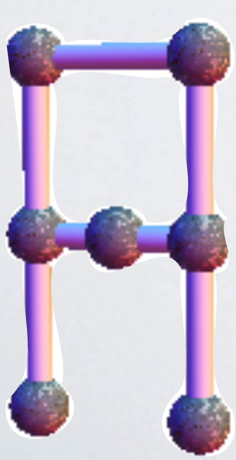
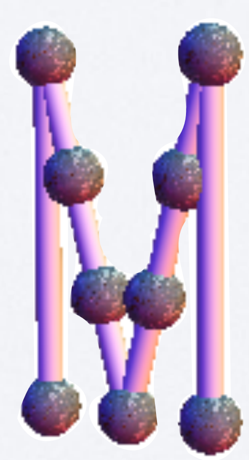
MATHEMATICAL STRUCTURES IN GRAPH THEORY

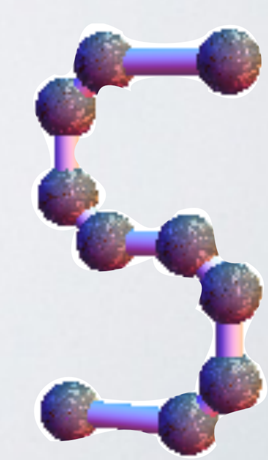
Oliver Knill
Harvard University



JAN 16



2014



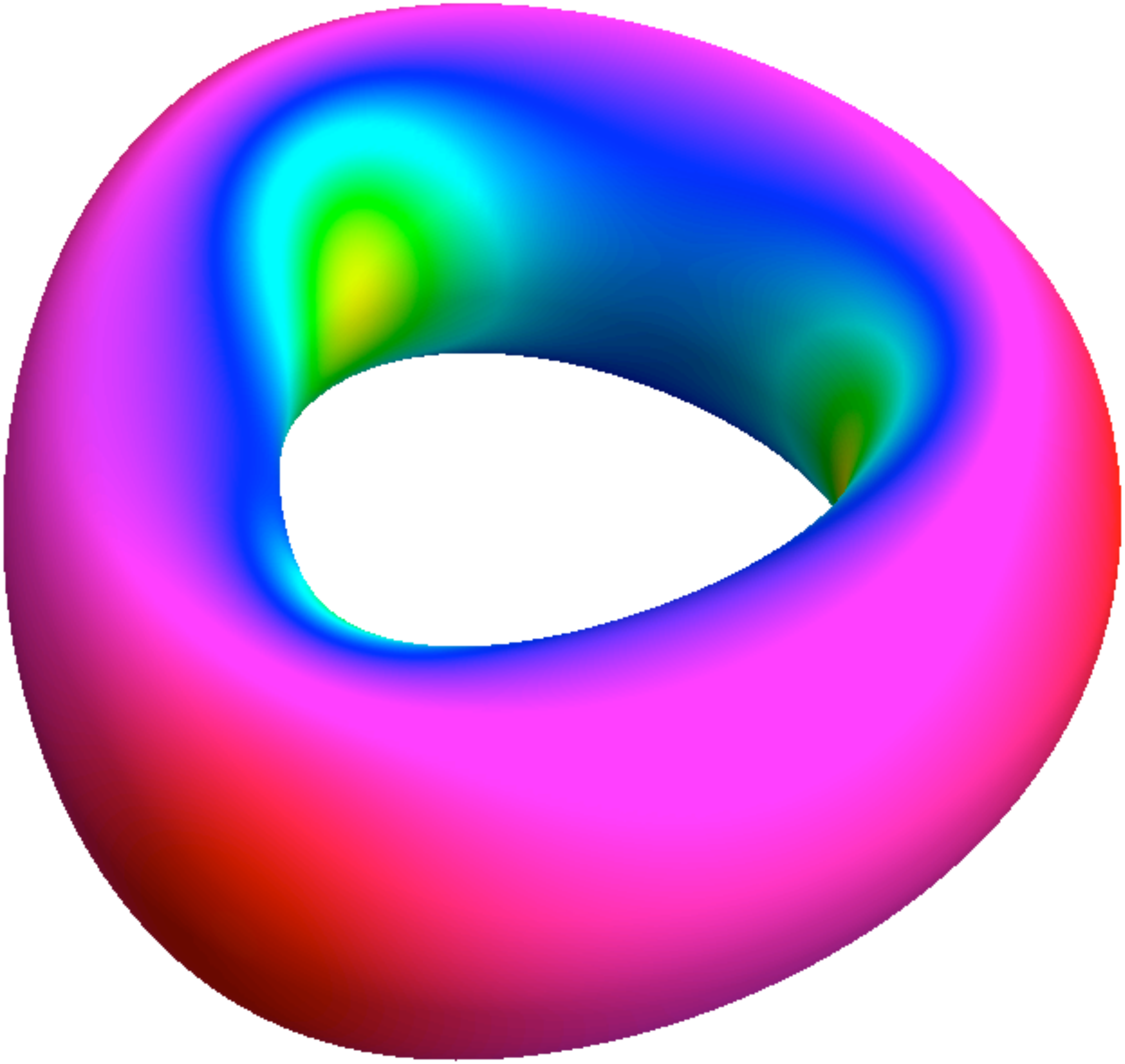
GEOMETRY

GAUSS BONNET CHERN

1777-1855

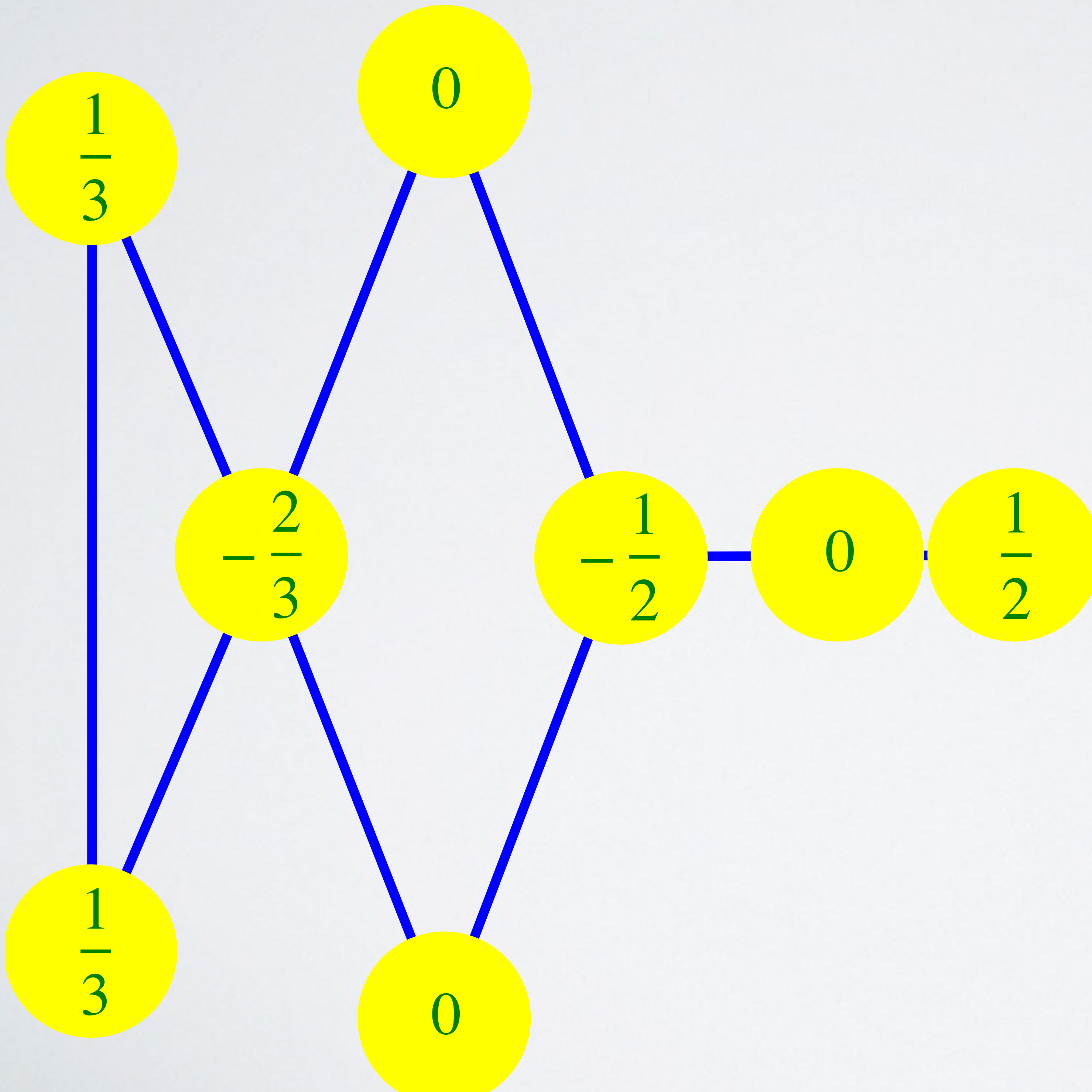
1819-1892

1911-2004



GAUSS BONNET CHERN

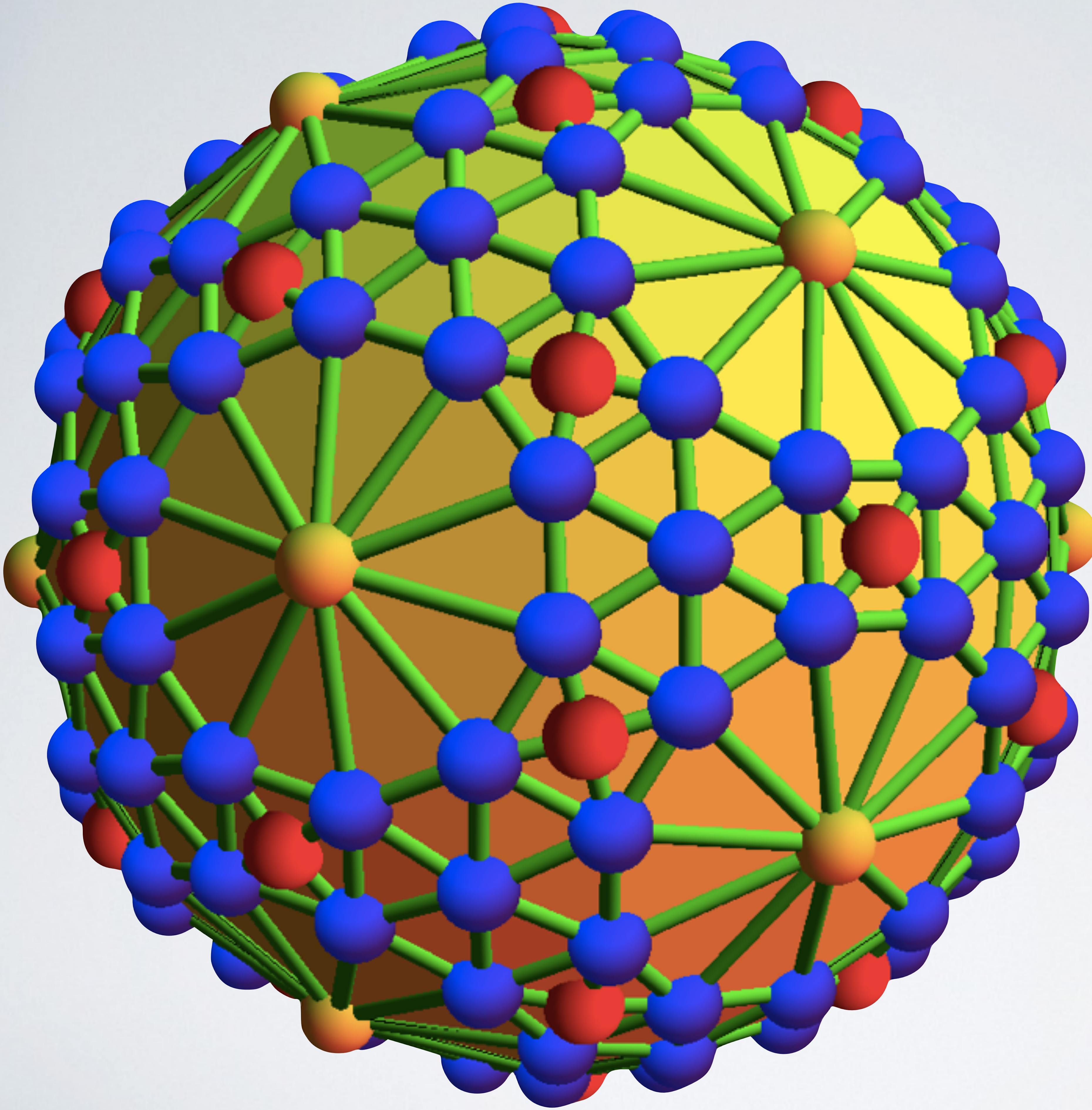
$$\chi(G) = \sum K(x)$$



$$K(x) = \sum_{k=0} \frac{(-1)^k V_{k-1}}{k+1}$$

$$= 1 - \frac{V_0}{2} + \frac{V_1}{3} - \ldots$$

$V_k = \# K_{k+1}$ subgraphs in sphere $S(x)$

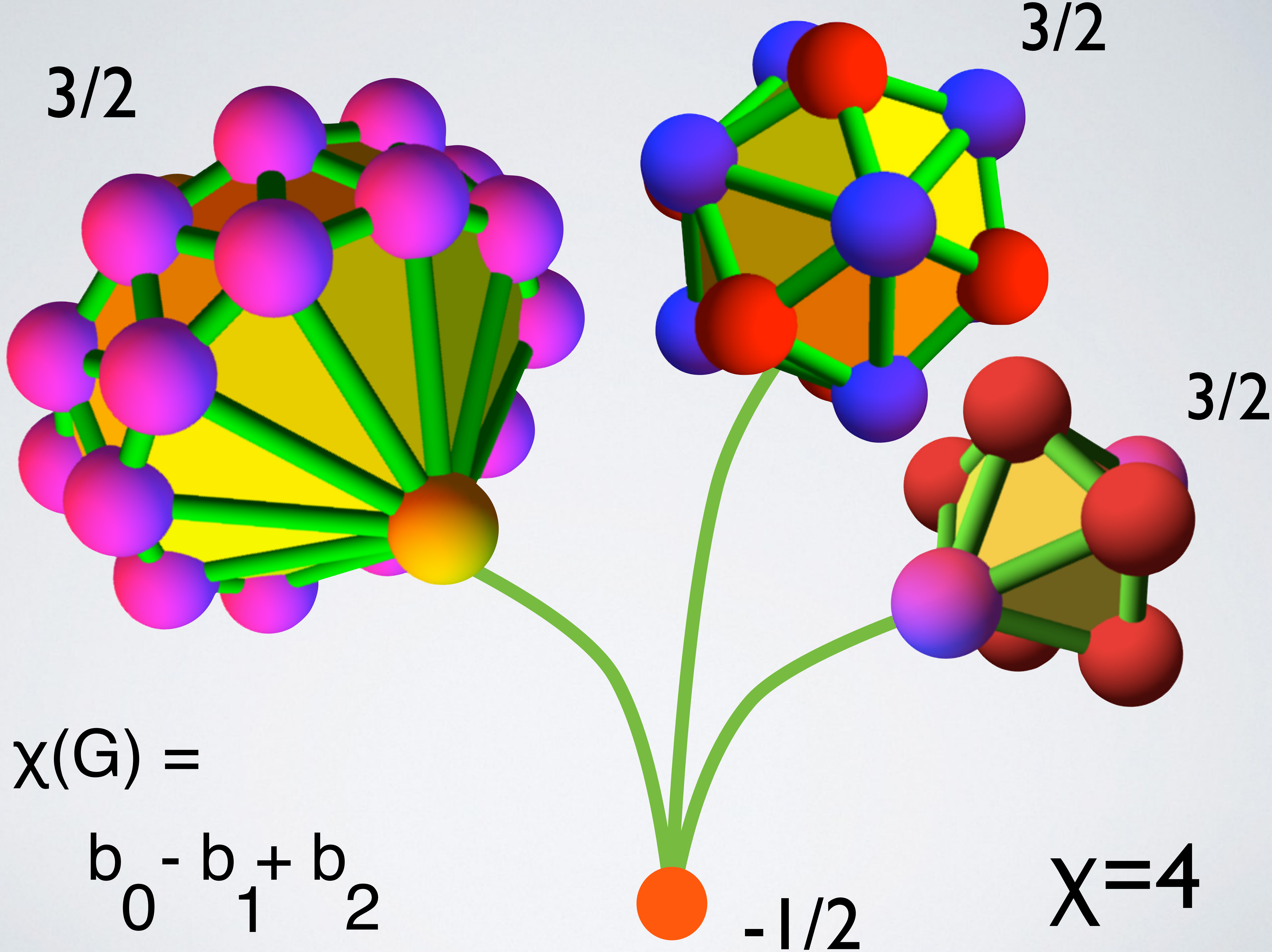


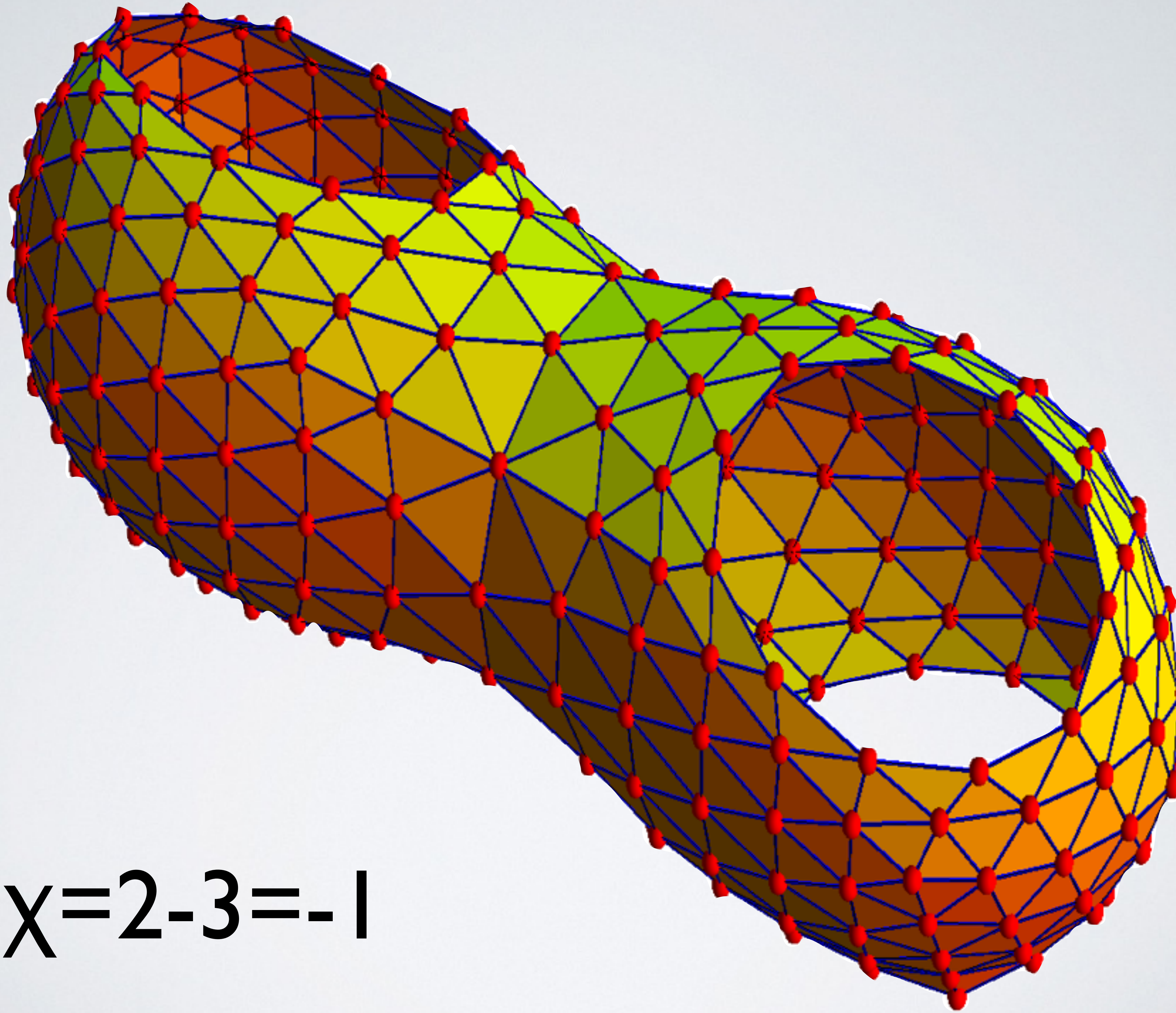
12 times -4/6

30 times +2/6

$$= 1 - \frac{d(x)}{6}$$

total curvature = 2


3/2
3/2
3/2
-1/2
$\chi(G) =$
$b_0 - b_1 + b_2$
$\chi=4$



$\chi = 2 - 3 = -1$

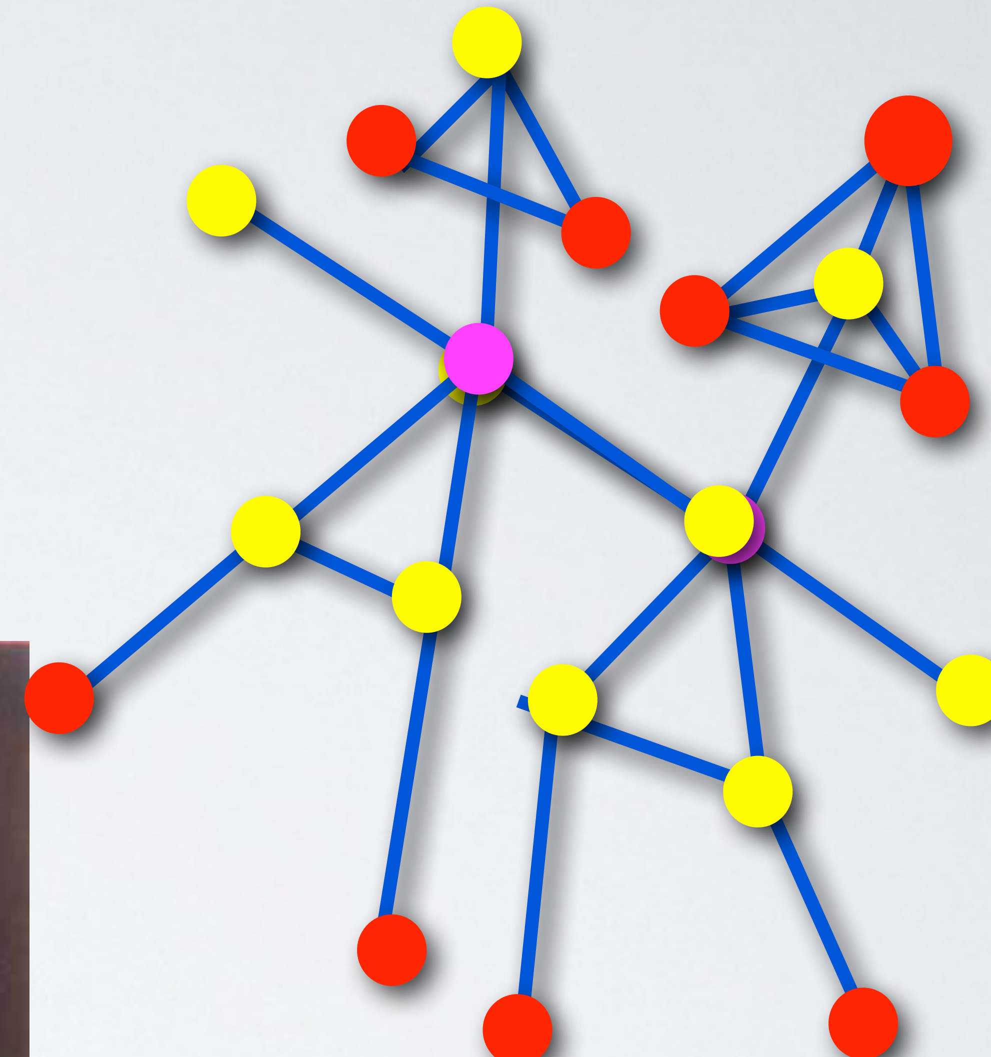
HANDSHAKING

$$\sum_{x \,\varepsilon\, V} V_{k-1}(x) = (k+1)\, v_k$$

k=0: vertex count

k=1: Euler:

every edge couns twice



k=2: every triangle gets counted three times

PROOF OF GAUSS-BONNET

$$\sum_{x \in V} K(x) = \sum_{x \in V} \sum_{k=0} (-1)^k \frac{V_{k-1}(x)}{k+1}$$ definition

$$= \sum_{k=0} (-1)^k \sum_{x \in V} \frac{V_{k-1}(x)}{k+1}$$ order of summation

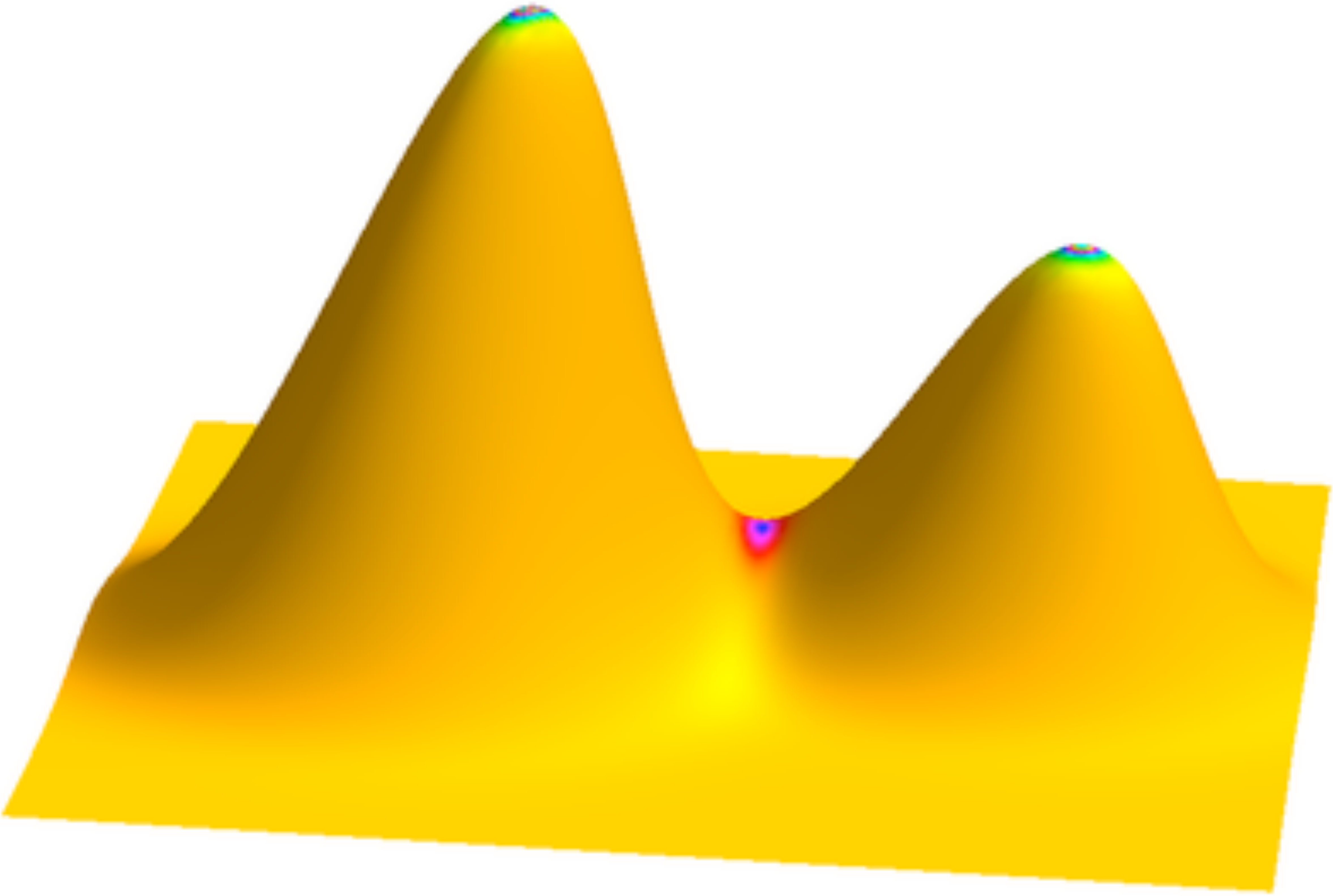
$$= \sum_{k=0} (-1)^k v_k$$ handshake

$$= X(G)$$ definition

POINCARE HOPF

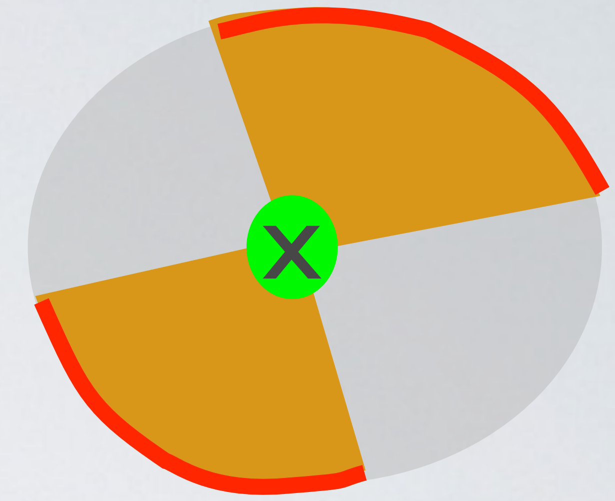
1854-1912

1894-1971

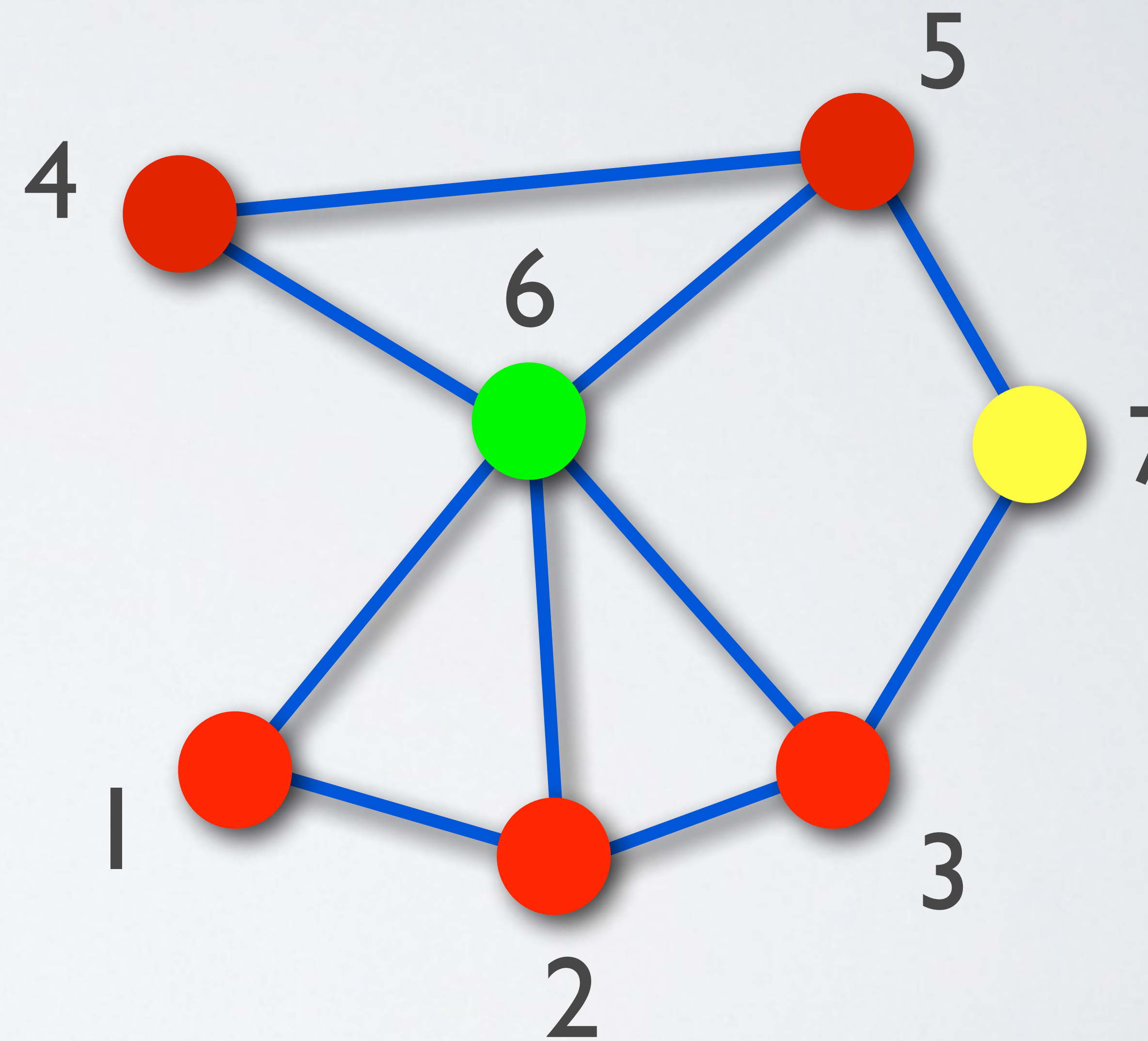


CRITICAL POINT

$\bar{S}(x) = S(x) \cap \{f(y)<f(x)\}$



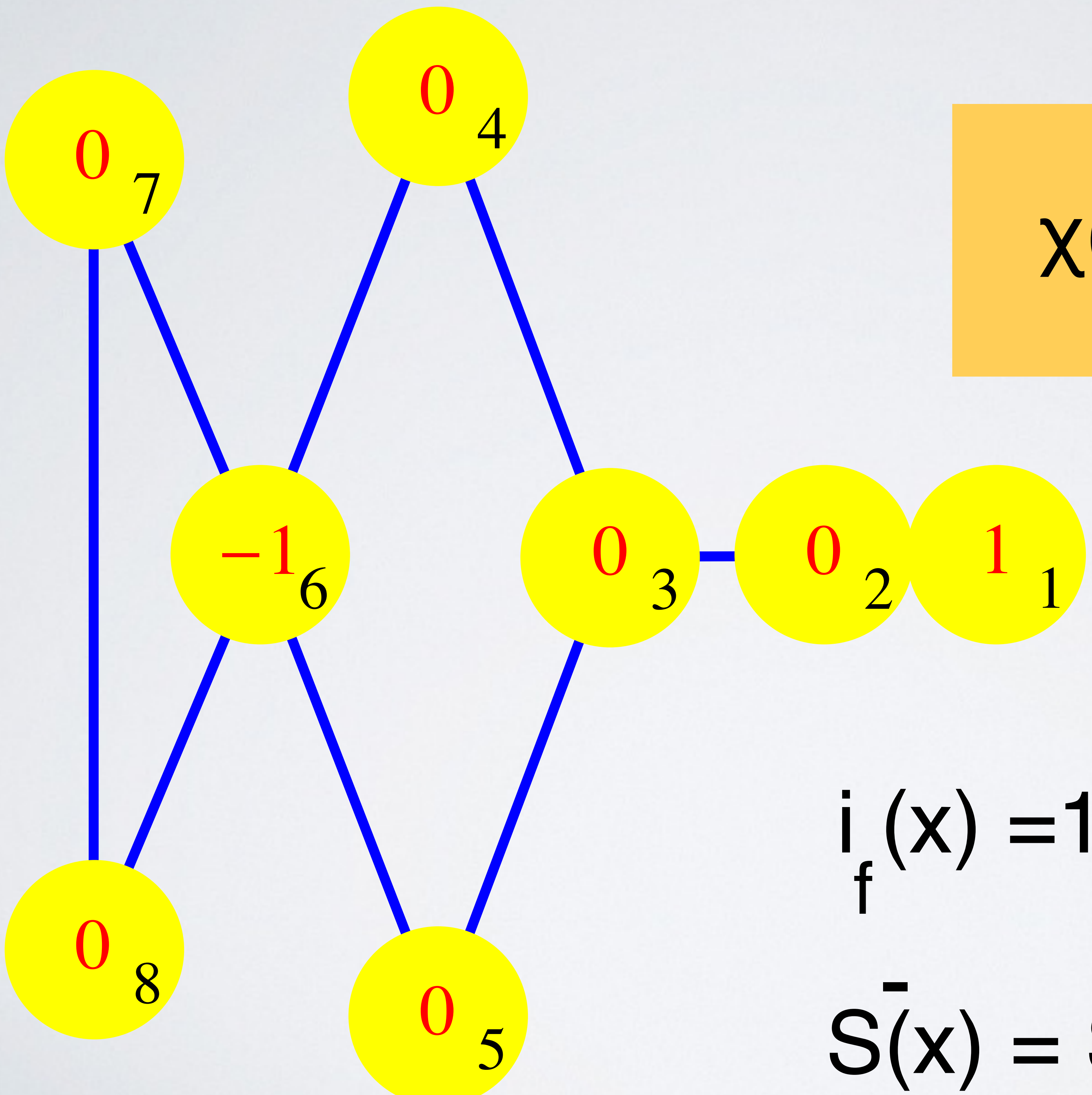
$\bar{S}(x)$ not contractible
critical point



$$1-\chi(\bar{S}(x)) = i_f(x)$$

$i_f(x)$, is zero at regular points

POINCARE HOPF



$$\chi(G) = \sum_x i_f(x)$$

$$i_f(x) = 1-\chi(S^-(x))$$

$$S^-(x) = S(x) \cap \{f(y)<f(x)\}$$

PROOF

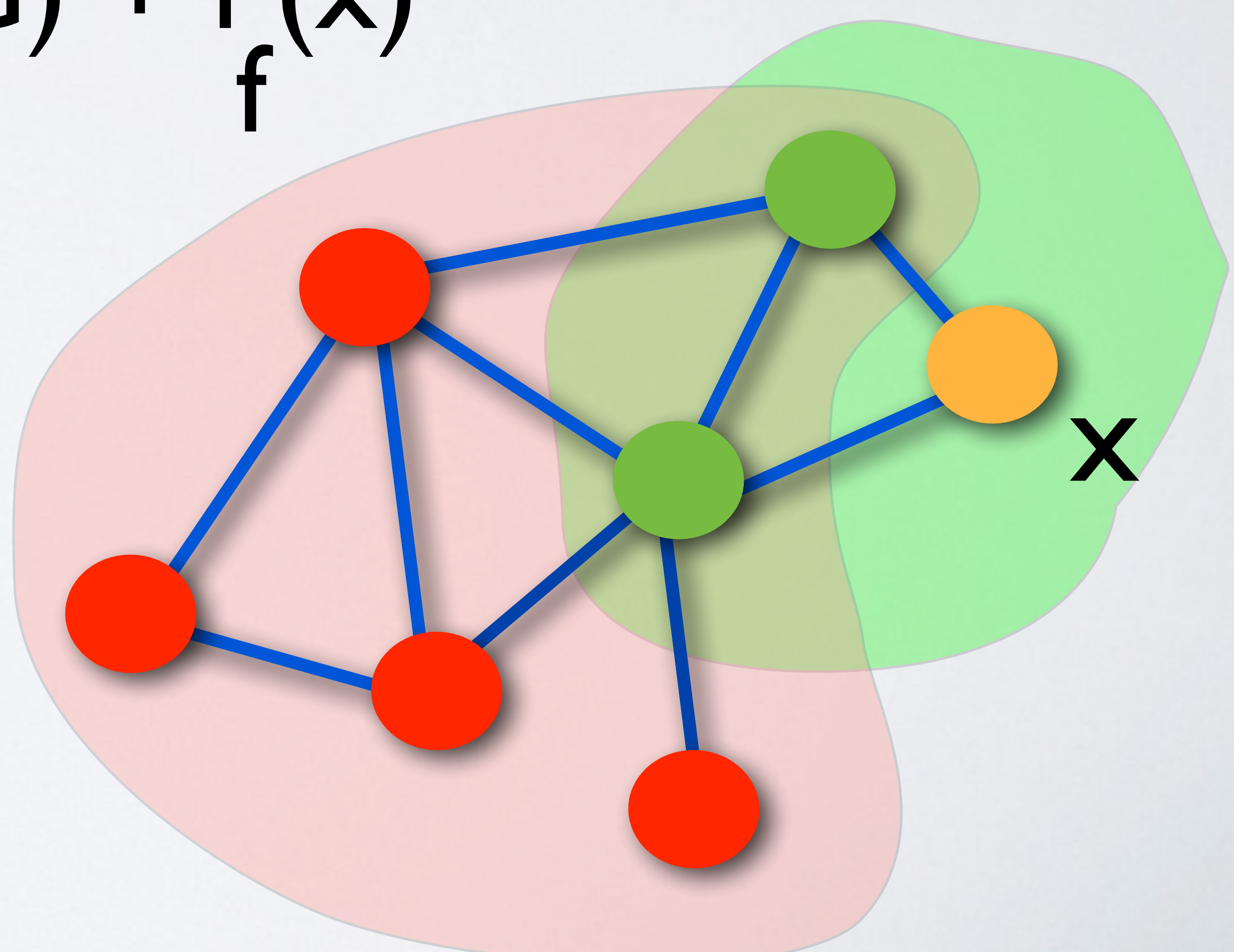
induction. Start at minimum, then add points

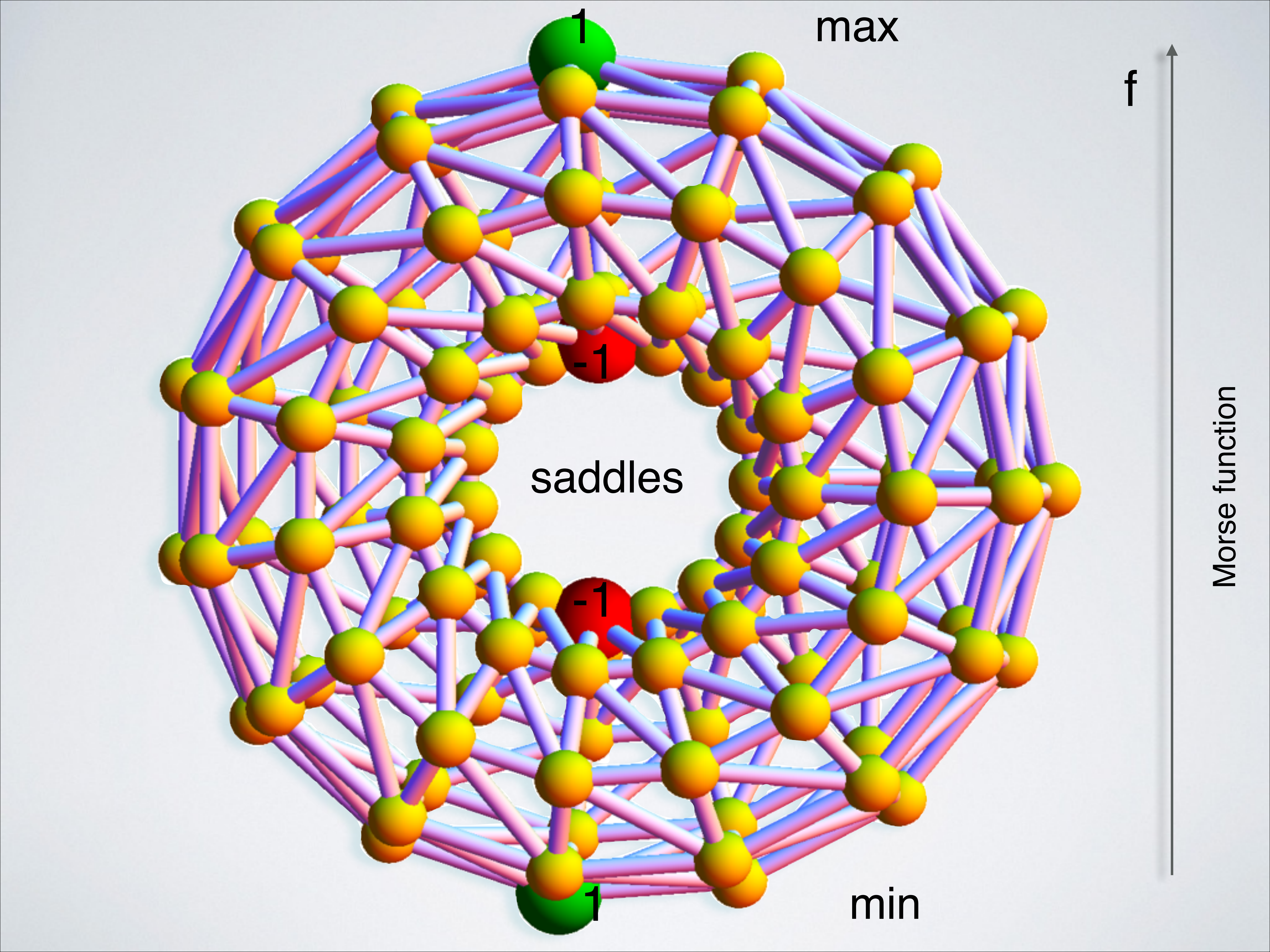
$$\chi(G \cup B(x)) = \chi(G) + \chi(B(x)) - \chi(\bar{S}(x))$$

