_0]$ for which $\mu(A_\lambda) = 0$ is smaller or equal than $C(3\lambda_0/2)/\log|\lambda_0|$.*

Proof. For $\lambda \in [\lambda_0/2, 2\lambda_0]$ we have $\mu(A_\lambda) \leq c_1 + \log(|\lambda_0|) + C_1/|\lambda_0|$ and $\int_{1/2}^2 \mu(A_{\lambda\cos})\, d\phi(\beta) \geq c_2 + \log(|\lambda_0|) - C_2/|\lambda_0|$. Therefore, for each λ_0 the normalized Lebesgue measure x of the subset $\{\lambda \in [\lambda_0/2, 2\lambda_0] \mid \mu(A_\lambda) = 0\ \}$ satisfies $(1-x) \geq (c_1 + \log(|\lambda_0|) - C_2/|\lambda_0|)/(c_2 + \log(|\lambda_0|) + C_1/|\lambda_0|)$ which implies that there exists a constant C such that $x \leq C/\log(\lambda_0)$ for some C. To get the Lebesgue measure, we have to multiply this by $|[\lambda_0/2, 2\lambda_0]| = 3\lambda/2$. ◻

Corollary (6.1) has an application in spectral theory:

Corollary 7.2 *Assume the transfer cocycle A_E of a discrete ergodic selfadjoint Jacobi matrix has a positive Lyapunov exponent at some energy $E \in \mathbb{R}$ for which the spectrum $\sigma(L)$ has positive Lebesgue density $\limsup_{\epsilon\to 0} |\sigma(L) \cap \{[E-\epsilon, E+\epsilon]\}|/(2\epsilon) > 0$. Then L has some spectrum which is not absolutely continuous.*

Proof. We apply Corollary (6.1) to the complex parameterization $E \mapsto A_E$. Let $S = \{E \in \mathbb{R} \mid \mu(A_E) > 0\}$. If $\mu(A_E) > 0$, then $\limsup_{\epsilon\to 0} |S \cap \{[E-\epsilon, E+\epsilon]\}|/(2\epsilon) = 1$. Therefore $S \cap \sigma(L)$ must have positive Lebesgue measure. Ishii-Pastur-Kotani theory (see [5, 4, 15]) tells that on $S \cap \sigma(L)$ the spectrum is not absolutely continuous. ◻

Appendix: On applications of subharmonic functions in ergodic theory and operator theory

The motivation to look at the fluctuation question are situations in dynamical systems theory and spectral theory, where subharmonic functions are relevant:

1) **Ergodic theory: Entropy of smooth diffeomorphisms**.
For a smooth diffeomorphism T on a two-dimensional Riemannian manifold M which preserves a volume measure ρ, the Kolmogorov-Sinai entropy is by Pesin's formula equal to the Lyapunov exponent

$$\lim_{n\to\infty} \int_M n^{-1} \log||dT^n(x)||\, d\rho(x)$$

(see i.e. [11]). The Jacobian $(x, n) \mapsto dT^n(x)$ is called a cocycle. If a complex parameter z is introduced into the problem such that $z \mapsto [A(x, z)]_{ij}$ is analytic, T stays fixed and $A(x, z_0) = dT(x)$, then

$$f(z) = \lim_{n\to\infty} \int_M n^{-1} \log||dT^n(x, z)||\, d\rho(x)$$

is subharmonic as an infimum of a monotonically decreasing sequence of subharmonic functions [10]. (That $\log||dT^n(x, z)||$ is a special case of a much more general lemma leading to Vasentinis theorem (see [16] Lemma 6,4.1) for Banach algebras). One often knows $f(z)$ at some point i.e. $z = 0$ and has therefore with the submean inequality a lower bound on $\int_{|z|=r} f(r\exp(i\theta))\, d\theta$. Because good upper bounds on $f(r\exp(i\theta))$ are usually known, estimating the entropy $f(z_0)$ from below leads to the problem of estimating the fluctuations.

2) **Complex dynamics: Green function of fractals**.
In computer generated pictures of Julia sets J of a polynomial map (see [3]) in the complex plane, one usually plots the potential $f(z) = \int_{\mathbb{C}} \log|z - w|\, d\mu$, where μ is the equilibrium measure supported on the usually fractal set J. This is a subharmonic function. The fluctuations of this function near the boundary of J gives information about the nature of the Julia set J. An other connection between the fields is stessed in Ransford ([16]), where the dimension of a Julia set is a subharmlonic function of the parameter.

3) **Spectral theory: the existence of bound states**.
A bounded selfadjoint Schrödinger operator L on a Hilbert space H defines a quantum mechanical unitary evolution $\psi \mapsto \exp(itL)\psi$. The spectral measure μ_ψ is a Borel measure in the complex plane with Fourier transform $\hat{\mu}(t) = (\psi, \exp(itL)\psi)$. The square $|\hat{\mu}(t)|^2$ is a quantum mechanical probability density. By a theorem of Wiener [12], if $\lim_{n\to\infty} n^{-1} \sum_{k=1}^n |\hat{\mu}(k)|^2 > 0$, where $f_n = n^{-1} \sum_{k=1}^n |\hat{\mu}(k)|^2$, then L has some eigenvalue. If $z \mapsto L(z)$ is a complex parameterization of the operator $L = L(z_0)$, the Cesaro averages $z \mapsto f_n(z)$ are subharmonic function. Consider for example the perturbation problem of an operator $L(z)\psi_n = z(\Delta\psi)_n + V_n\psi_n$ on $l^2(\mathbb{Z}^d)$, where Δ is a discrete Laplacian. We know that $\lim_{n\to\infty} f_n(0) > 0$ because $L(0)$ is a diagonal operator. We would like to understand $f_n(\epsilon)$ for small ϵ because $\lim_{n\to\infty} f_n(\epsilon) > 0$ would imply that the operator $\Delta + (1/\epsilon)V$ had bound states. Estimating the fluctuations of f_n could shed light on that perturbation problem.

4) **Ergodic operators: the absolutely continuous spectrum**.
The density of states $d\mu$ of an ergodic one-dimensional Jacobi operator L is a Borel measure in the complex plane which is supported by the almost sure spectrum of L. By the Thouless formula, the subharmonic potential $f(z) = \int_{\mathbb{C}} \log|z - w|\, d\mu(w)$ is equal to the Lyapunov exponent of the transfer cocycle of the complex parameterized operator $L(z) = L - z$, where z is the energy. Ishii-Pastur-Kotani theory (see [5, 4, 15]) relates the Lyapunov exponent with the absolutely continuous spectrum of L. For example, if the $f(z) > 0$ on some interval I in the real line, then there is no absolutely continuous spectrum on I with probability one.

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