MATHEMATICAL STRUCTURES IN GRAPH THEORY II

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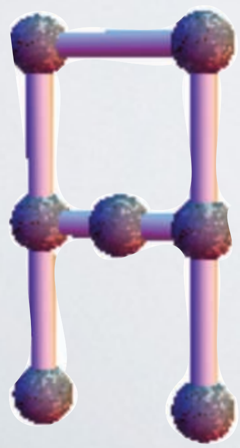
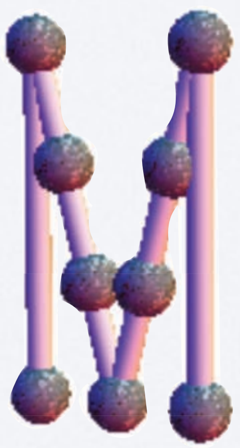
Harvard University



JAN 16



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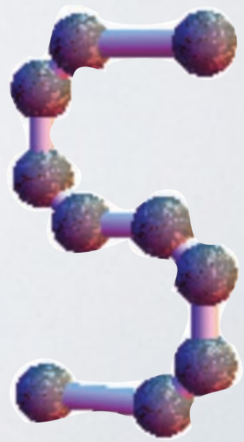


These 67 slides were not delivered on the 16th and are still in a rough state or partly recycled from a previous talk. It had become clear that there would be no time for them in 20 minutes.

delivered

slides I

not delivered

slides II

1 GEOMETRY

Differential Geometry,
Algebraic Geometry, Integral Geom.

2 TOPOLOGY

Homotopy, Dimension,
Graph Topology, Fixed points

3 ANALYSIS

Category, Calculus

4 ALGEBRA

Cohomology, Hodge theory Matrix Tree

5 DYNAMICS

Discrete PDE’s, Integrable Systems

6 SPECTRAL PROBLEMS

Zeta function

ANALYSIS

LJUSTERNIK SCHNIRELMANN

Ljusternik 1899 - 1981

Schnirelmann 1905-1938

LJUSTERNIIK SCHNIRELMANN THEORY

A prototype theory which has a couple of implementations in the discrete. The version done together with Josellis is by far closest to the continuum and natural since all notions are parallel and the results are the same than in the continuum except for minor modifications.

Essentially the only difference is that graphs can be too small to make an embedded graph contractible. The equator circle in an octahedron G is not contractible in G. But we can expand G first and then contract so that the circle deforms to a point.