$$= \chi(G) + 1 - \chi(\bar{S}(x))$$

$$= \chi(G) + i_f(x)$$

$$\chi(G \cup H) = \chi(G) + \chi(H) - \chi(G \cap H)$$




1
max
f
-1
saddles
-1
Morse function
1
min

INTEGRAL GEOMETRY

Blaschke 1885-1962

Chern 1911-2004

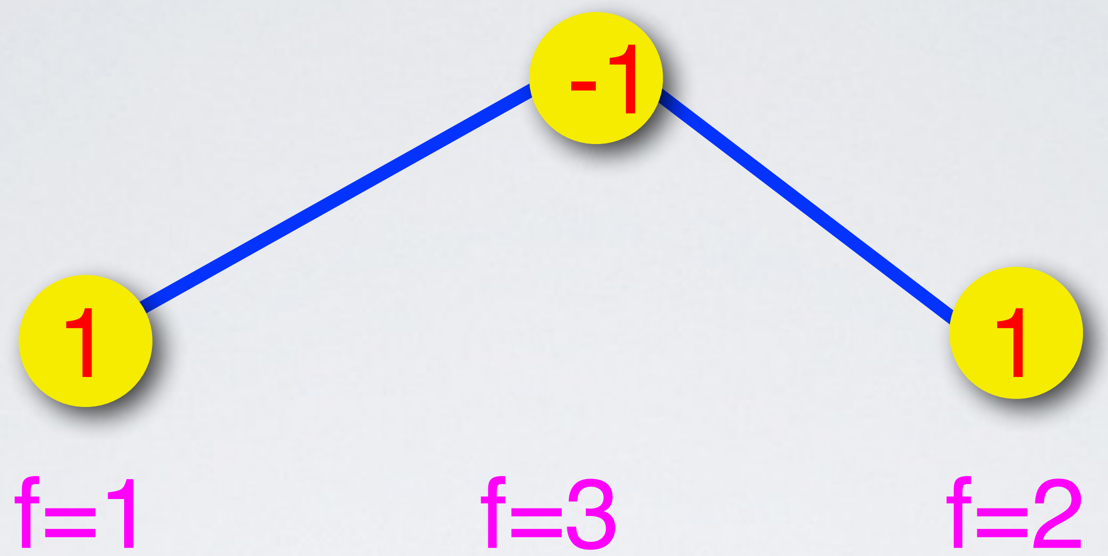
Banchoff

$$\chi(G) = \sum_x i_f(x)$$

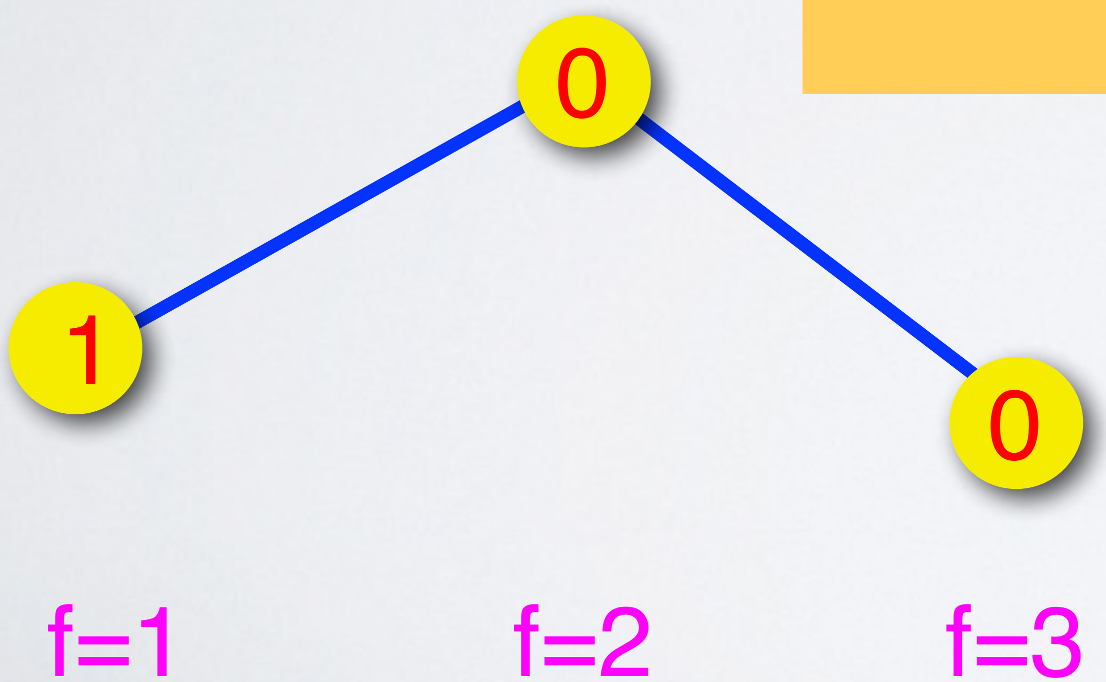
averaging over f

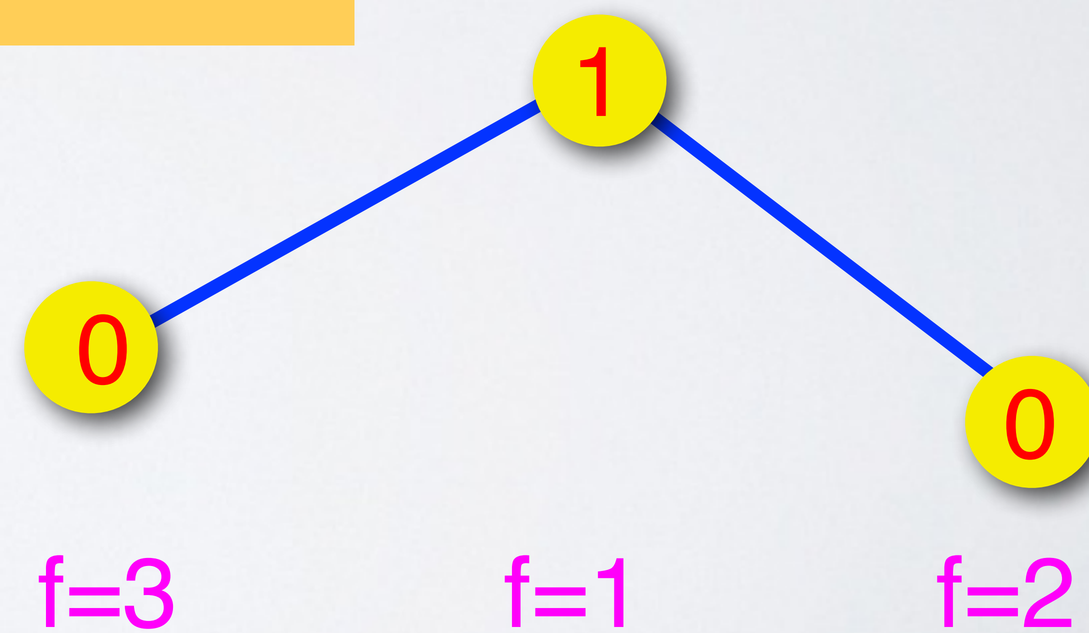
$$\chi(G) = \sum K(x)$$

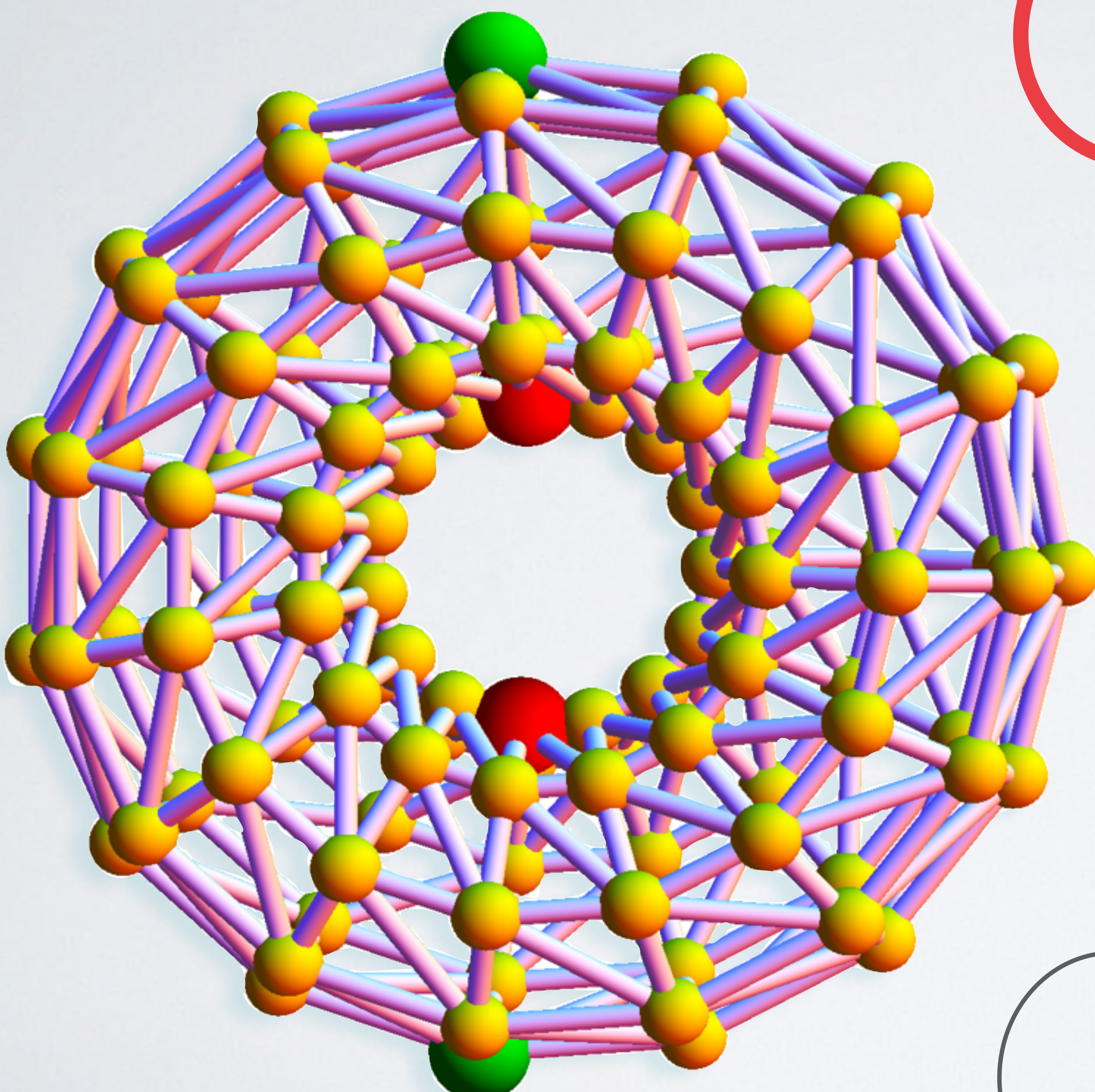
INDEX AVERAGE



$$K(x) = E[\, i_f(x) \,]$$

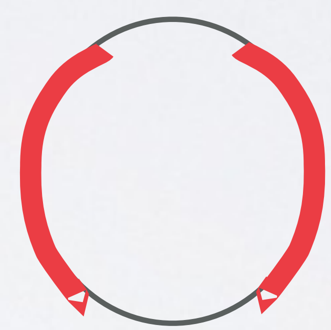






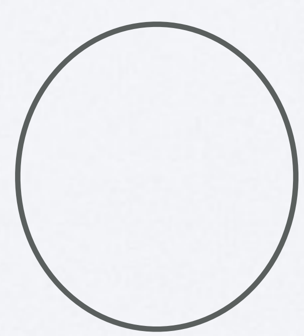
$S \cap \{f(y)<f(x)\} = S^-$

positive curvature



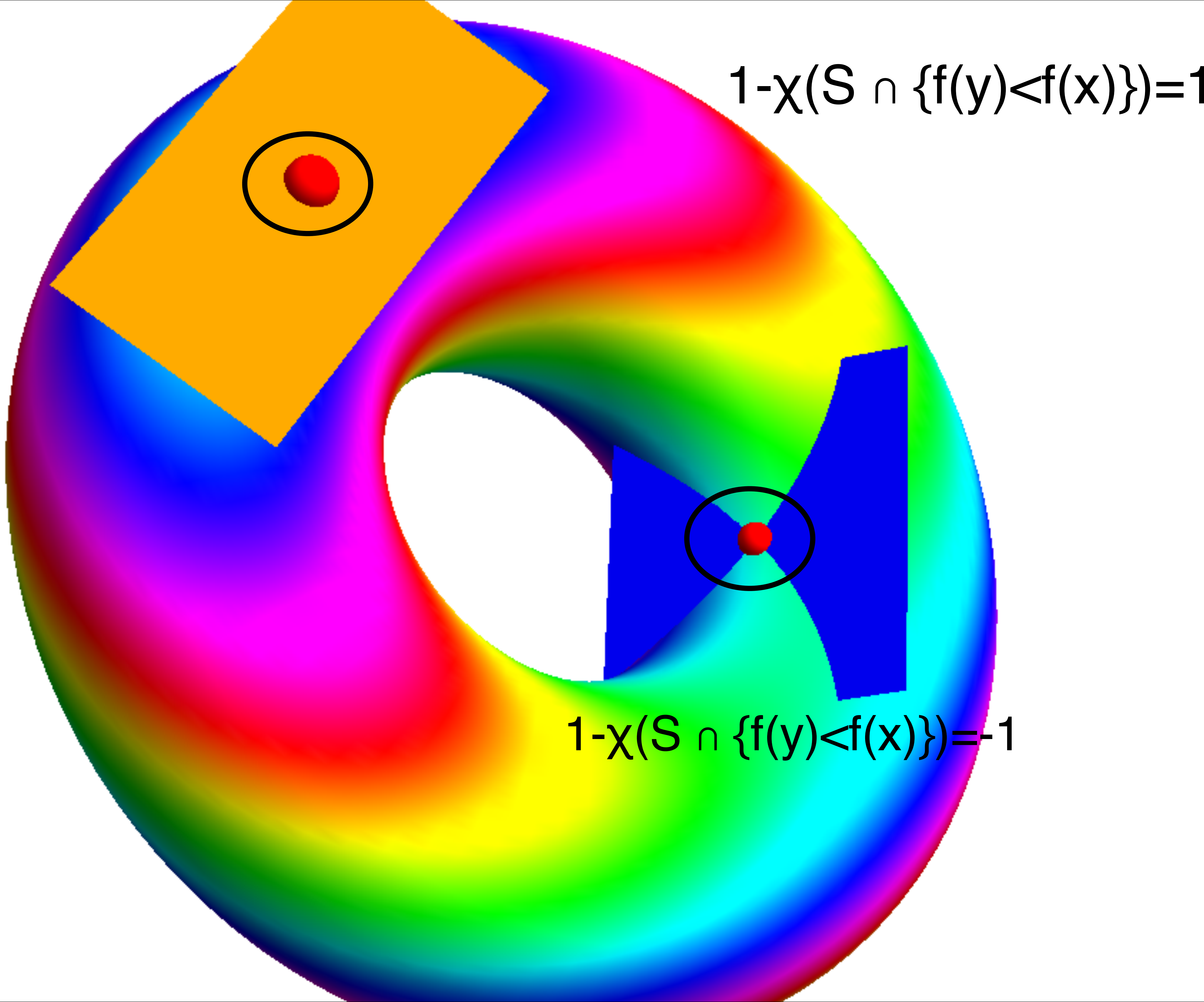
$S \cap \{f(y)<f(x)\} =$ two arcs

negative curvature



$S \cap \{f(y)<f(x)\} = \emptyset$

positive curvature


1-χ(S ∩ {f(y)<f(x)})=1
1-χ(S ∩ {f(y)<f(x)})=-1

RIEMANN ROCH

1826-1866

1839-1866

RIEMANN ROCH

Paper of Baker-Norine 2007

G graph triangle free

D divisor

K canonical divisor

L=Laplacian

$$K = \sum (\deg(x)-2)\,(x)$$

$$r(D) - r(K-D) = \chi(G) + \deg(D)$$

principal divisors: (L f)

Jacobi Group=Zero divisors/Principal divisors

|D| linear system: effective divisors ~ D

r(D) = dimension

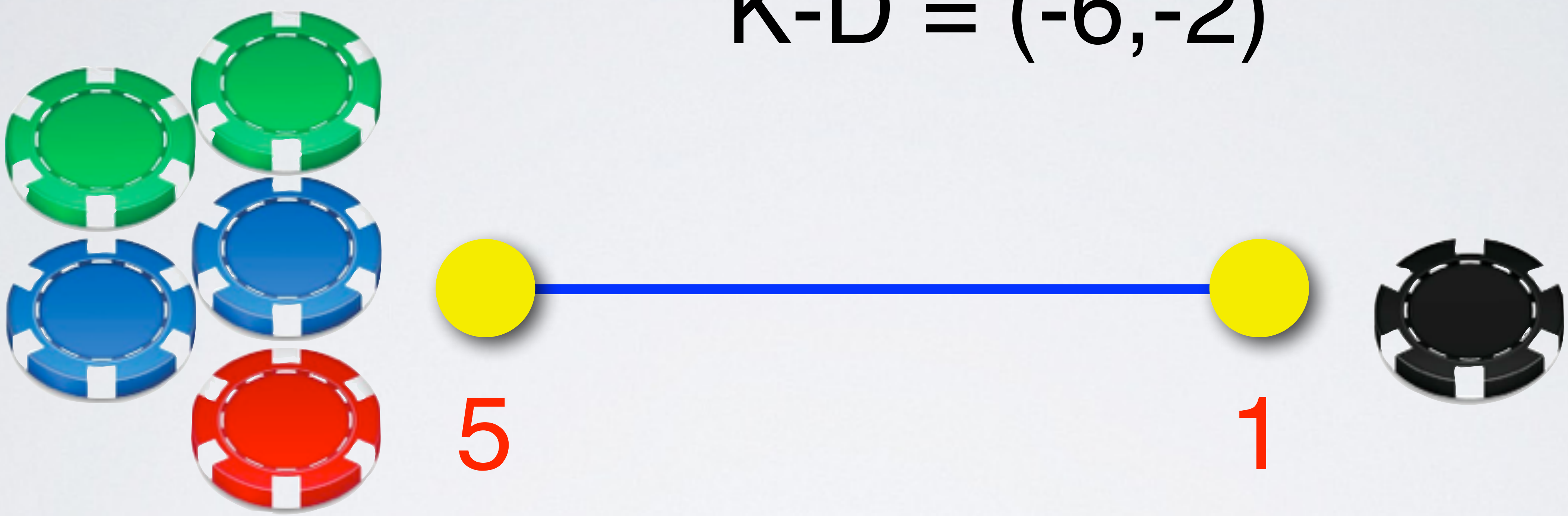
$\chi(G) = 1$

$\chi(D) = 6$

$D=(5,1)$

$K = (-1,-1)$ principal divisor

$K-D = (-6,-2)$



$r(D) = 6$

$r(K-D)=-1$

$\chi(G) + \deg(D) = 7$

$\chi(G)=1-g=1-0=1$

$\deg(D)=5+1=6$

RIEMANN HURWITZ

1826-1866

1859-1919

RIEMANN-HURWITZ

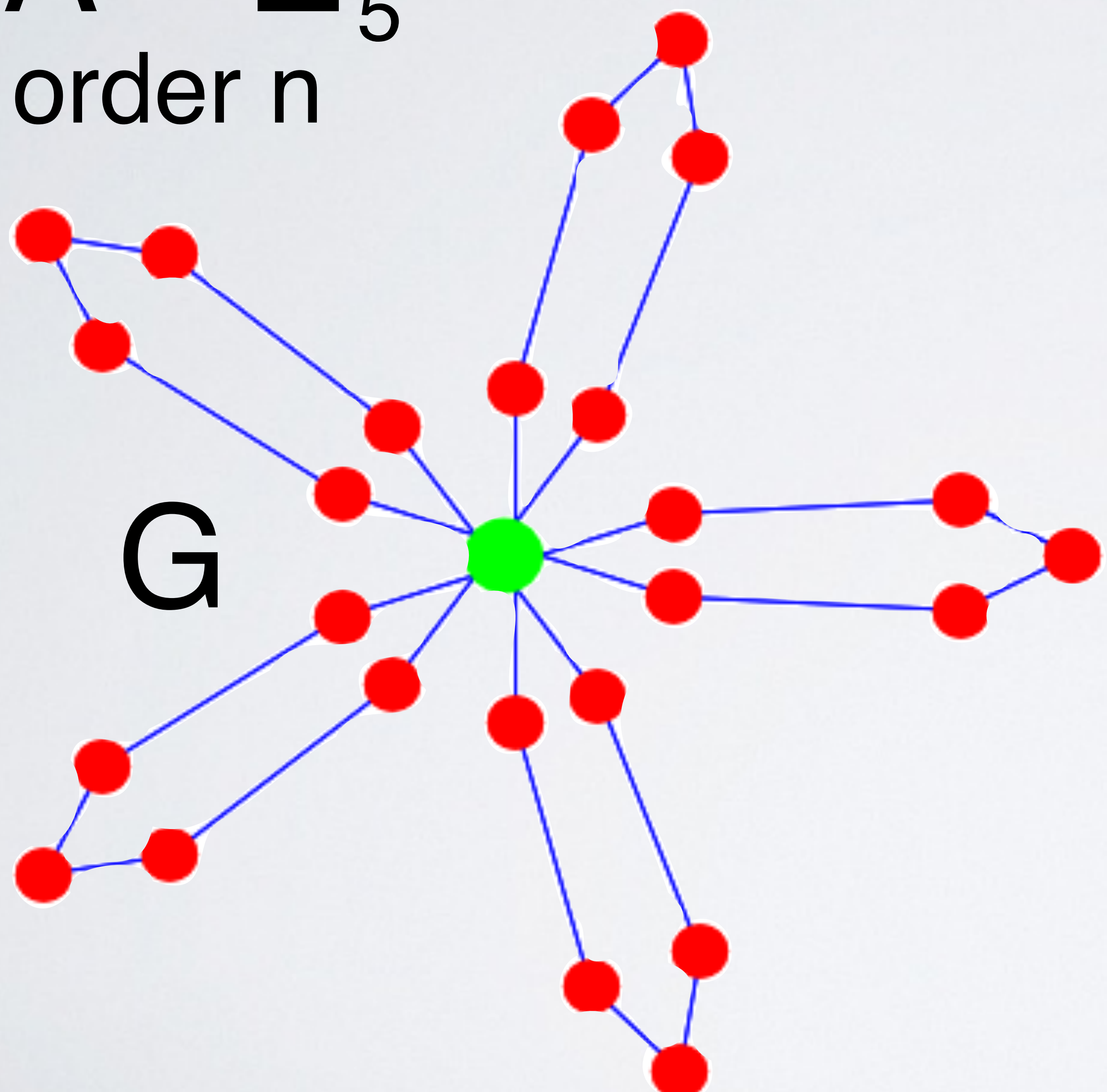
Work with Thomas Tucker

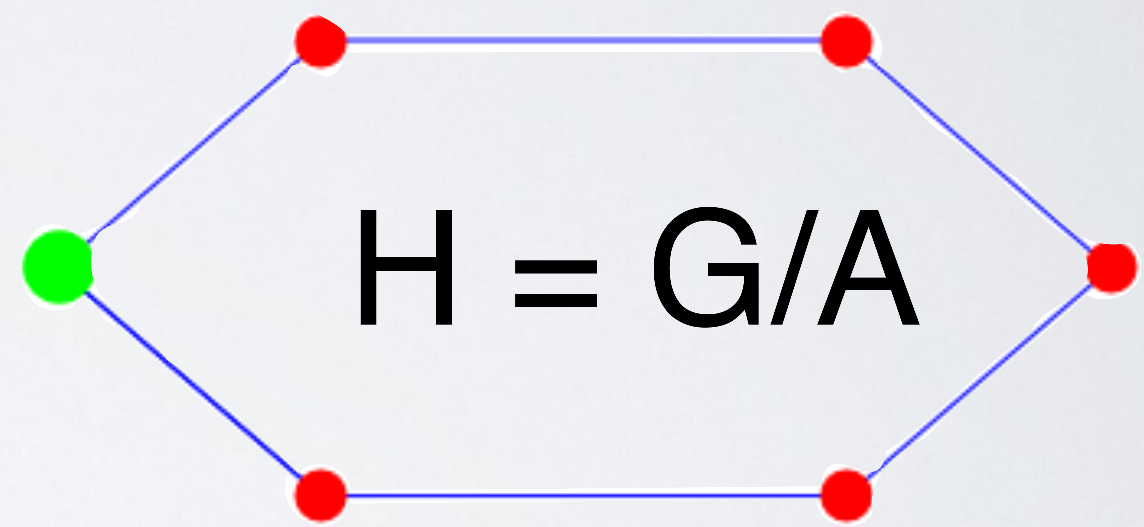
$$\chi(G) = n\,\chi(H) - \sum_x (e_x - 1)$$