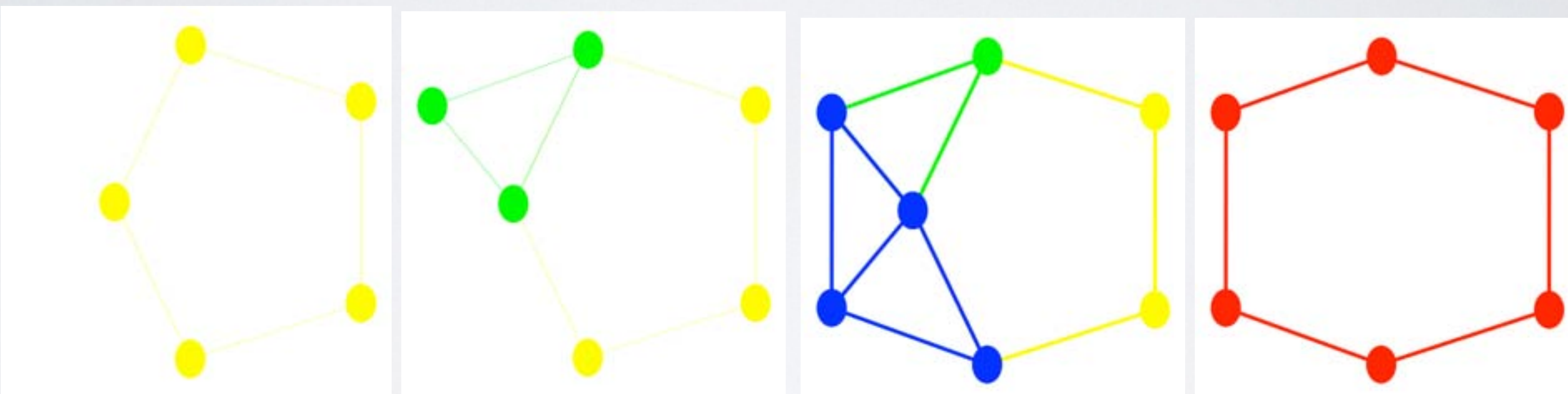
HOMOTOPY

Two graphs are homotopic, if they can be transformed into each other by homotopy steps.

Ivashchenko
1994 also has edge operations

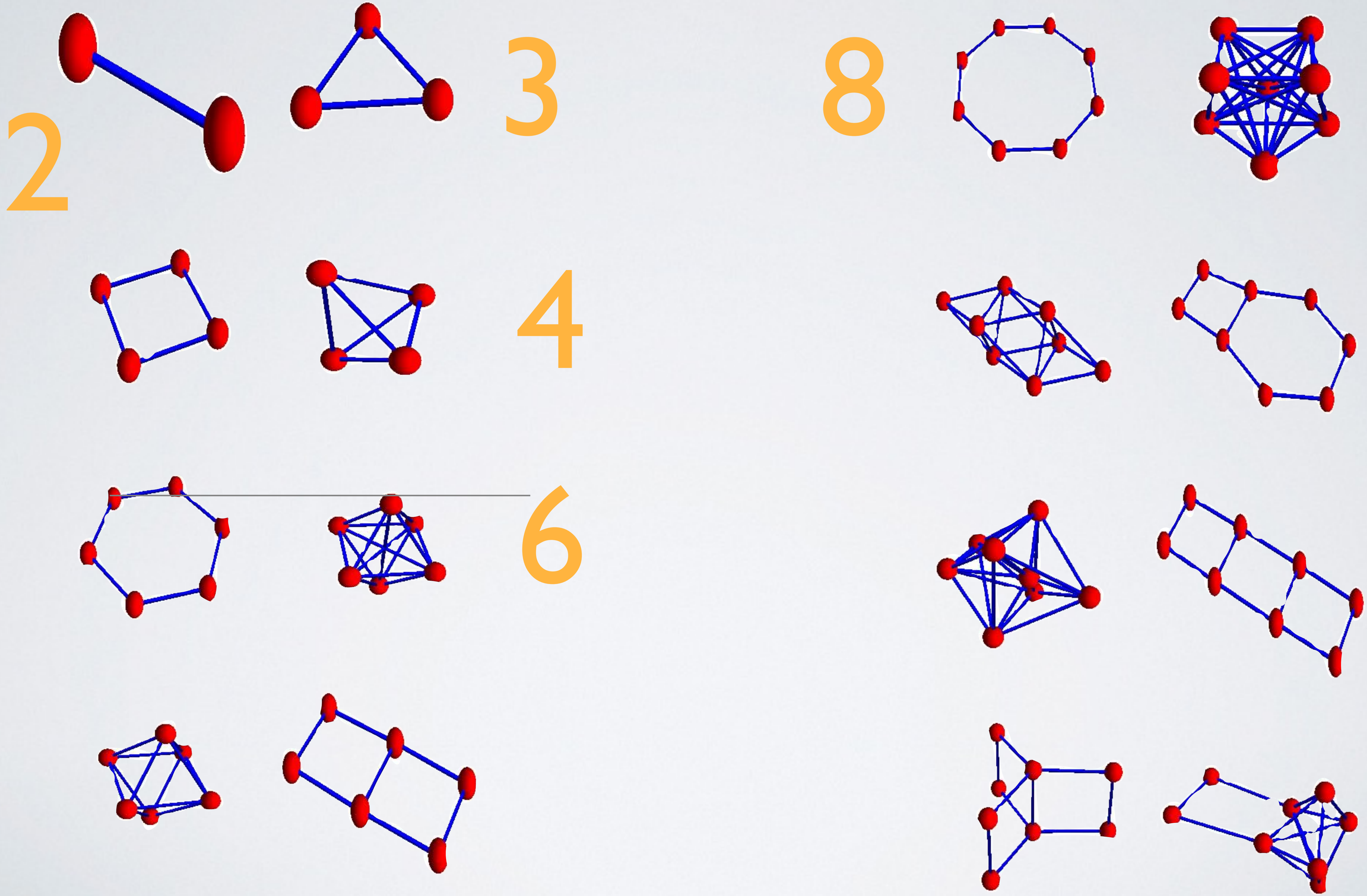
ChenYauYeh
2001:

edge operations not needed

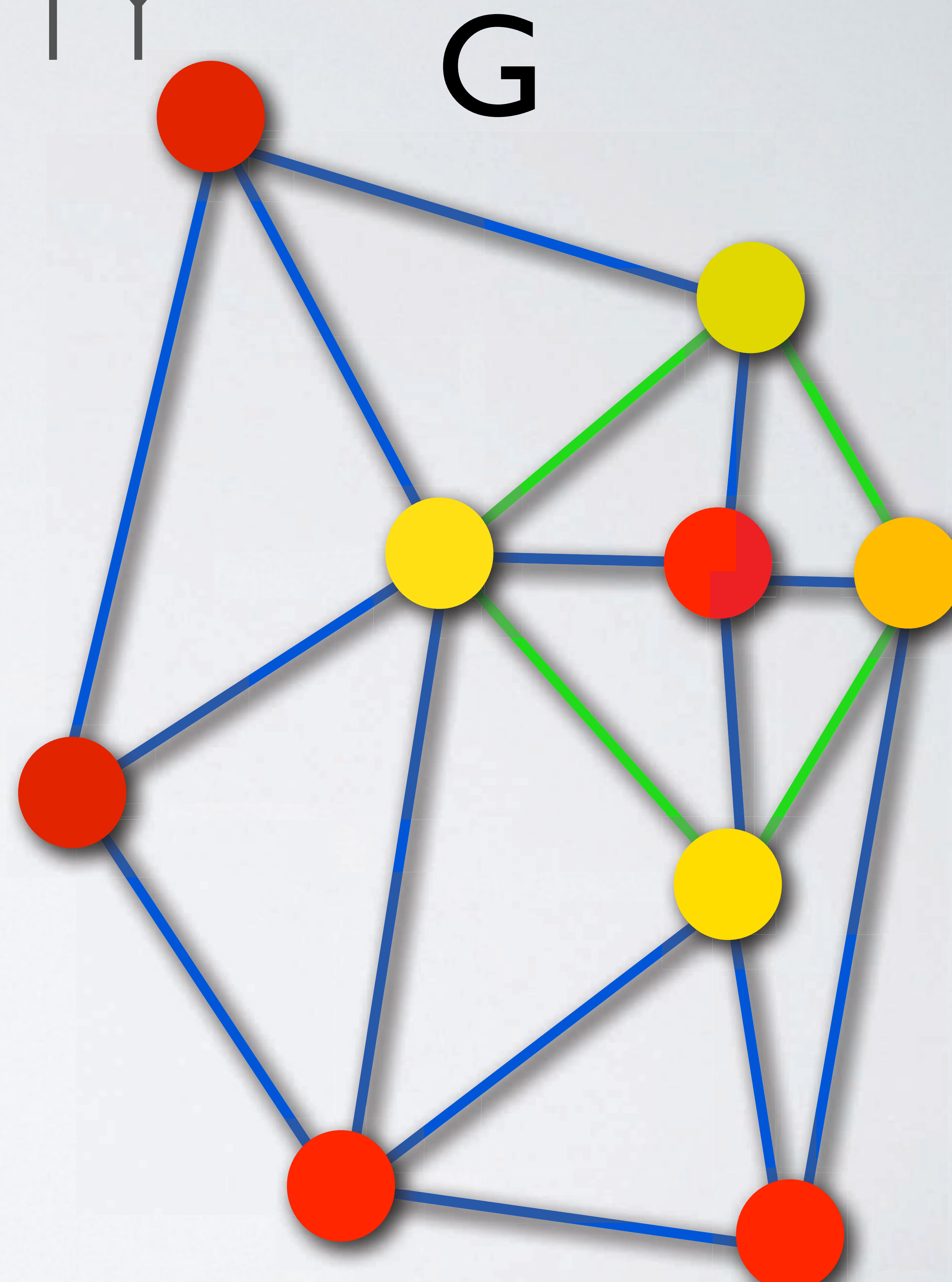


C_5 C_6

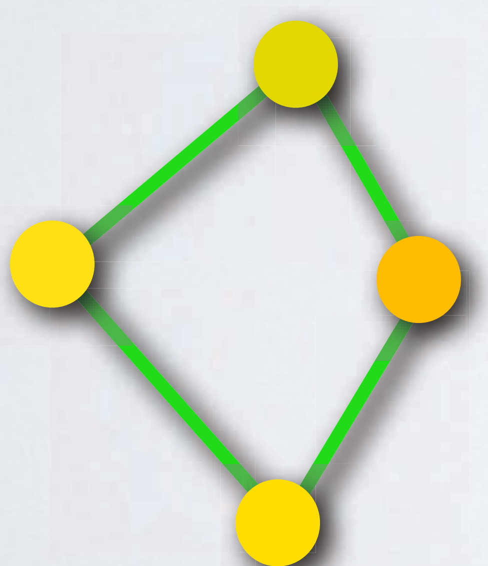
HOMOTOPY CLASSES



CONTRACTIBILITY WITHIN G



Contraction of G which shrinks H to a point of if H is contractible in itself.



is contractible in G but not in itself.

LJUSTERNIK SCHNIRELMANN

work with Frank Josellis

tcat(G) = min # in G contractible sets

crit(G) = min # of critical points

cup(G) = cup length

cup ≤ cat ≤ cri

Algebra Topology Analysis

cat and cri are homotopy invariant versions of tact and crit.

CALCULUS

Calculus on graphs is surprisingly simple. Stokes theorem is almost tautological in arbitrary dimensions.

Surprising is also that this has already been done by Poincare. Poincare defined grad, curl, div etc in arbitrary dimensions as concrete matrices and used this to define cohomology.