$A = Z_5$

order n

$$-4 = 5*0 - (5-1)$$

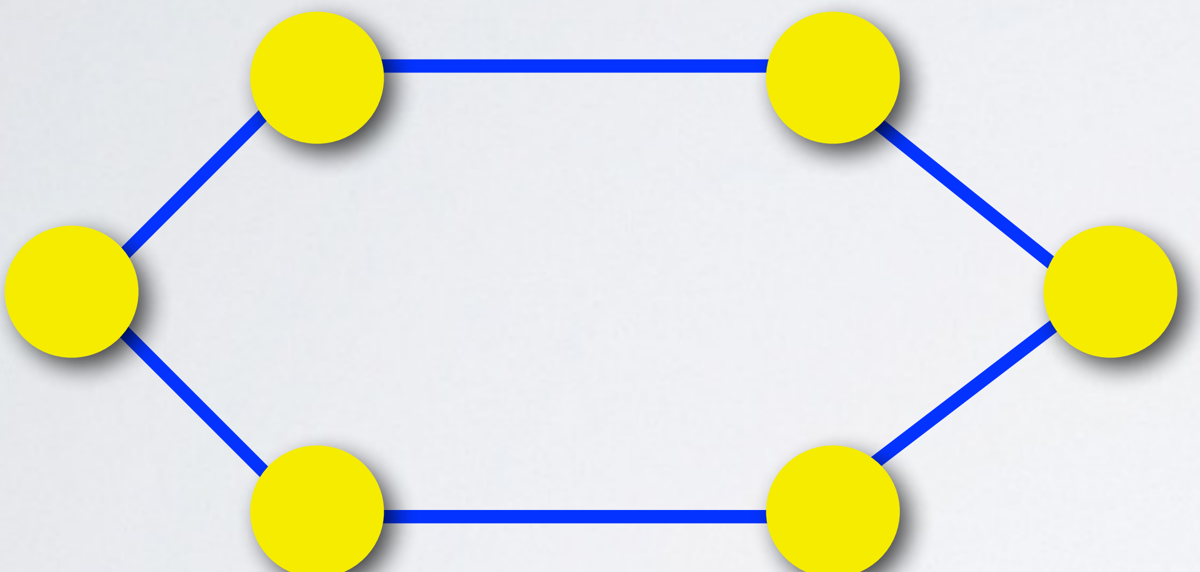




$$e_x - 1 = \sum_{a(x)=x} (-1)^{k(x)}$$

$$\chi(G) = 2\,\chi(H) - \sum_{x \text{ fixed}} (e_x - 1)$$

$A = Z_2$



G

H=G/A

A COROLLARY OF RH

Assume G has no edges.

$$|G| = n\,|G/A| - \sum_x e_x - 1$$

$$0 = n\,|G/A| - \sum_x e_x$$

$$|G/A| = \frac{1}{n} \sum_x e_x$$

$$|G/A| = \frac{1}{n} \sum_a |X^a|$$

$$\sum_x e_x$$

$$\sum_x \sum_a X^a(x)$$

$$\sum_a \sum_x X^a(x)$$

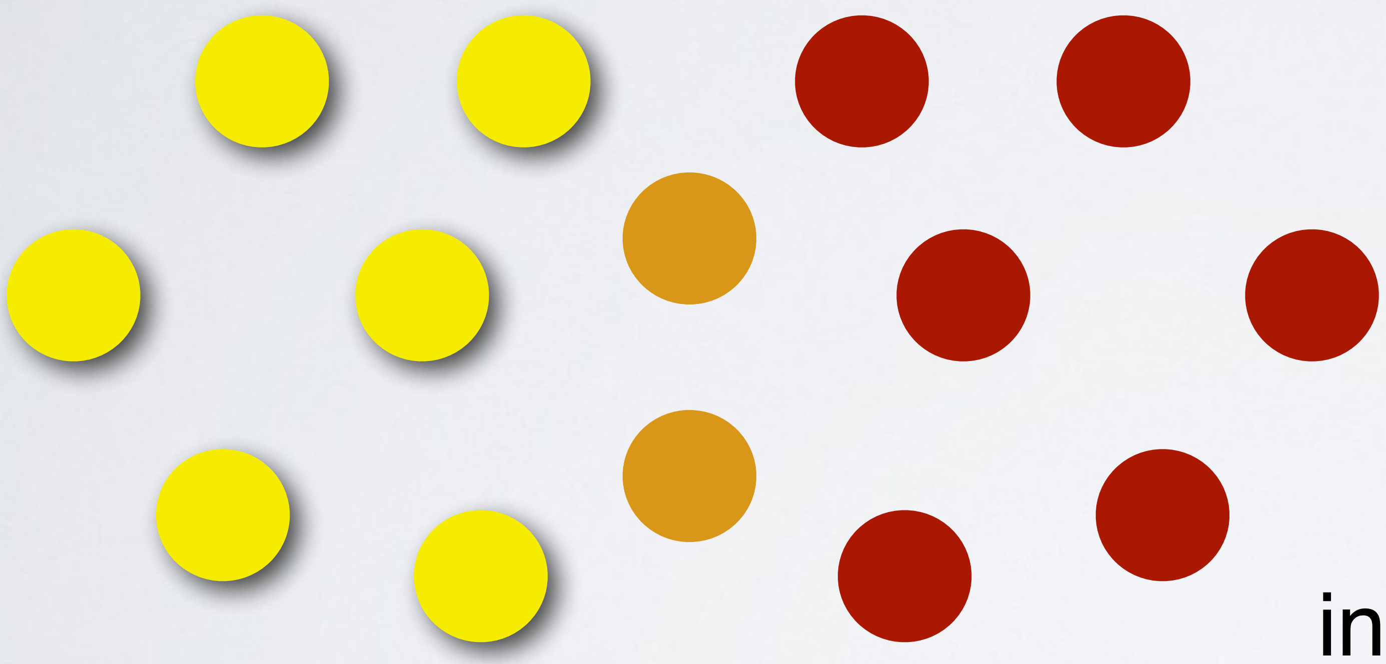
$$\sum_a |X^a|$$

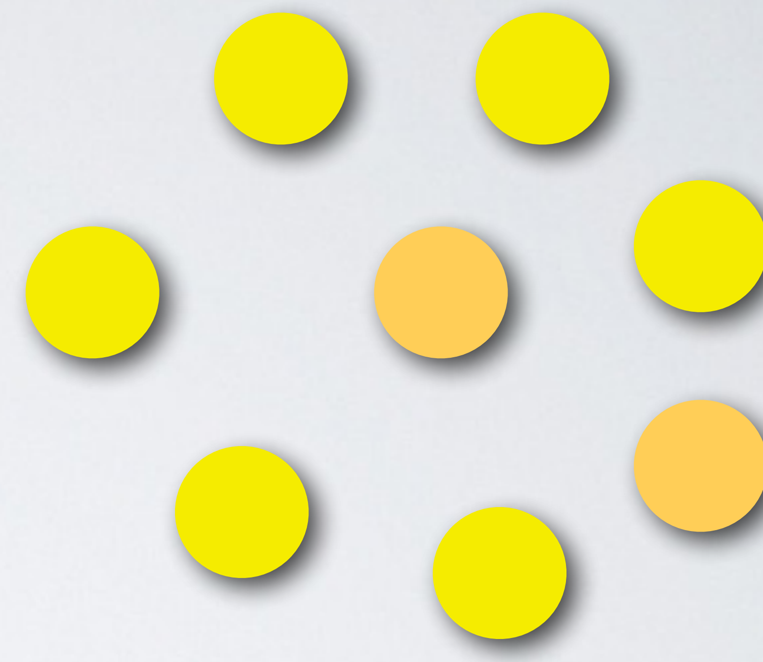
Burnside Lemma follows from Riemann Hurwitz!

A DEEPER APPLICATION OF RH

G set

$A = Z_2$





$H = G/A$

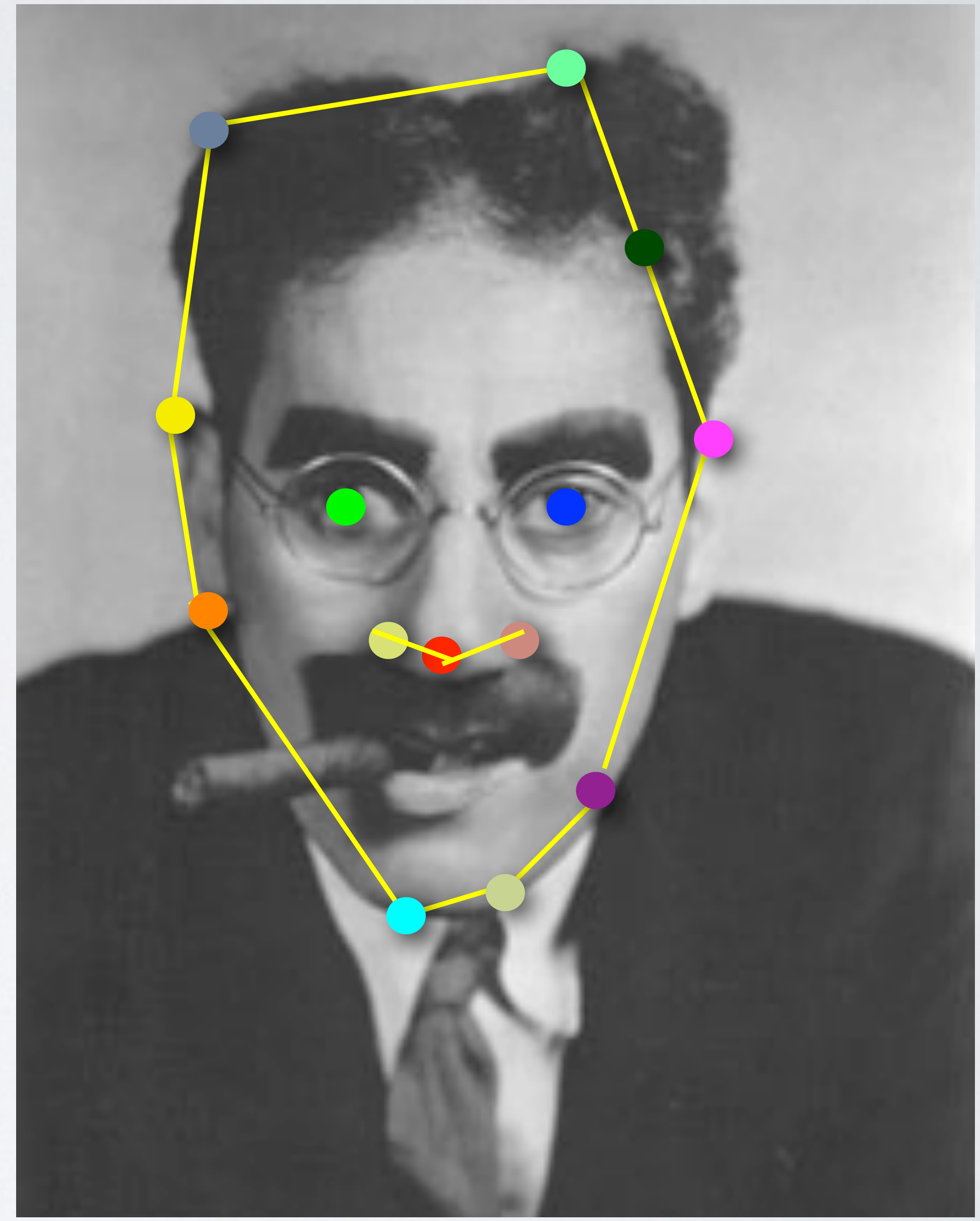
intersection points: ramifications

$$\chi(G) = 2\,\chi(H) - \chi(U)$$

HURWITZ

1859-1919

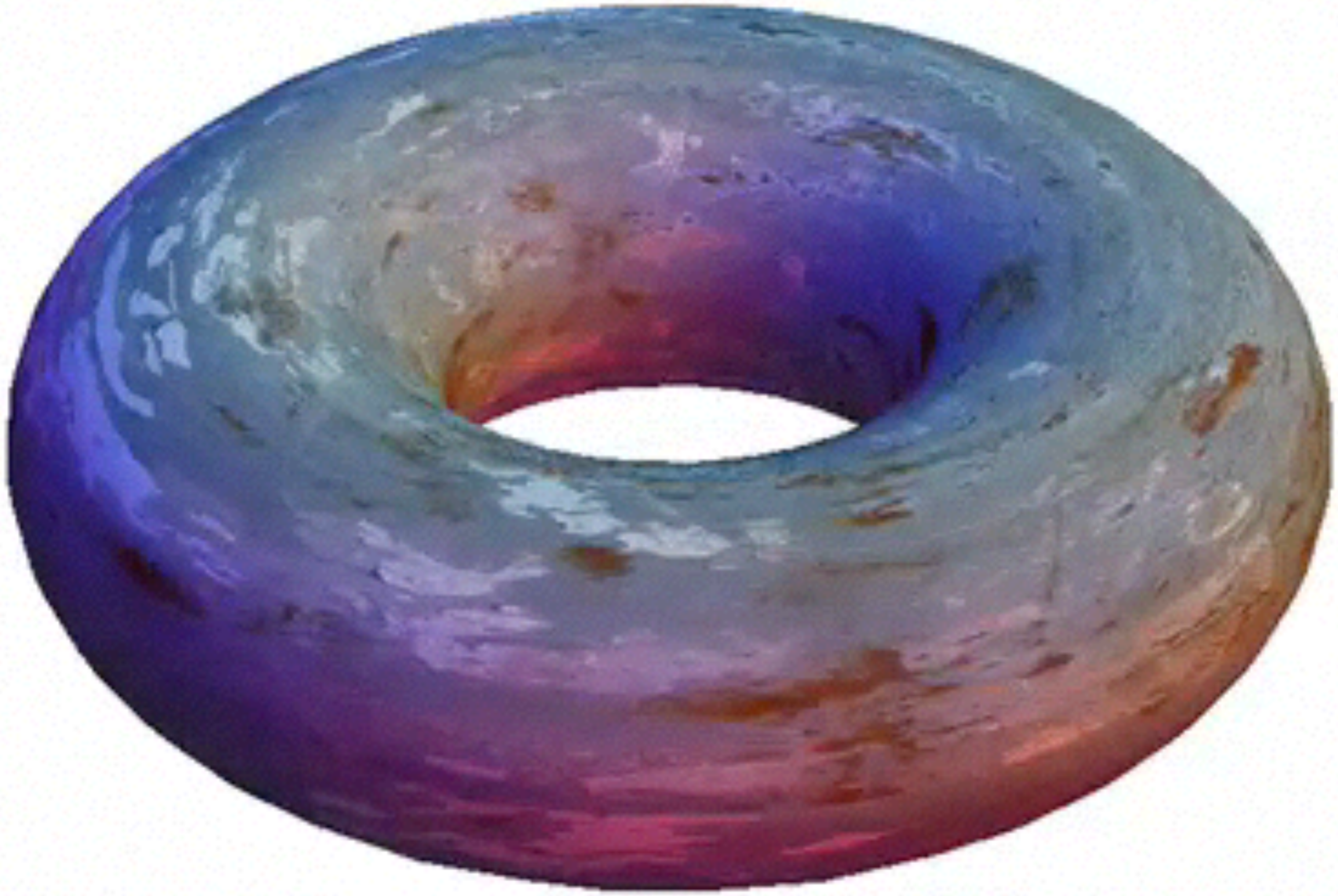
A HOMEOMORPHISM:

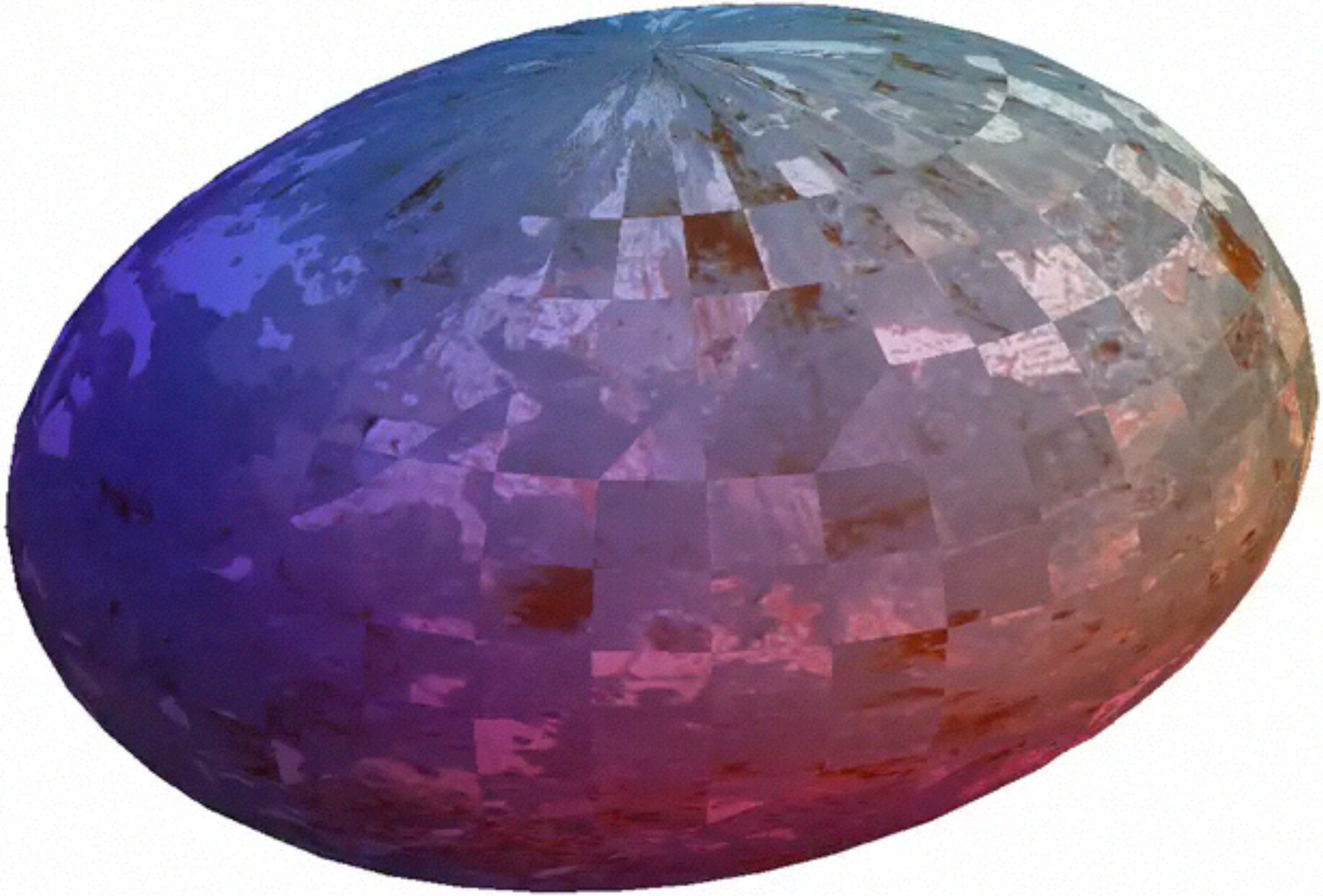


THE END

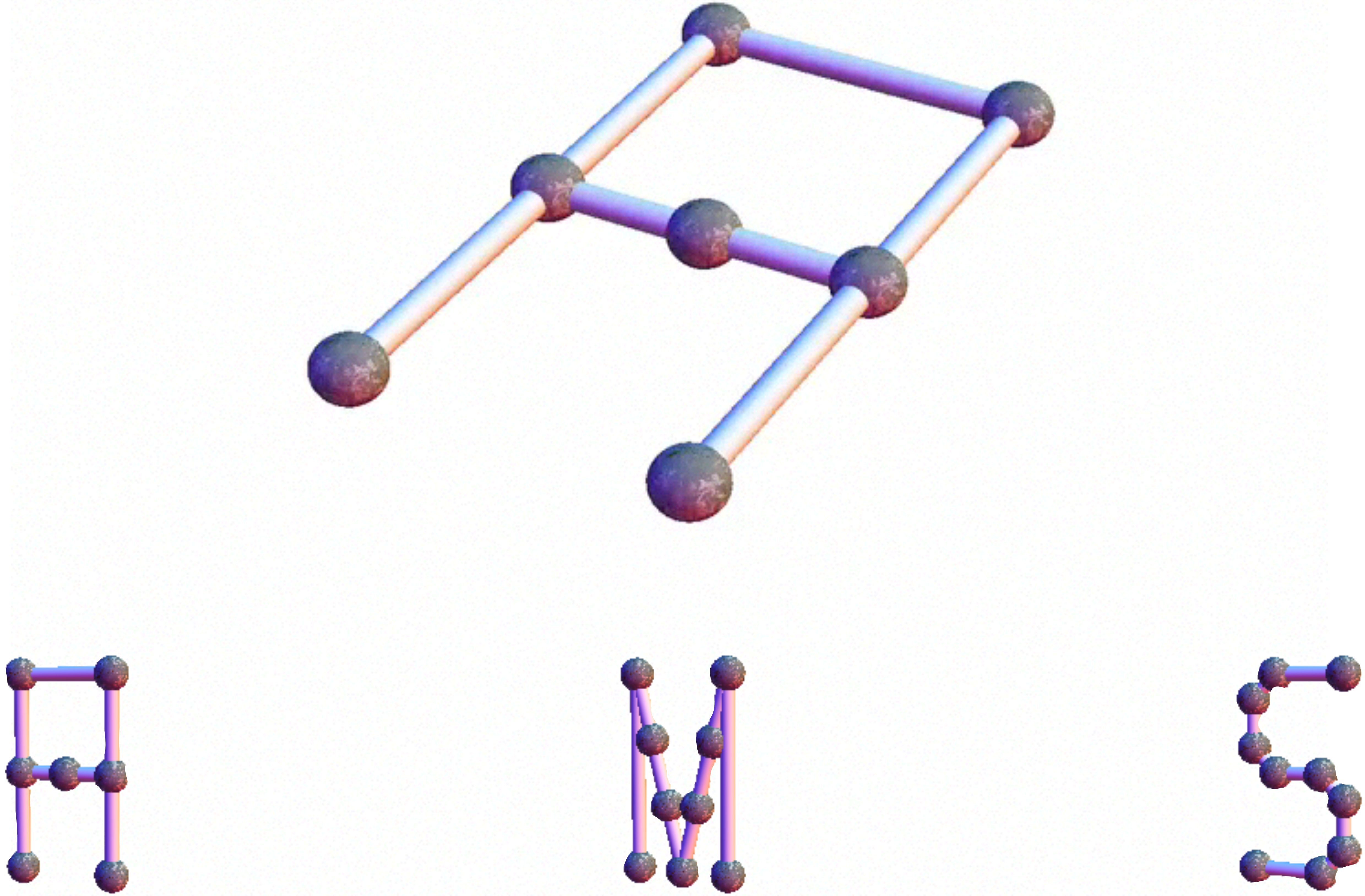
TOPOLOGY

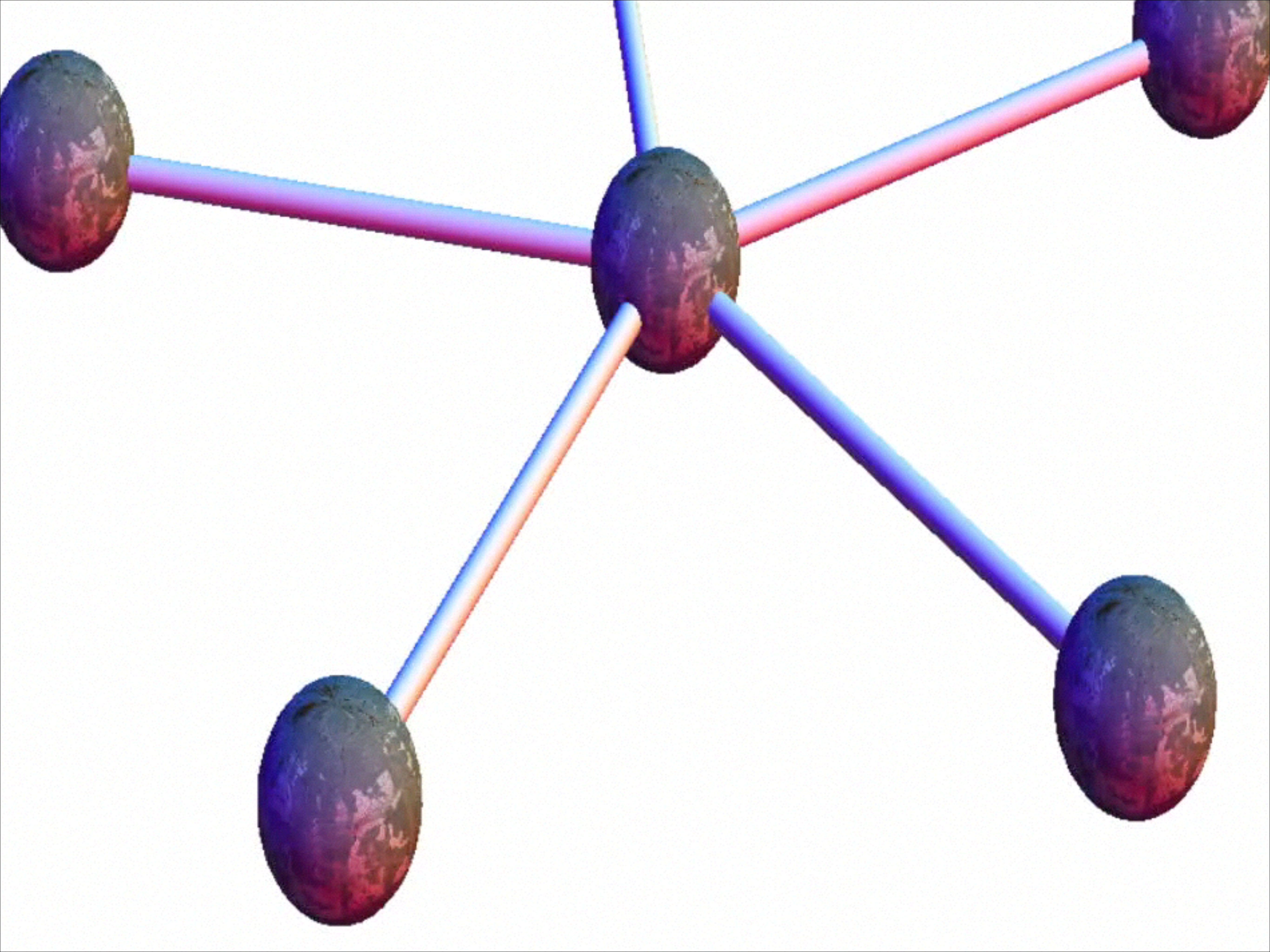
There had been no time to deliver this part in 20 minutes also.