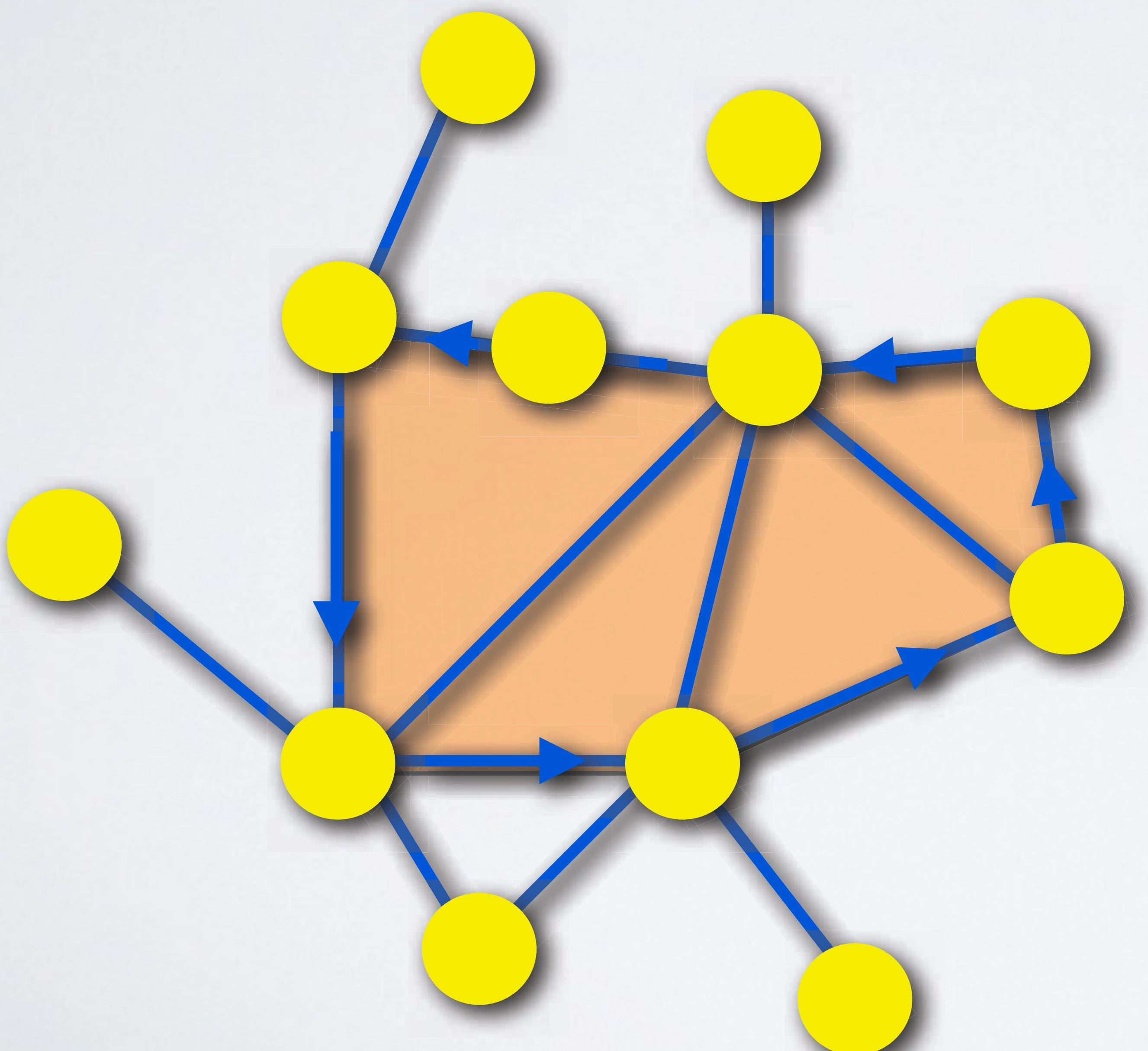
STOKES-GAUSS

Stokes

Gauss

GREEN STOKES GAUSS

$$\sum_{x \,\varepsilon G} df(x) = \sum_{x \,\varepsilon\, \delta G} f(x)$$



d_0 grad

d_1 curl

d_0^* div

Δ = div grad

STOKES

$$\int_S curl(F)\, dS = \int_C F dr$$

MORSE REEB

MORSE THEORY

Morse theory also has several implementations in the discrete. Our approach does not need additional structure on the graph.

We have seen that Poincare Hopf works with Morse ideas. What is needed for Morse theory is that functions are so nice that adding a point only changes one entry by 1 in the Betti vector.

Then the Morse inequalities etc hold as in the continuum.