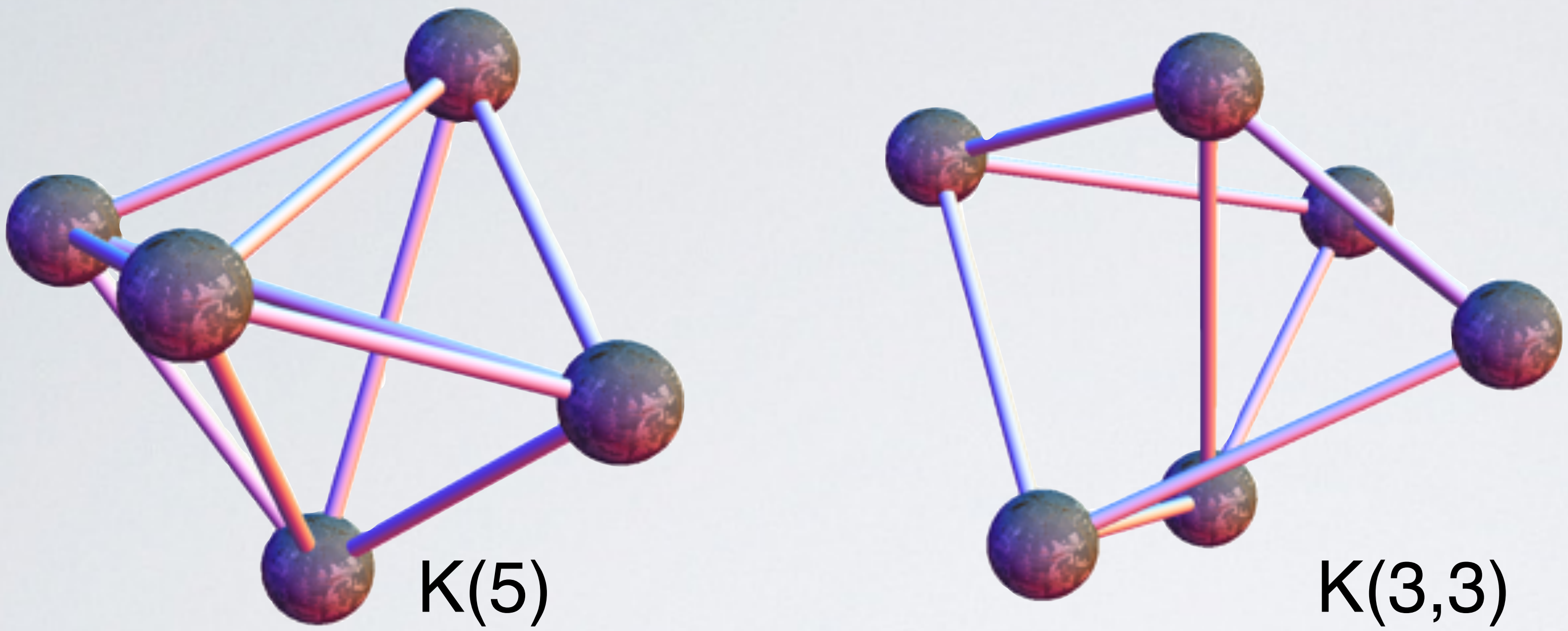


FOR GRAPHS?


A
M
S

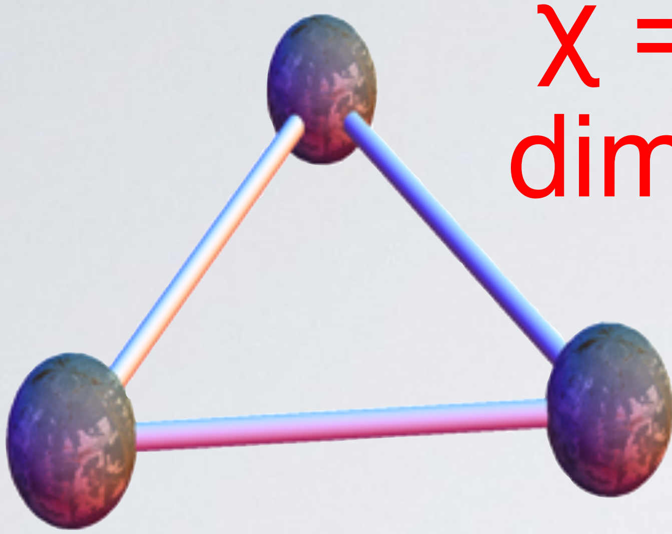


KURATOWSKI



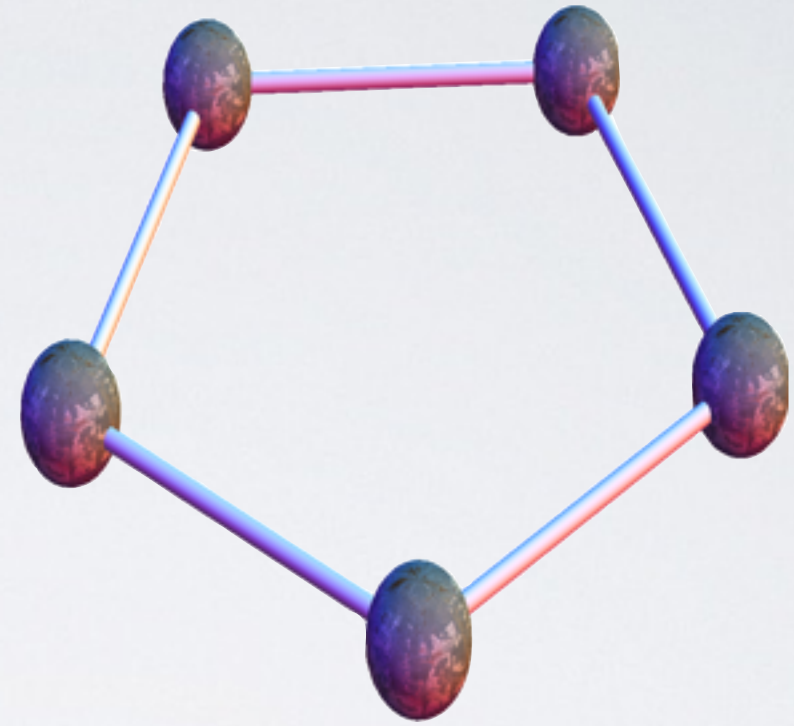
G planar ⟷
G contains no subgraph H
1-homeomorphic
to K(5) or K(3,3)

WHAT DO WE WANT?



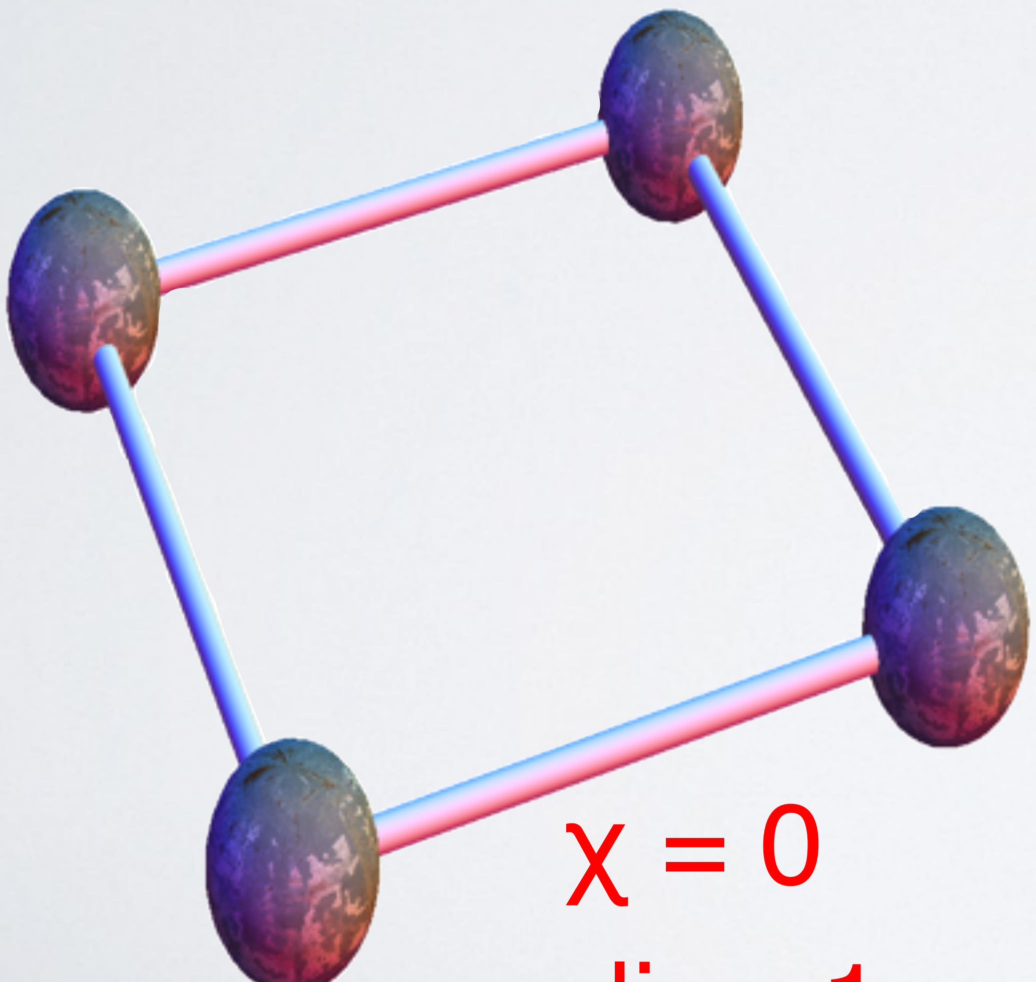
$\chi = 1$

dim=2



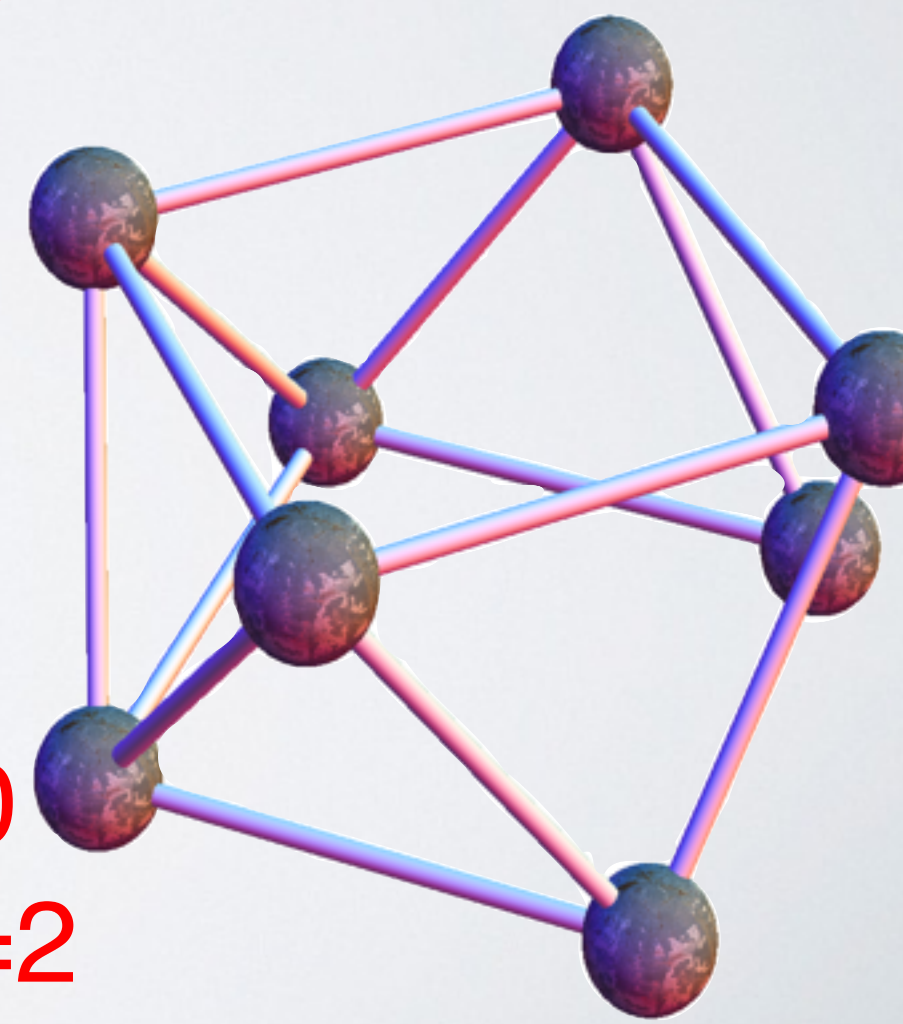
$\chi = 0$

dim=1



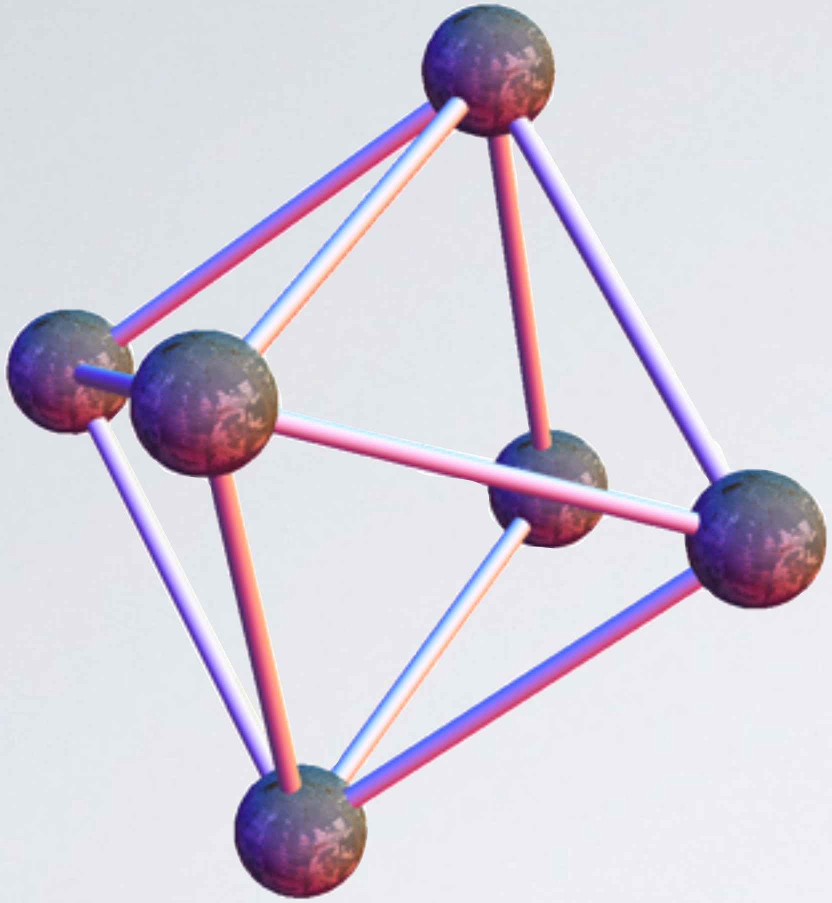
$\chi = 0$

dim=1

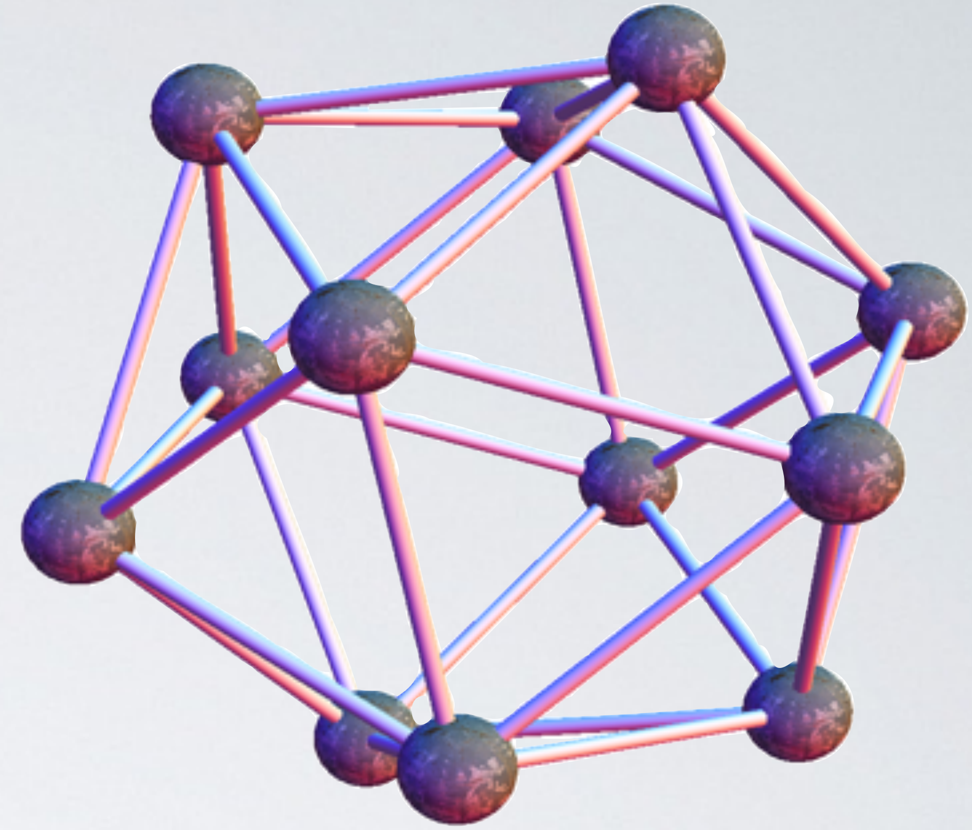


$\chi = 0$

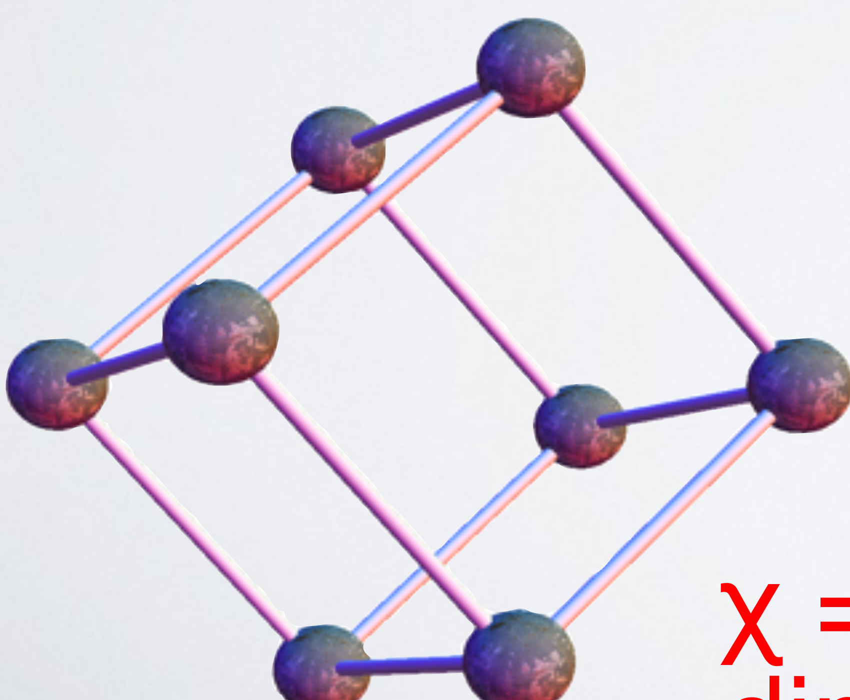
dim=2



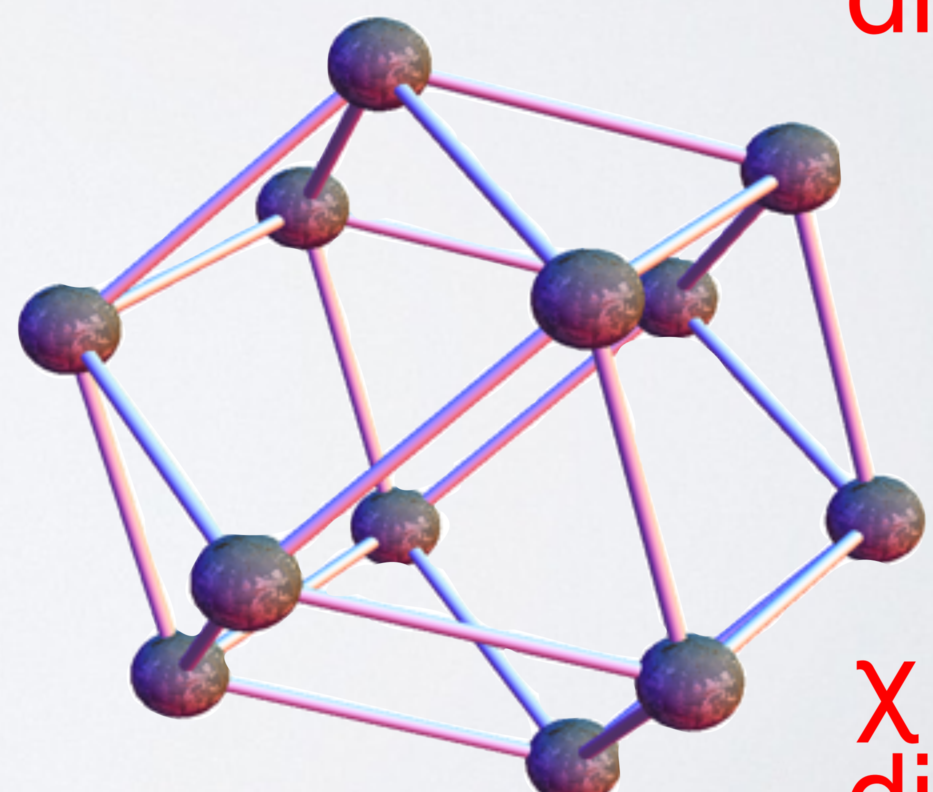
$\chi = 2$
dim=2



$\chi = 2$
dim=2



$\chi = -6$
dim=1



$\chi = -6$
dim=2

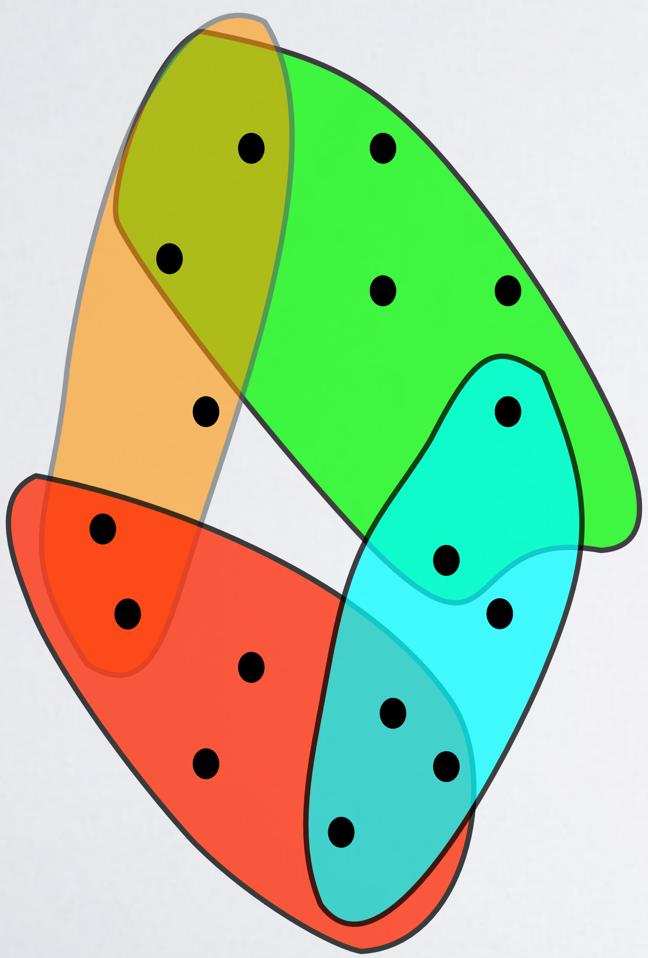
TOPOLOGY

$\mathcal{O}$ is finite topology:

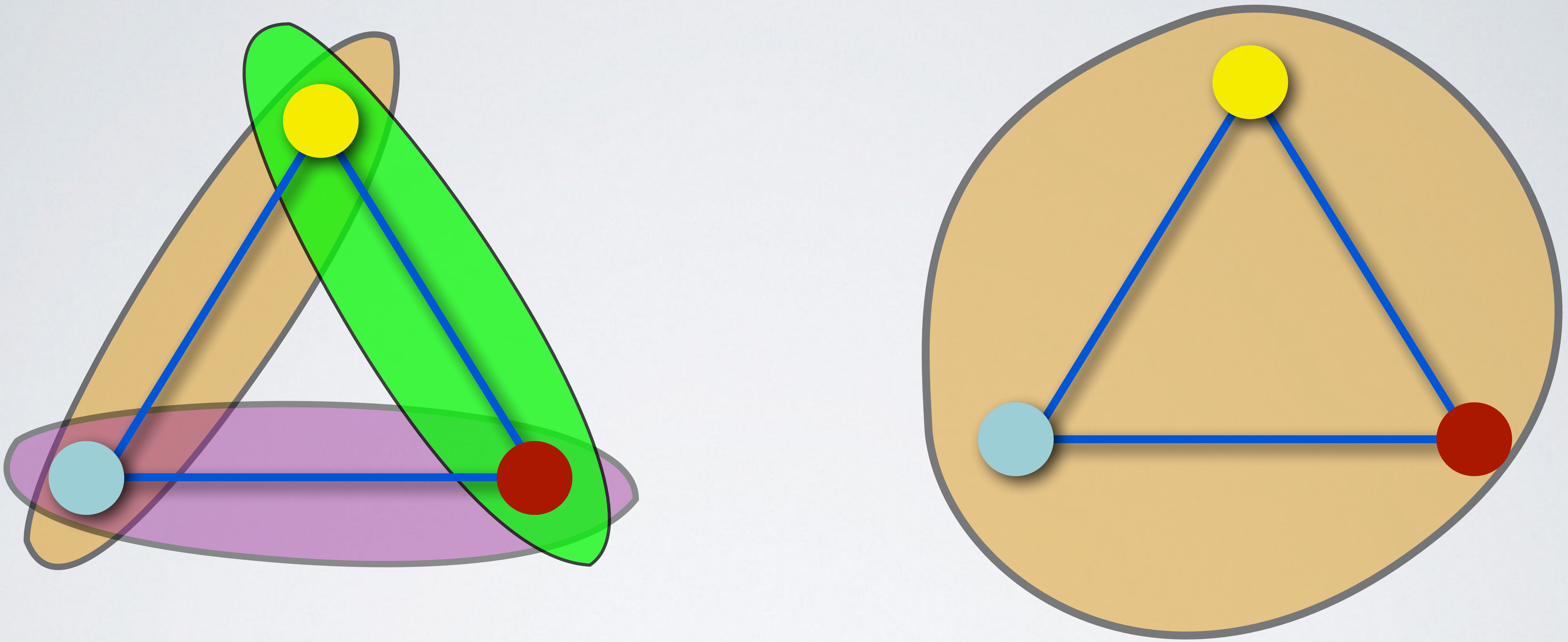
$\varnothing$, X are in $\mathcal{O}$

$\bigcap_{B \in \mathcal{B}} B$ are in $\mathcal{O}$

$\bigcup_{B \in \mathcal{B}} B$ are in $\mathcal{O}$



CHALLENGE:

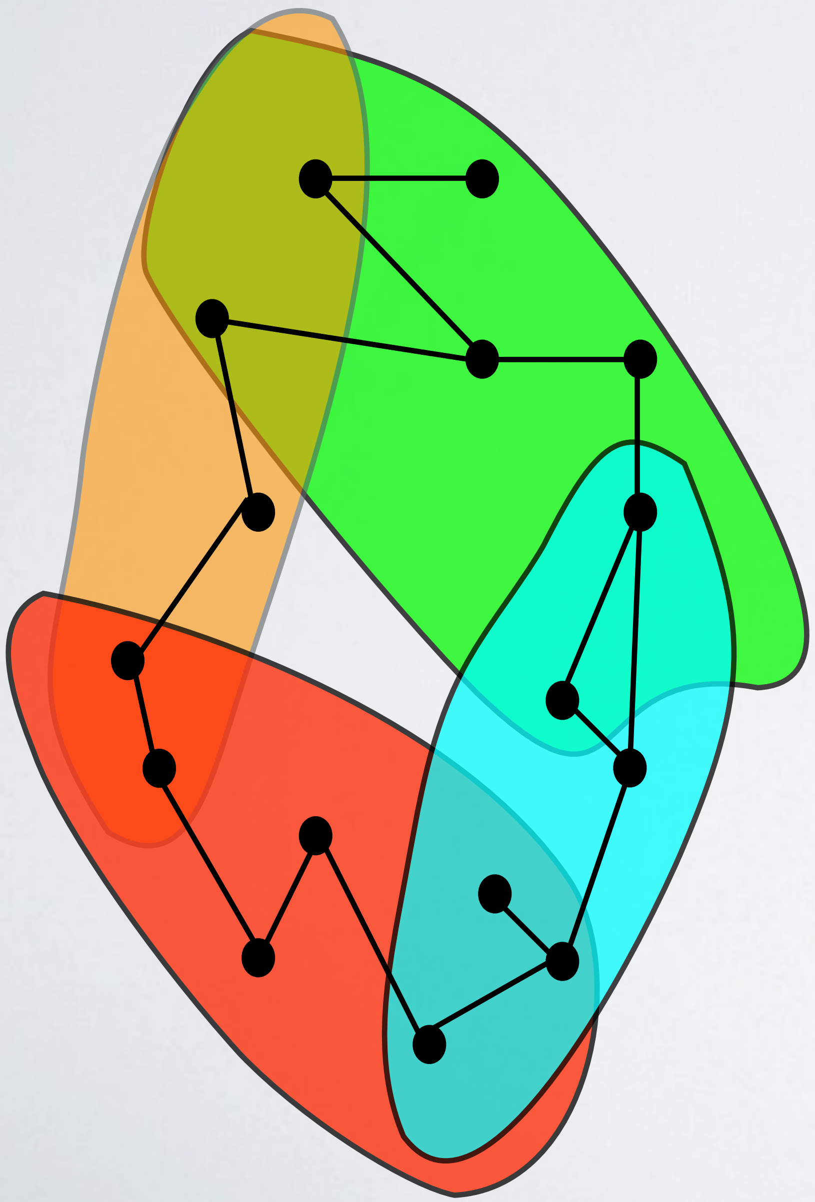


O is often discrete topology: graph is completely disconnected.

GRAPH TOPOLOGY

G = (V,E) finite simple graph

$\mathcal{O}$ is graph topology:



subbase $\mathcal{B}$ of contractible sets

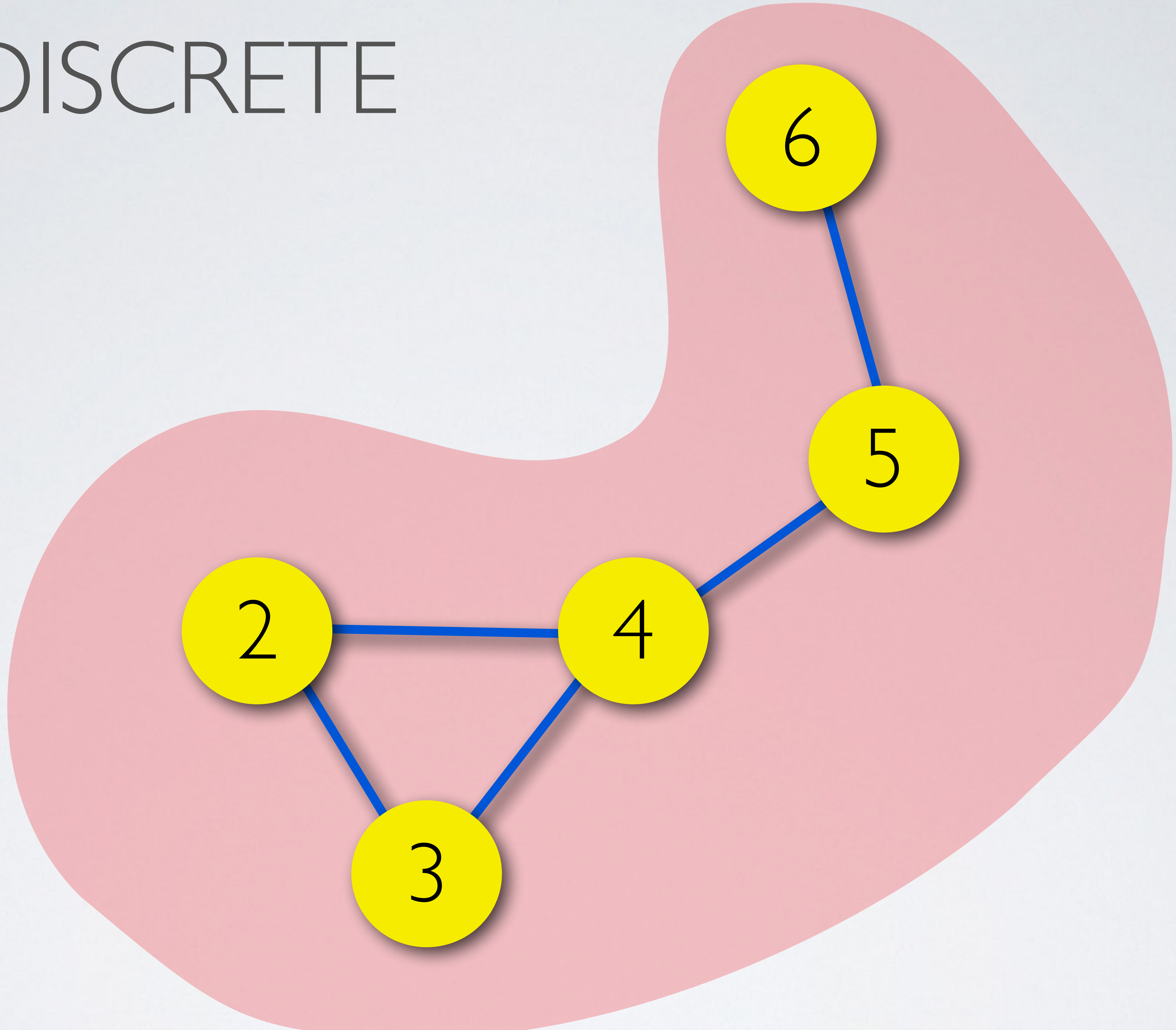
intersections in $\mathcal{B}$ are contractible if

dim(A ∩ B) ≥ dim(A) or dim(B)

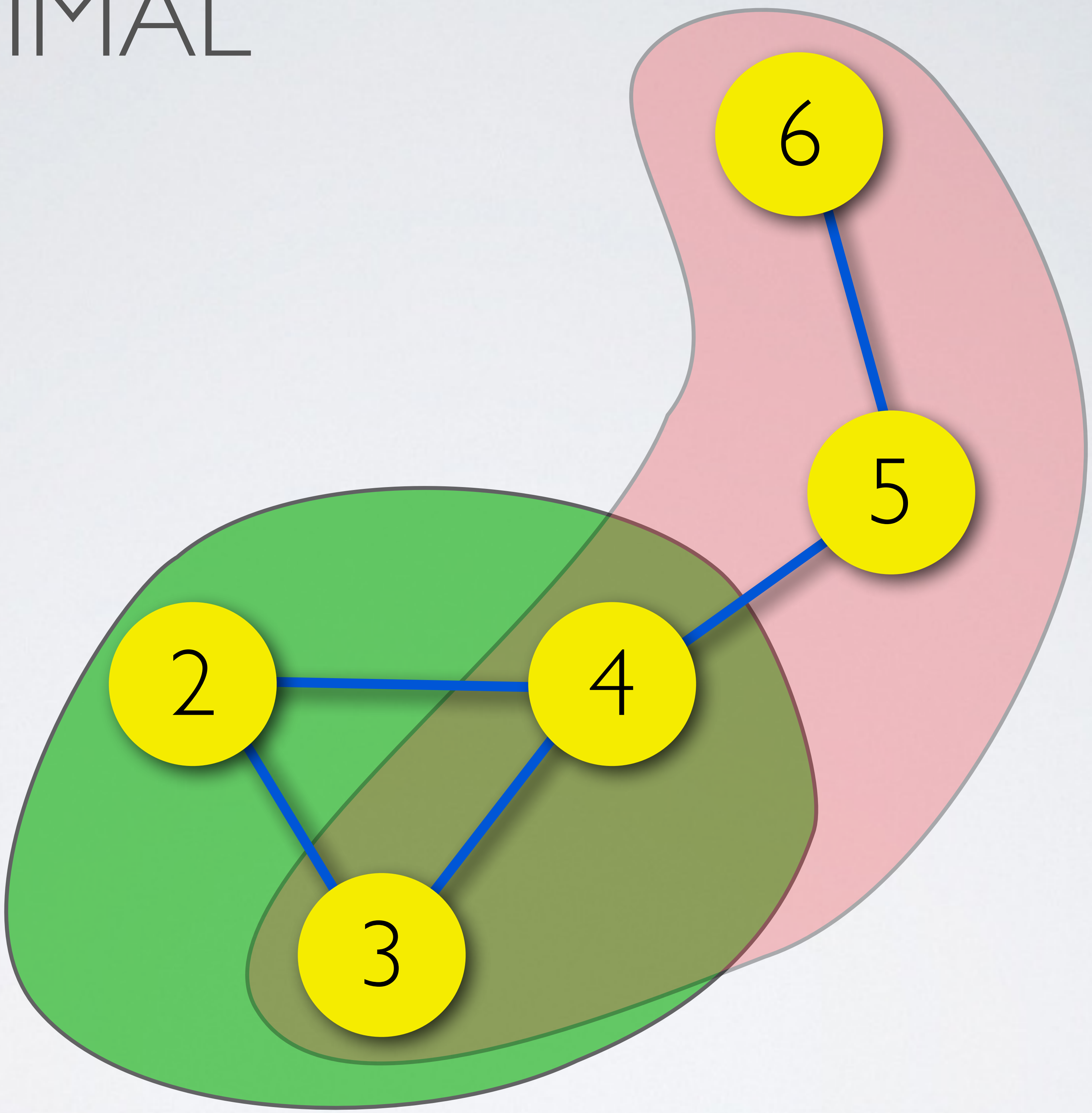
every edge is contained in an A in $\mathcal{B}$

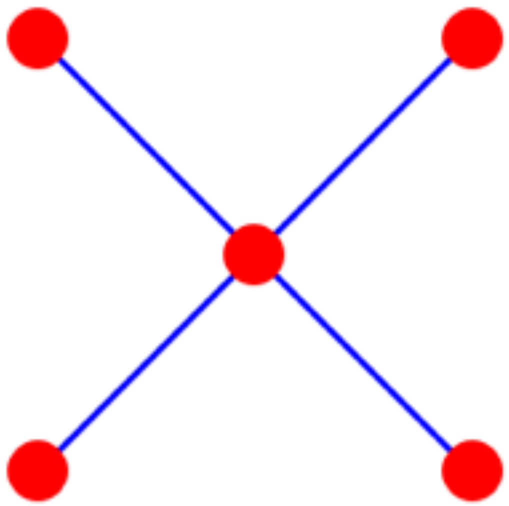
nerve graph is homotopic to G

INDISCRETE

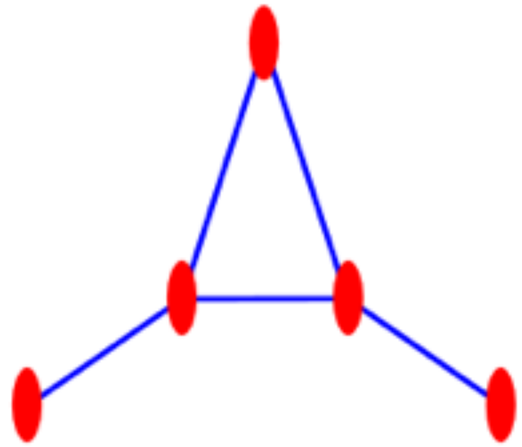


OPTIMAL

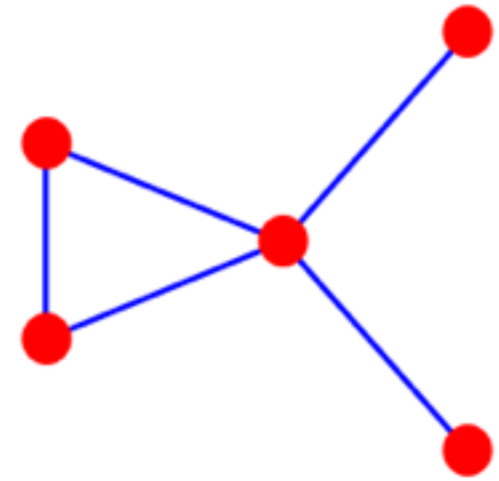




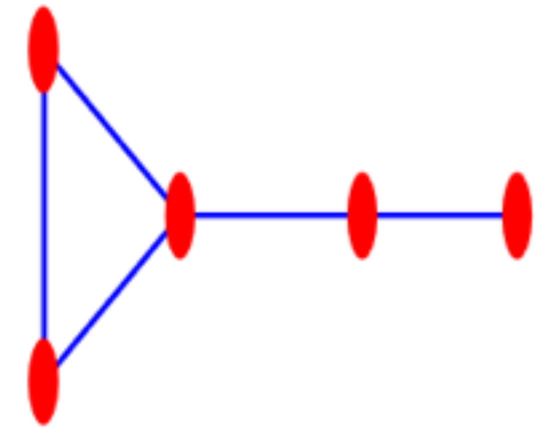
1



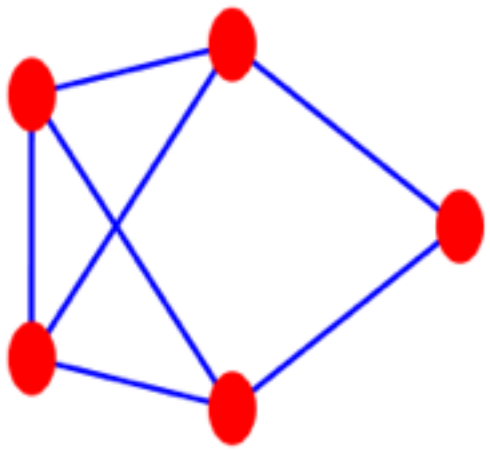
$\frac{22}{15}$



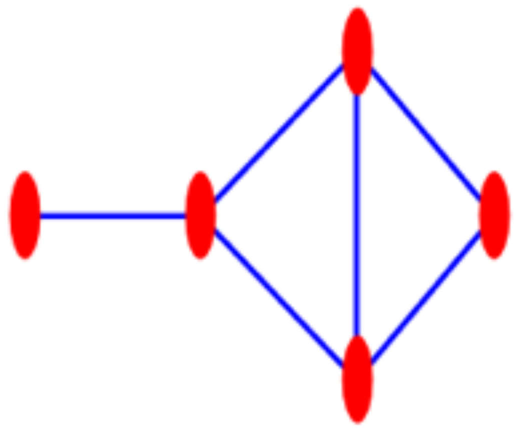
$\frac{3}{2}$



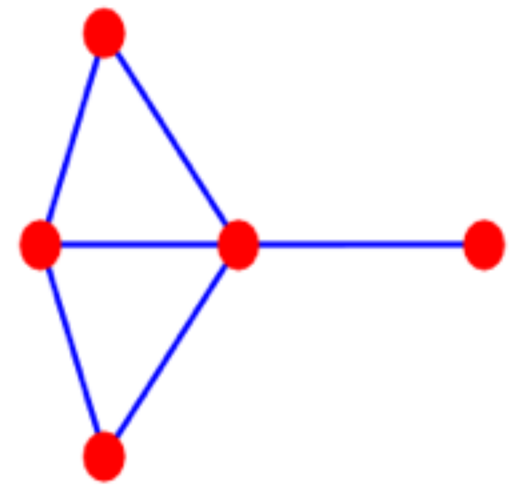
$\frac{23}{15}$



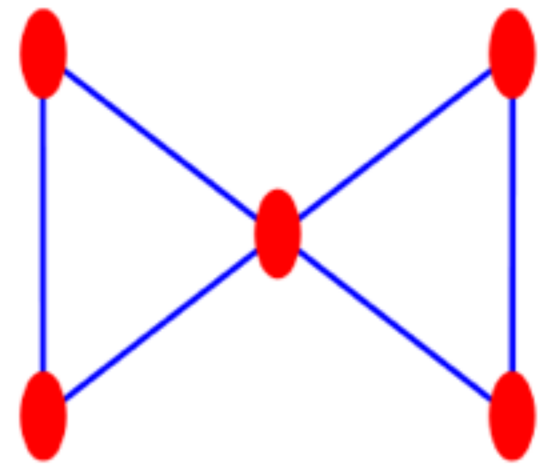
$\frac{5}{3}$



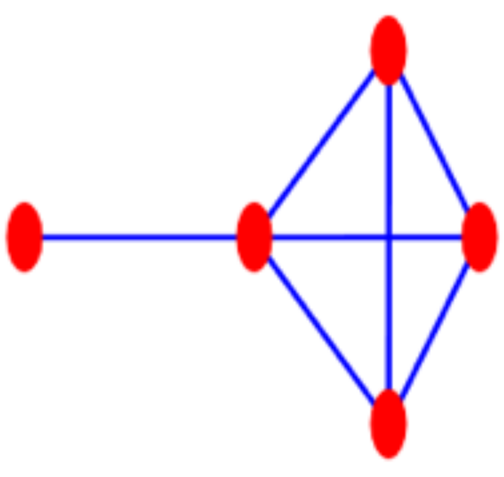
$\frac{26}{15}$



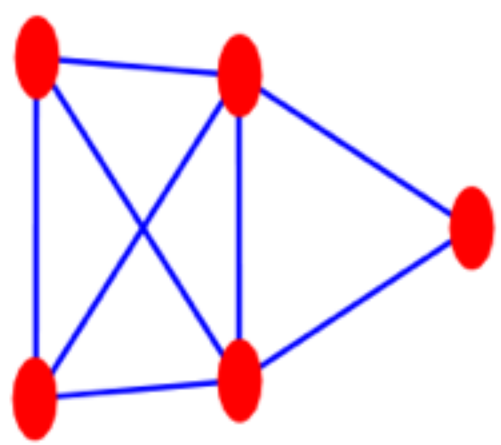
$\frac{7}{4}$



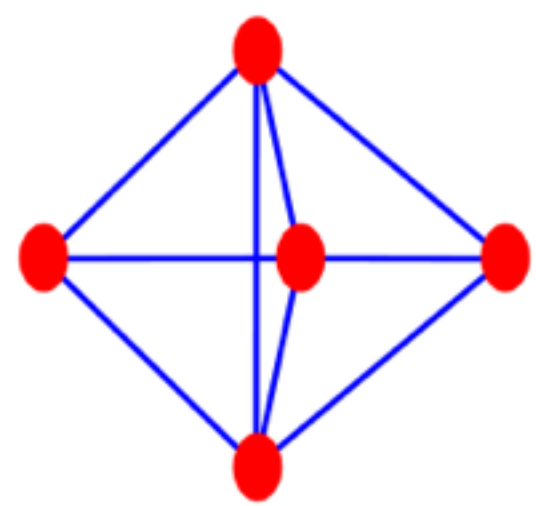
2



$\frac{5}{2}$



$\frac{8}{3}$



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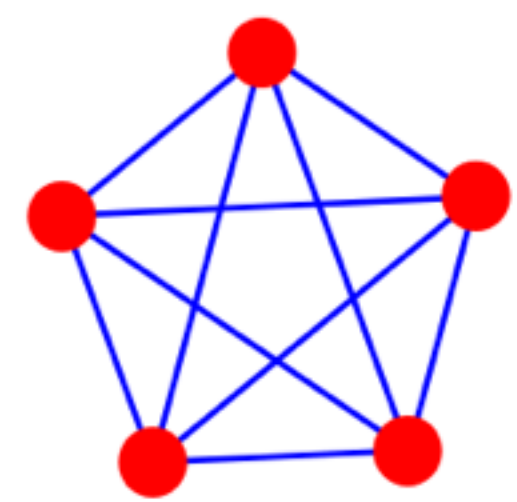
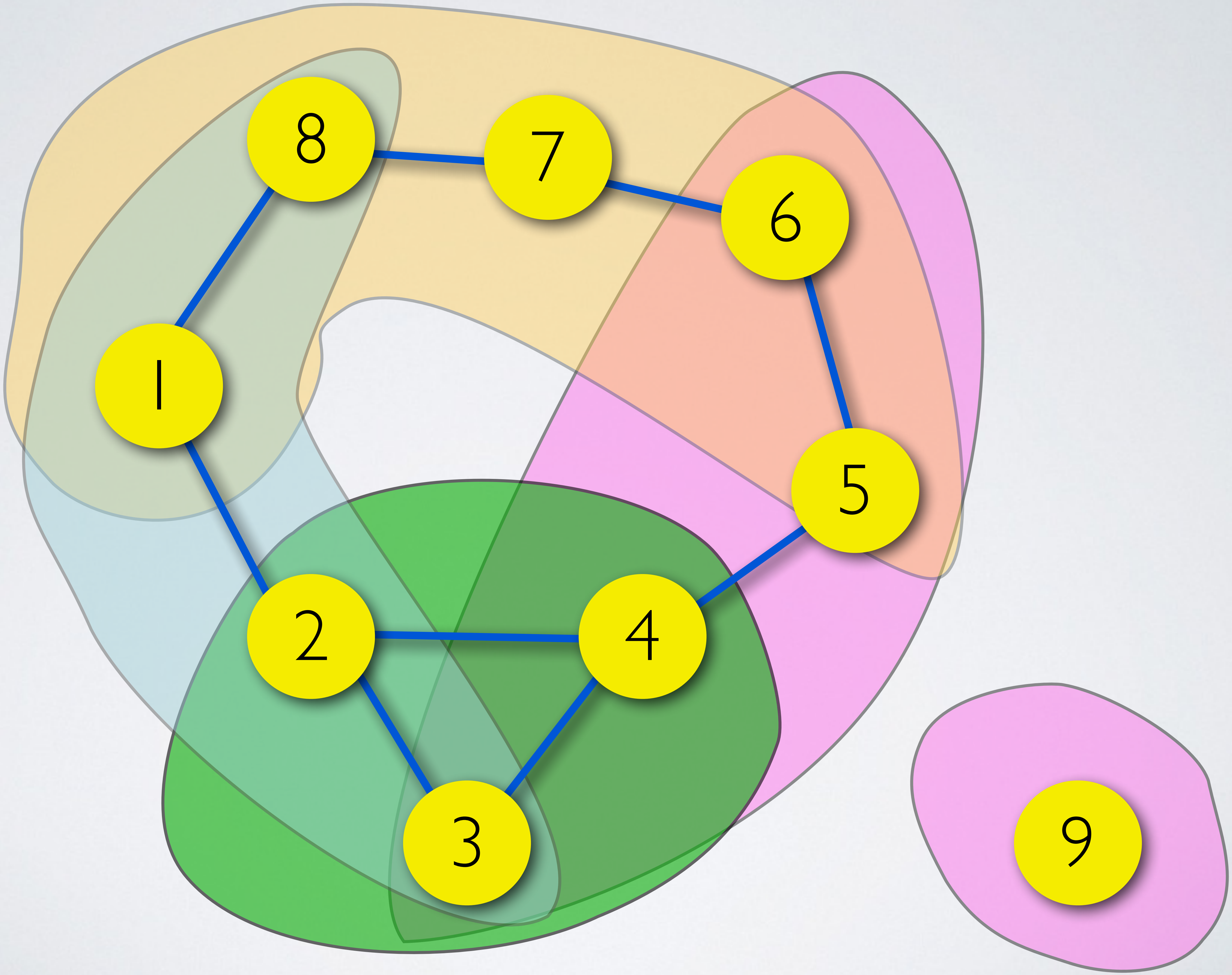
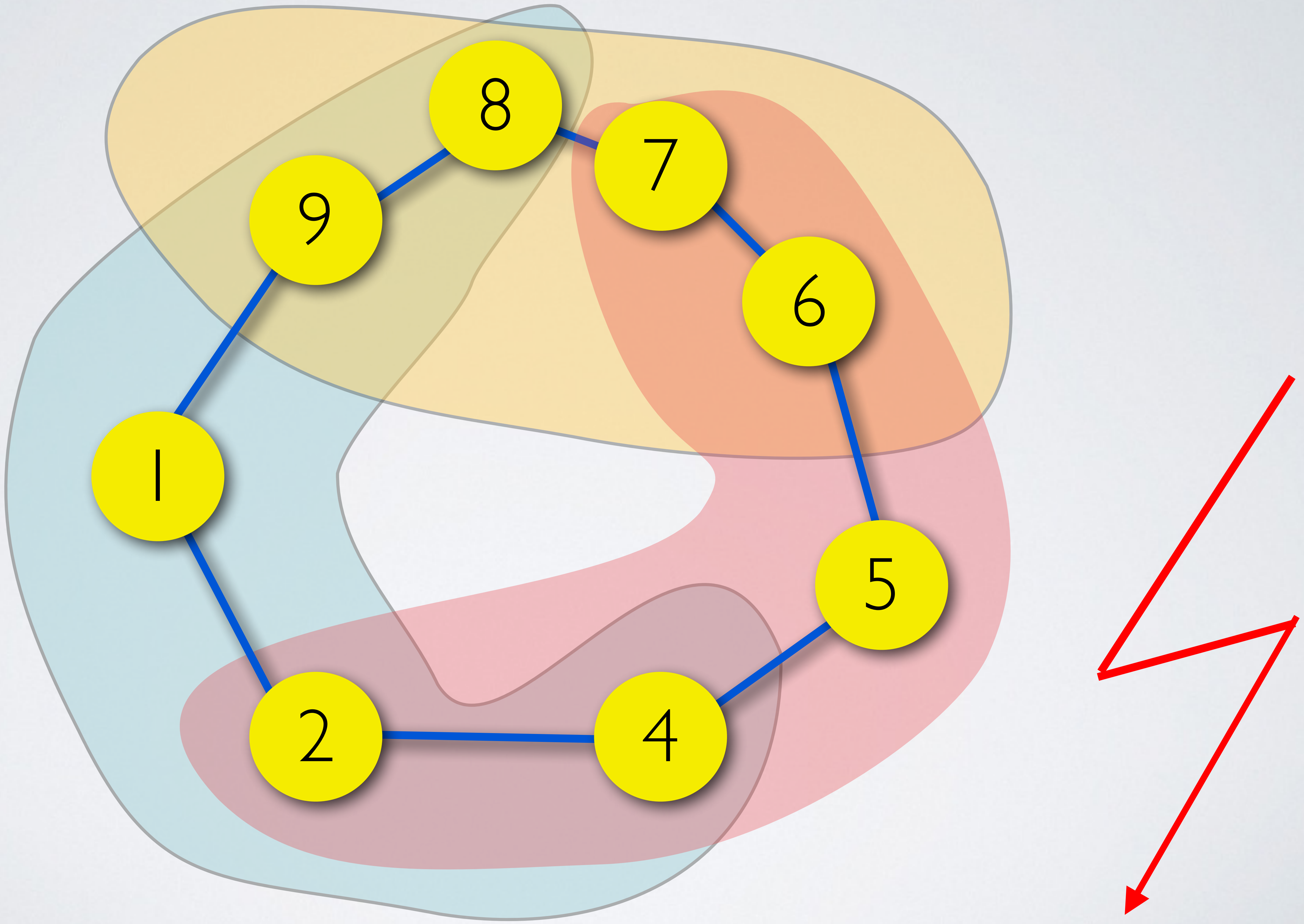


4

DIMENSION ASSUMPTION



NERVE NOT HOMOTOPIC



GRAPH HOMEOMORPHISM

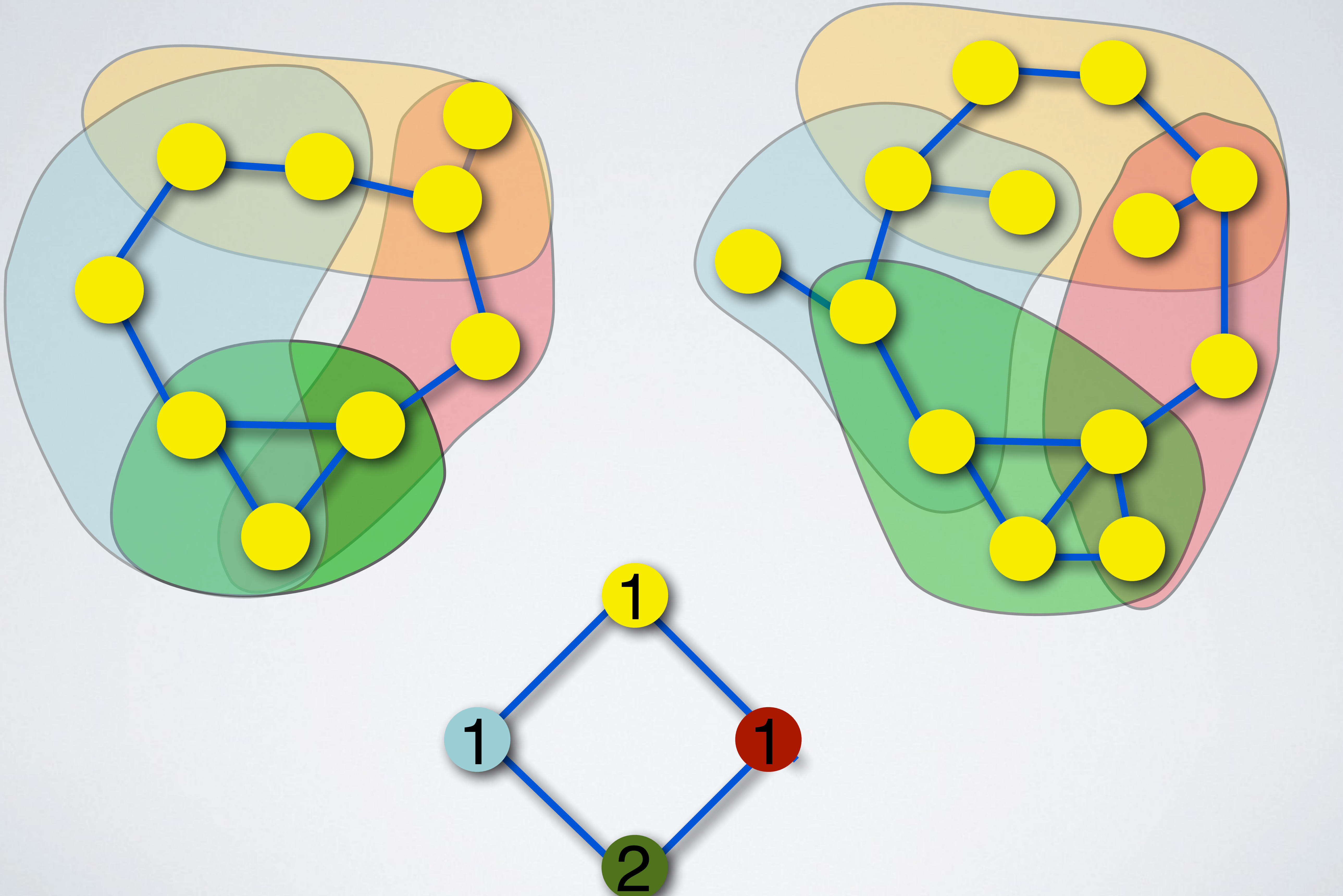
There exist graph topologies

defined by subbase $\mathcal{B}$ and C such that

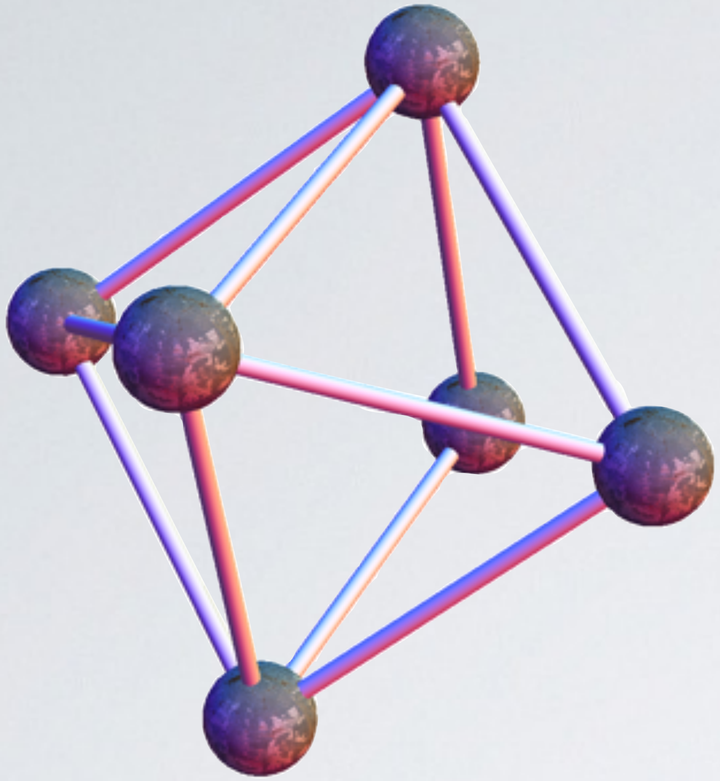
φ:$\mathcal{B}$ -> C is lattice isomorphism