MORSE REEB

MORSE

$$\sum (-1)^k c_k = \chi$$

$$b_k \leq c_k$$

Algebra Analysis

REEB

2 critical points -> sphere

Analysis Topology

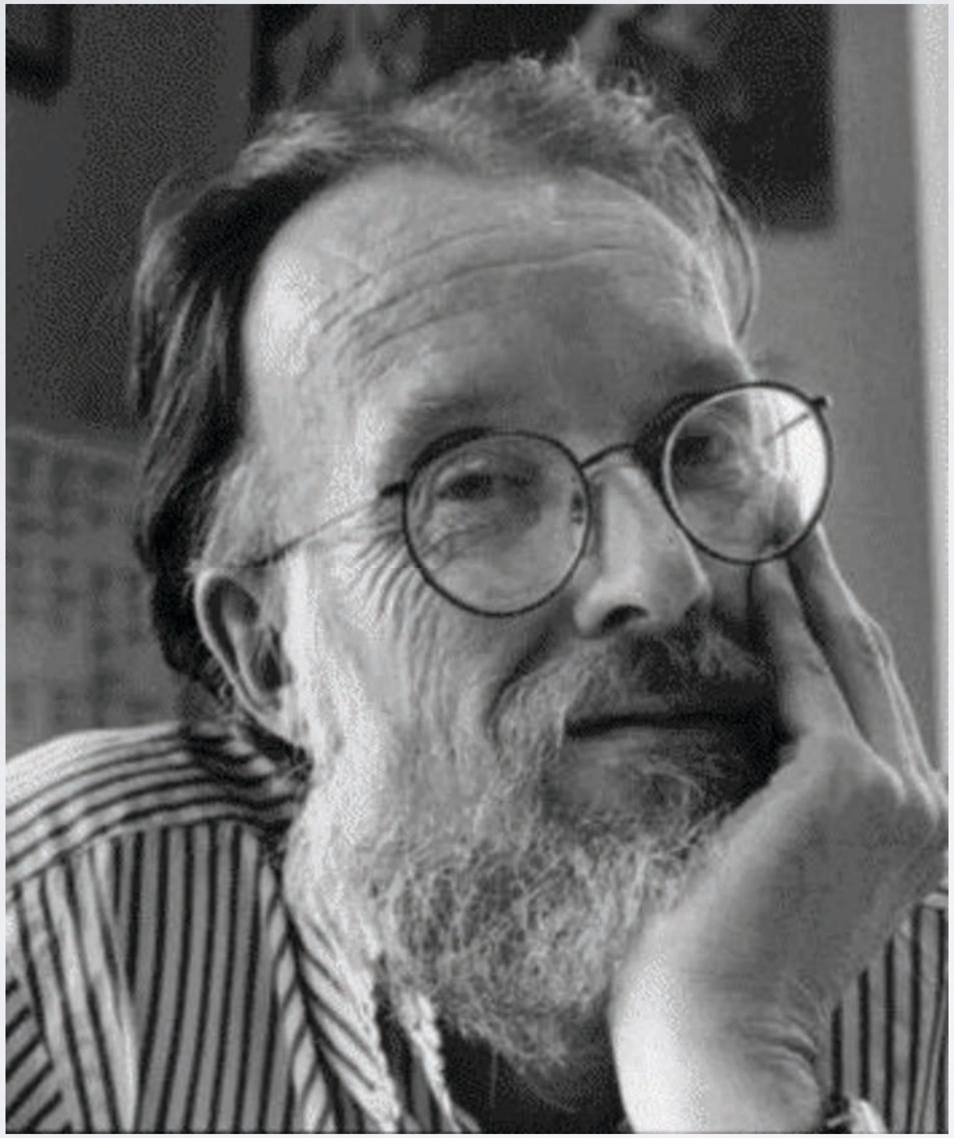
THE END

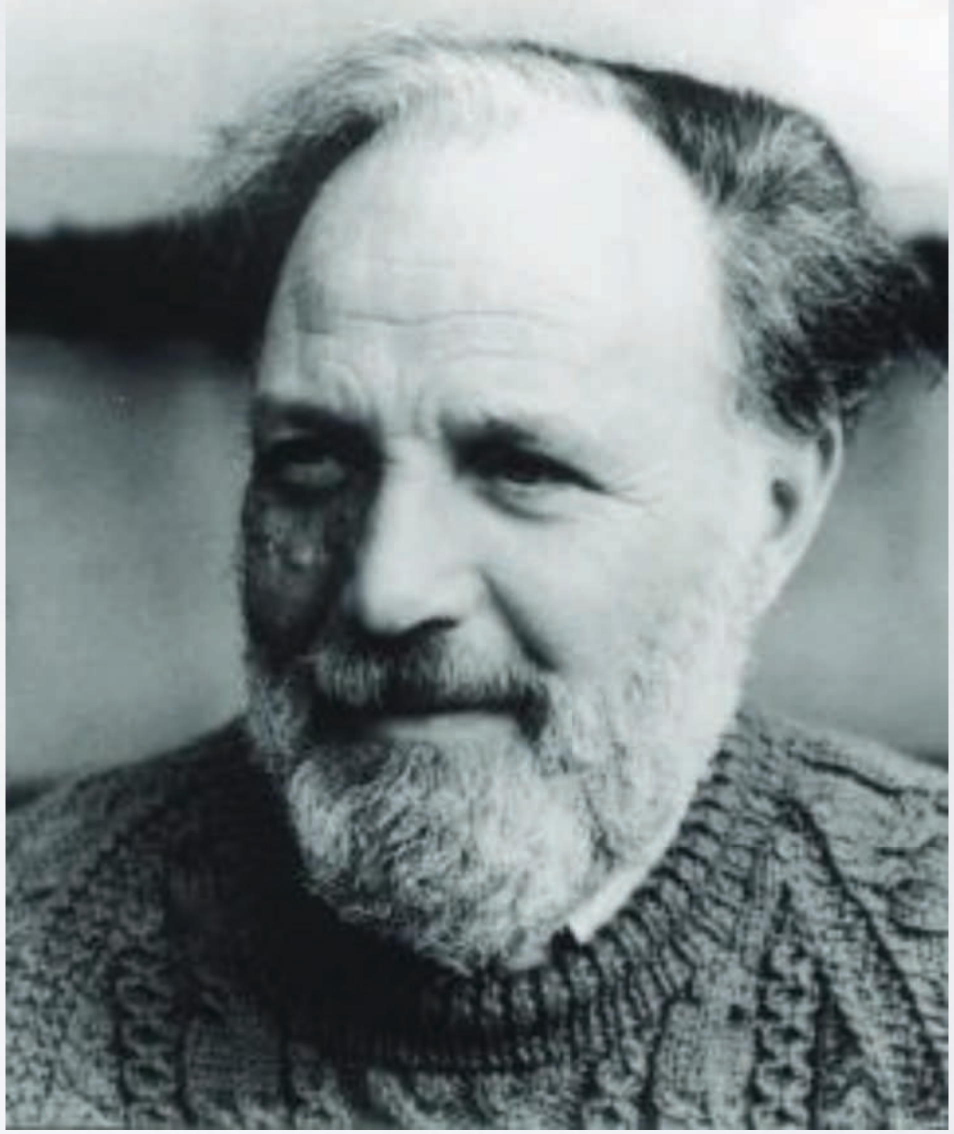
ALGEBRA

ALGEBRA