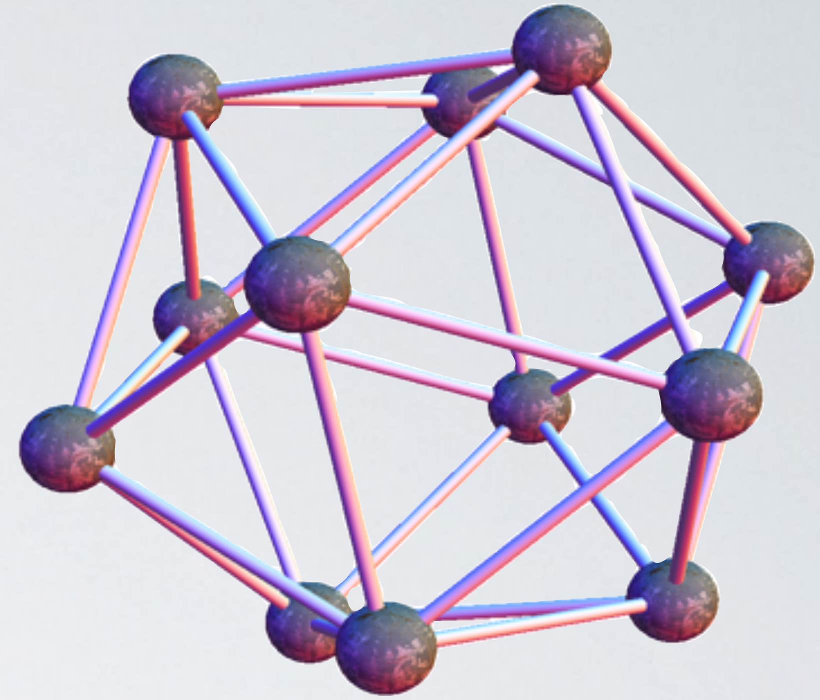
φ preserves dimension

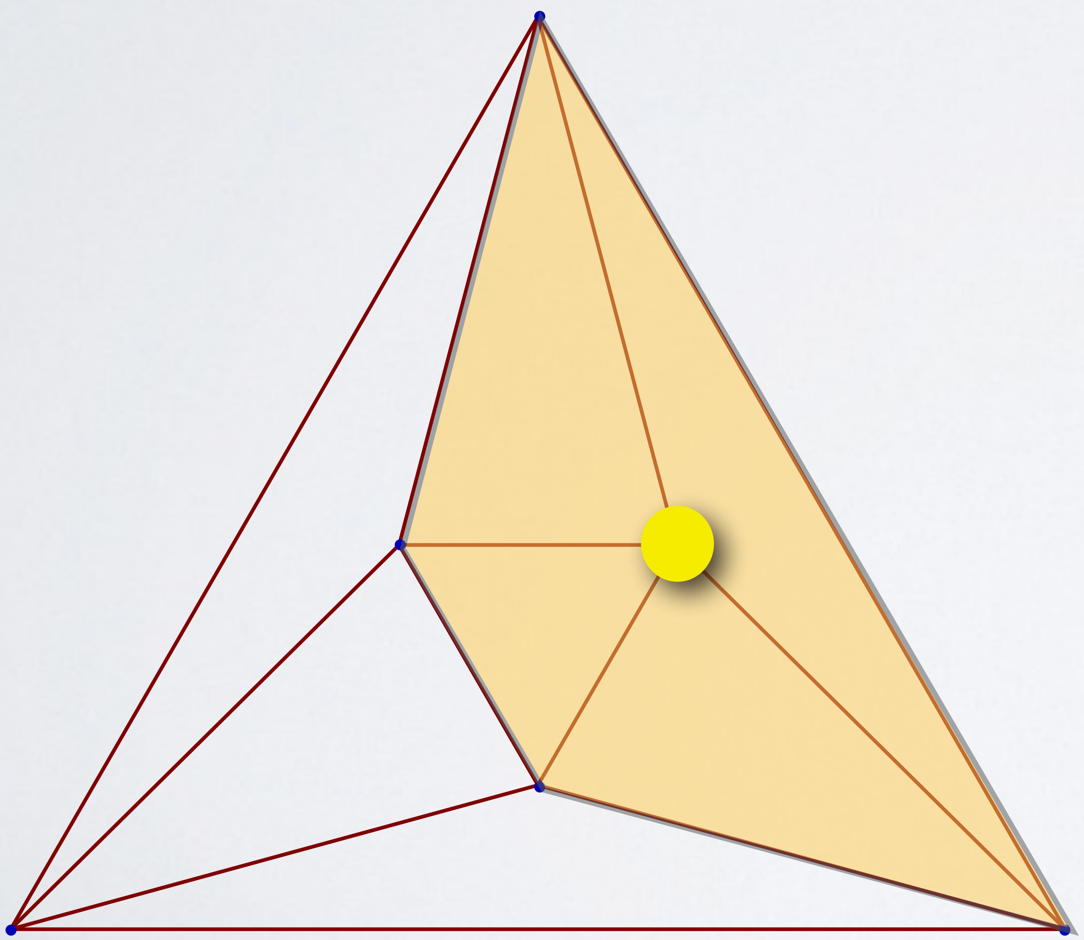
EXAMPLE



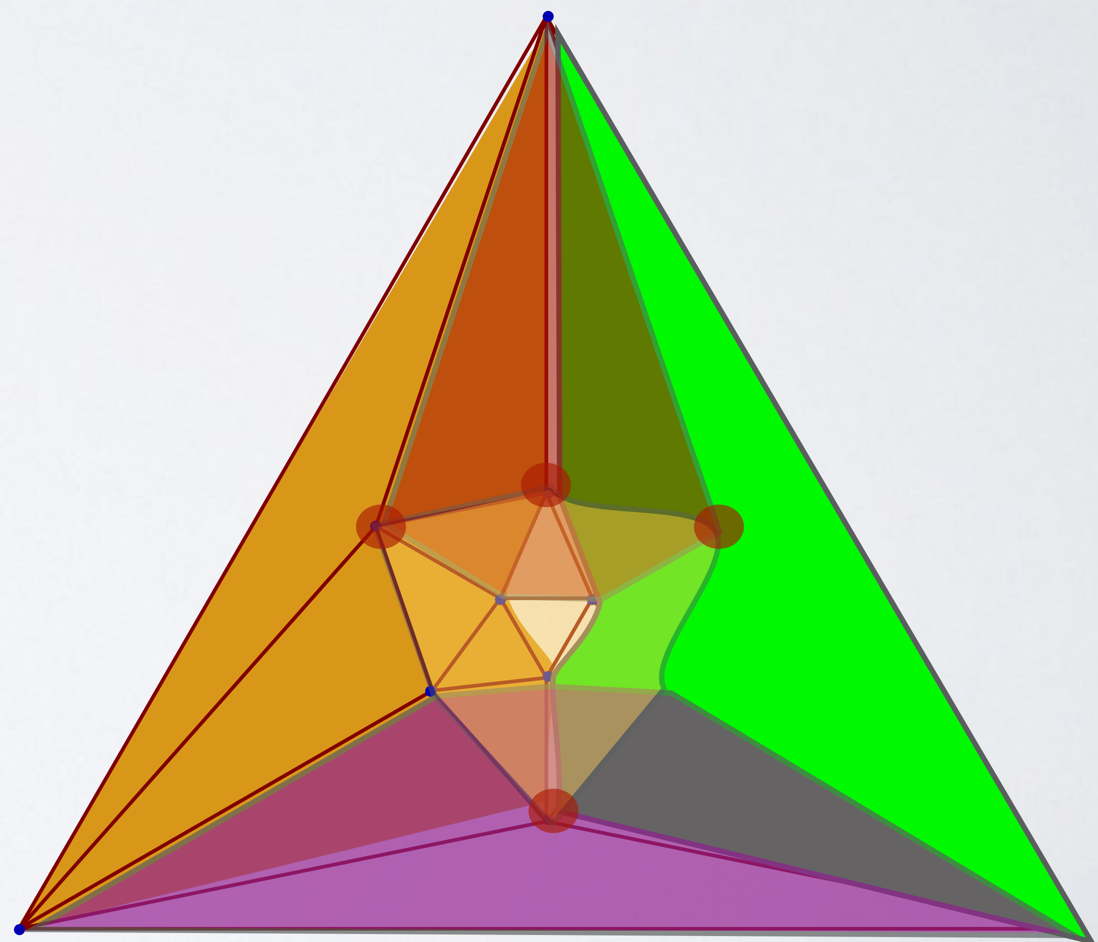
EXAMPLE







6 unit balls



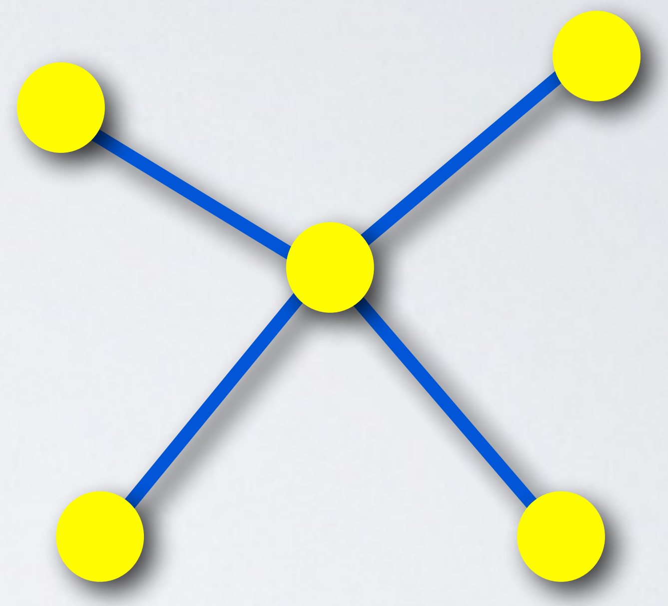
6 balls

FACTS

1. Homeomorphic graphs are homotopic.
2. Homeomorphic graphs have same cohomologies.
3. Isomorphic graphs are homeomorphic
4. Triangularizations of M are homeomorphic
5. 1-homeomorphisms are homeomorphisms
6. homeomorphism with nonzero L has contractible fixed set

GRAPH BROWER

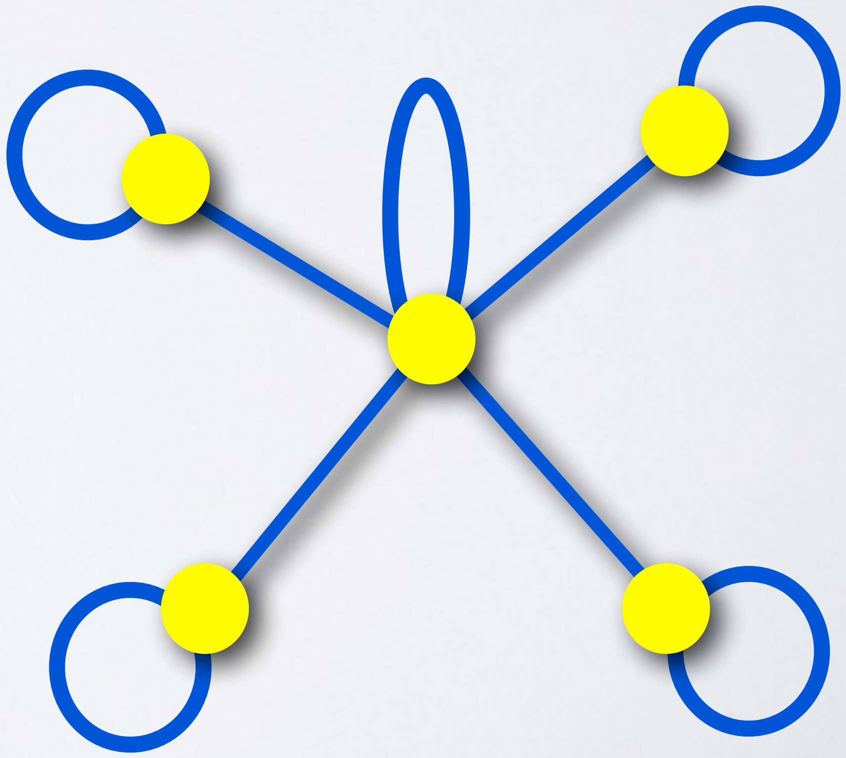
A graph endomorphism of a contractible graph has a fixed clique.



Generalizes edge theorem of

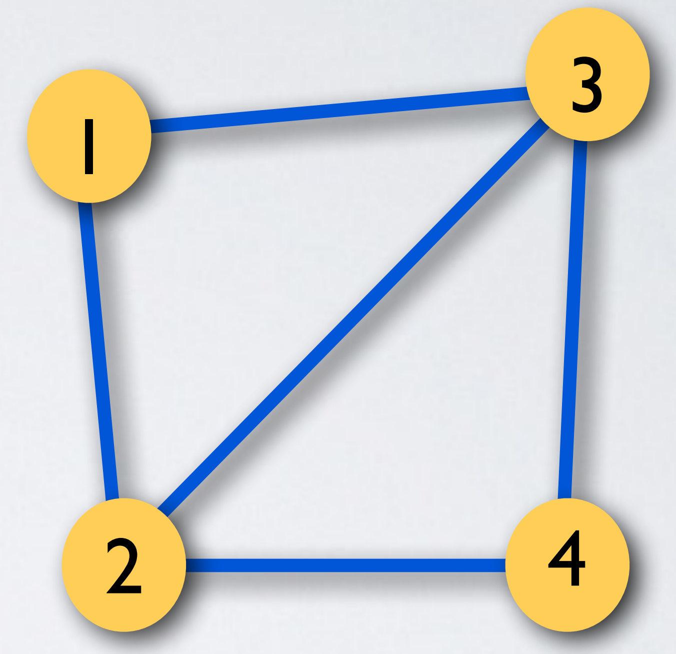
Nowakowski and Rival from 1979.

Add a loop at every vertex. Then every graph endomorphism has a fixed edge.



LEFSCHETZ

$$L(T) = \sum_{x \,\varepsilon\, Fix} i_T(x)$$

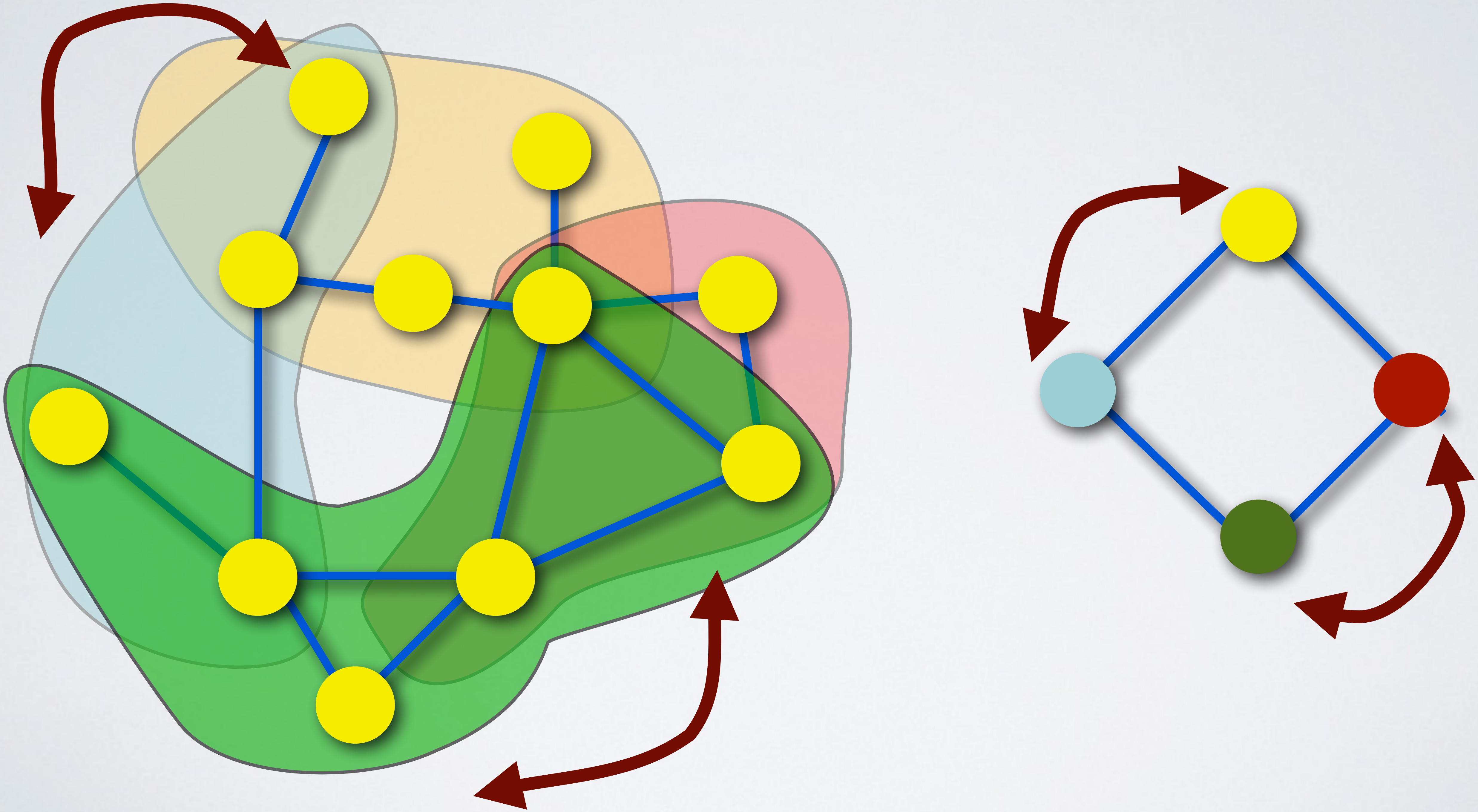


L(T) Lefschetz number: super trace on cohomology

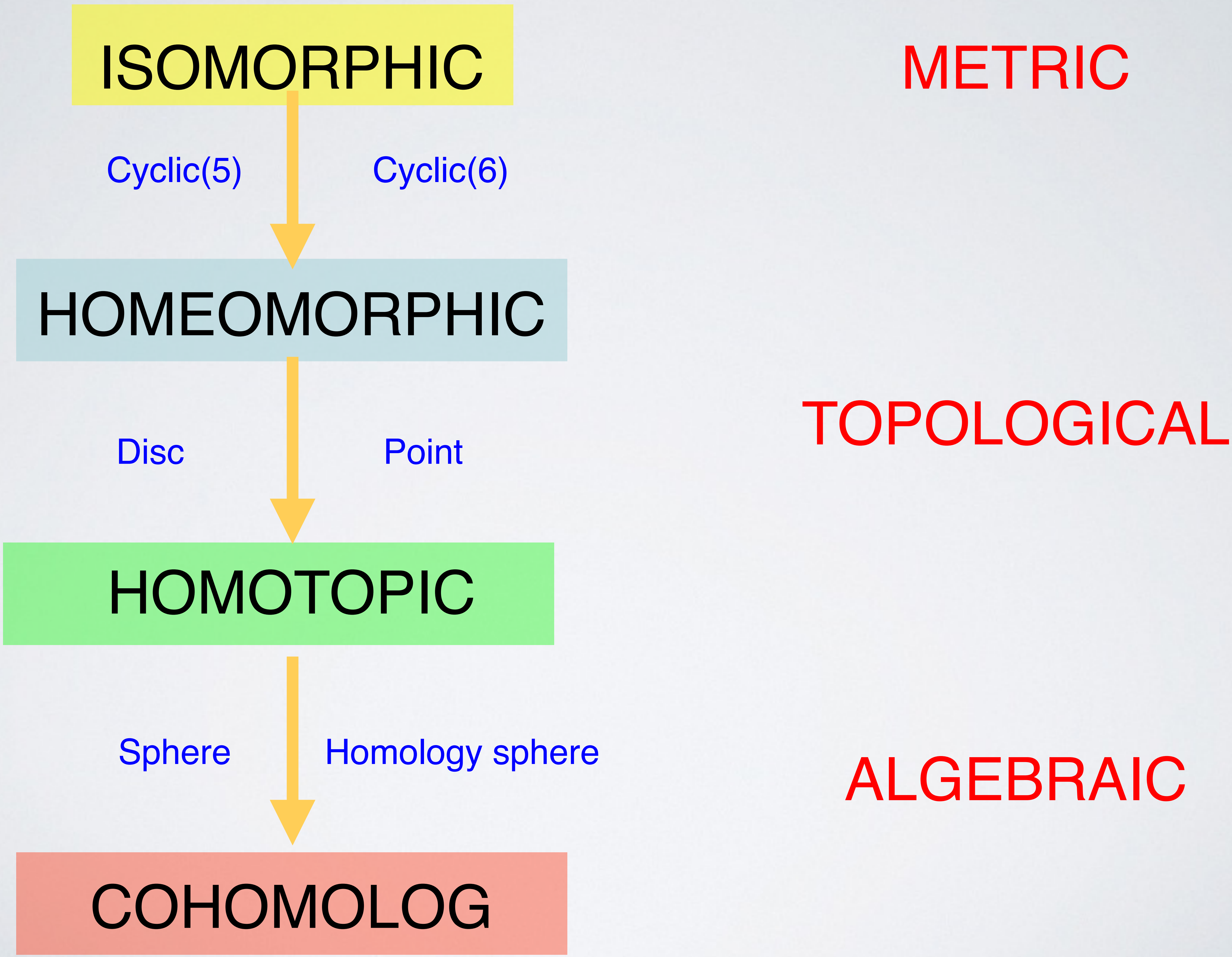
Example: $T = (14)(23)$ $L(T) = 1 - 0 + 0$

$x=(2,3) \,\varepsilon\, Fix$ with $i_T(x)=1$

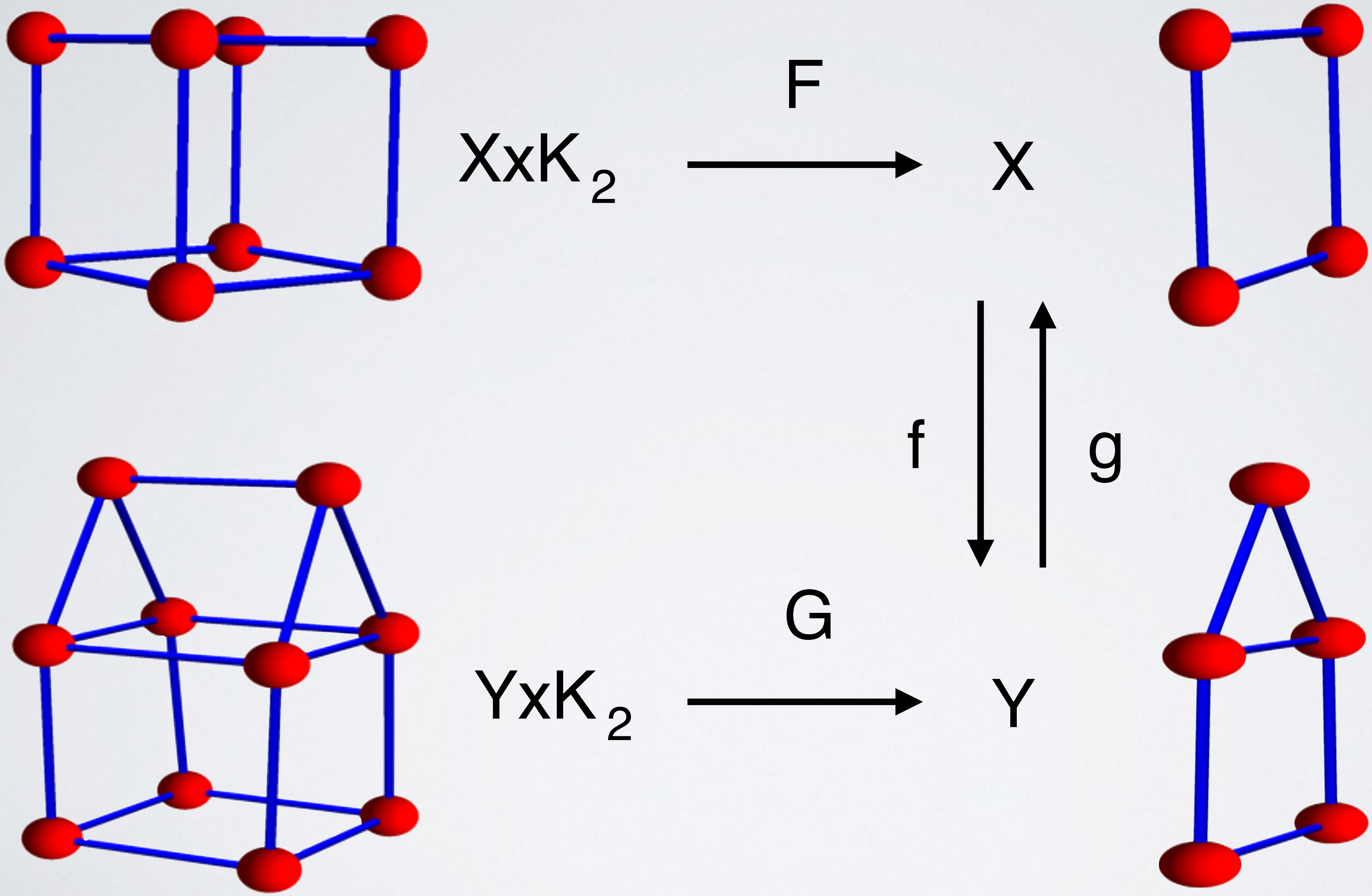
KAKUTANI



EQUIVALENCE RELATIONS

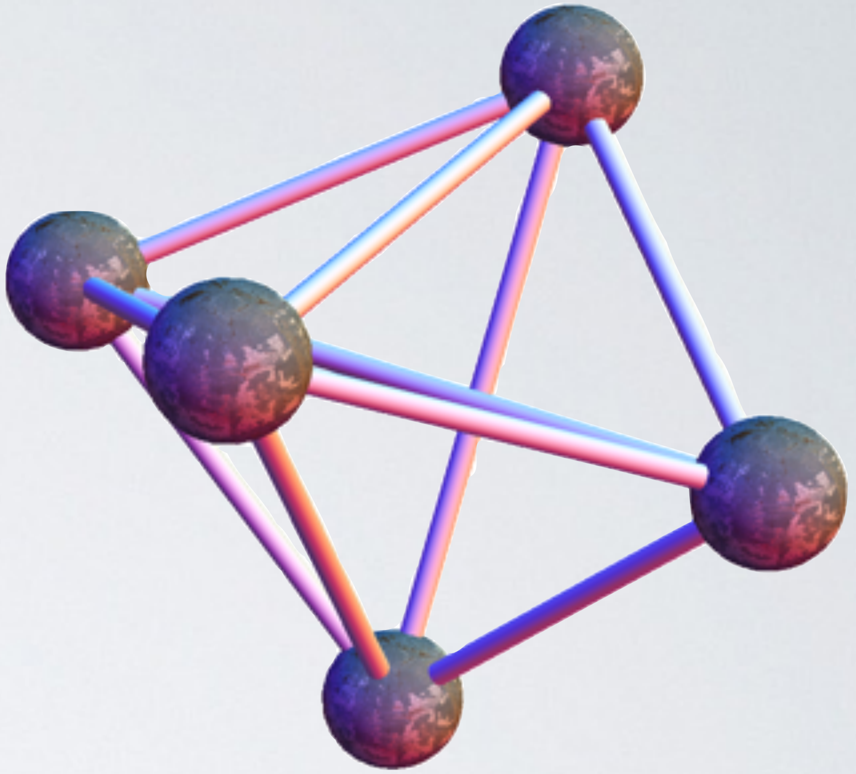


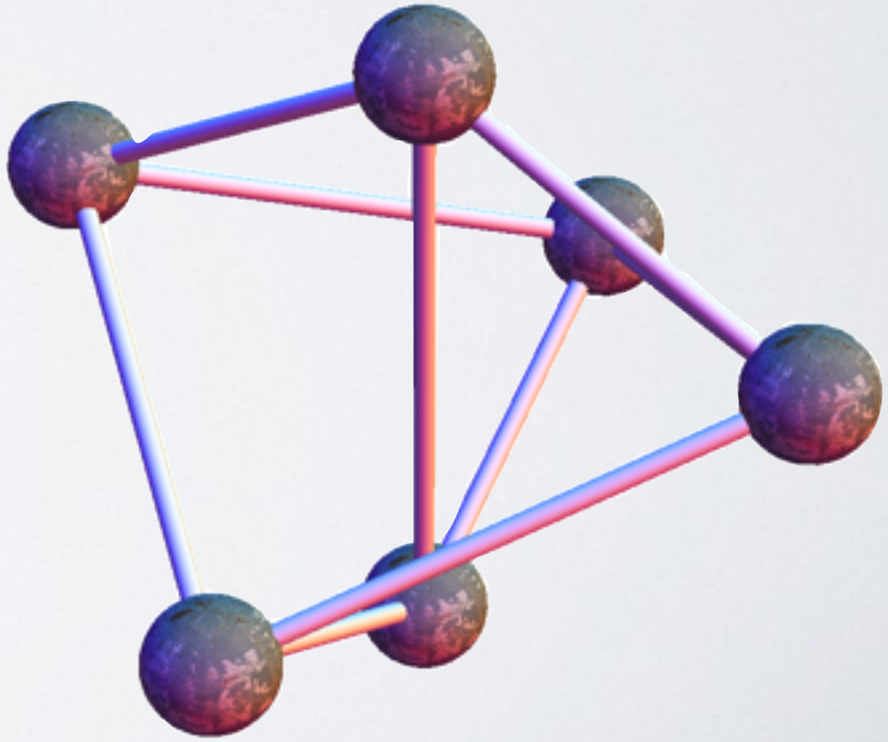
HOMOTOPY



PROBLEM

If G is homeomorphic to H and G is planar, is H planar?





THE END