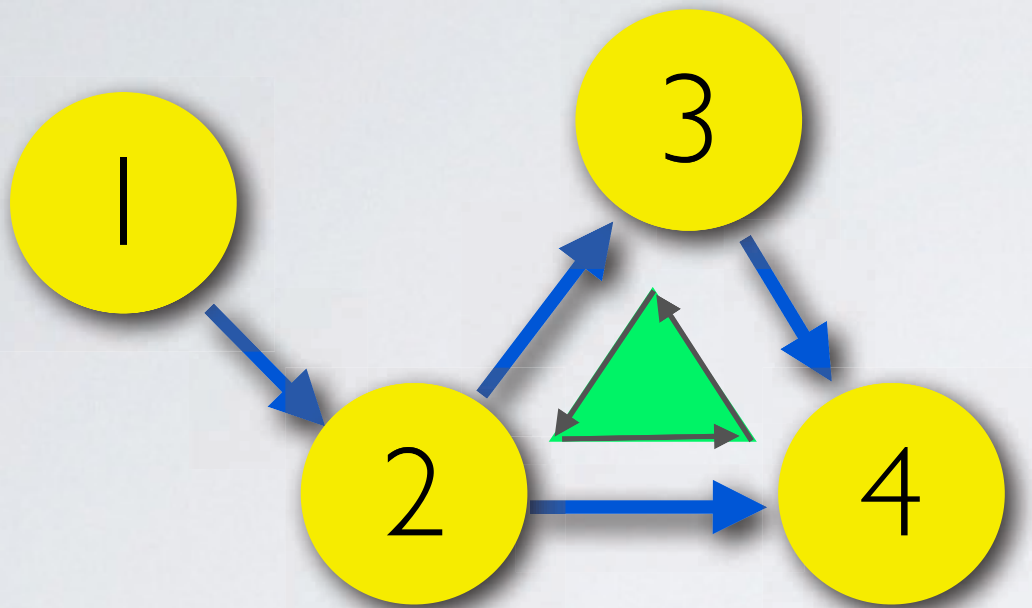
Some of the following slides are from my ILAS talk which had obviously more focus on linear algebra. I recycle here some slides.

MCKEAN SINGER





DIRAC OPERATOR



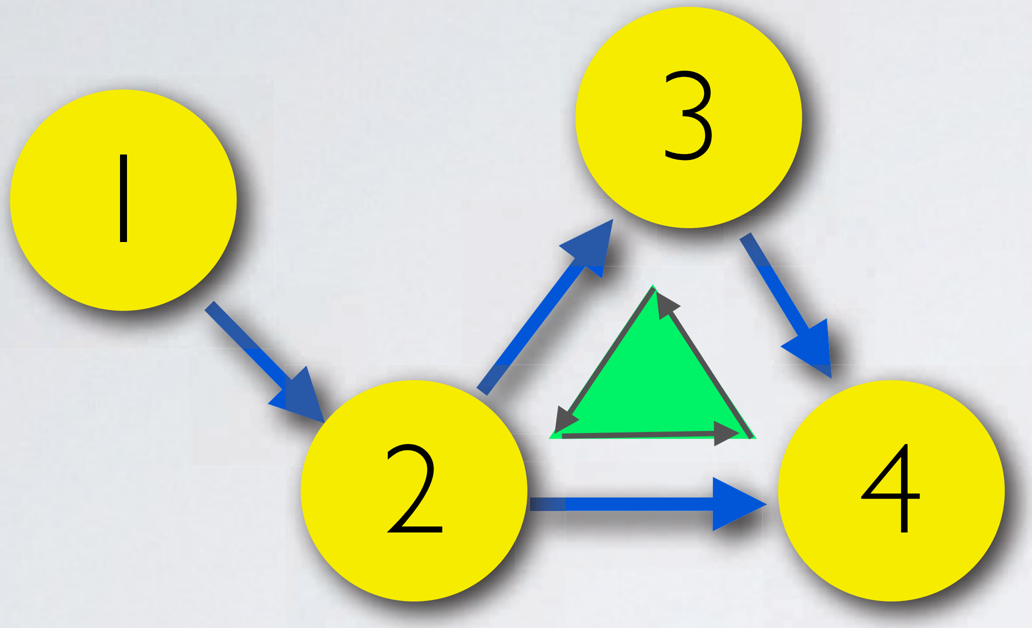
$$D = d + d^*$$

0	0	0	0	-1	0	0	0	0
0	0	0	0	1	-1	-1	0	0
0	0	0	0	0	0	1	-1	0
0	0	0	0	0	1	0	1	0
-1	1	0	0	0	0	0	0	0
0	-1	0	1	0	0	0	0	1
0	-1	1	0	0	0	0	0	-1
0	0	-1	1	0	0	0	0	-1
0	0	0	0	0	1	-1	-1	0

$v_0 = 4$
$v_1 = 4$
$v_2 = 1$

$\chi(G)=1$

LAPLACE-BELTRAMI MATRIX

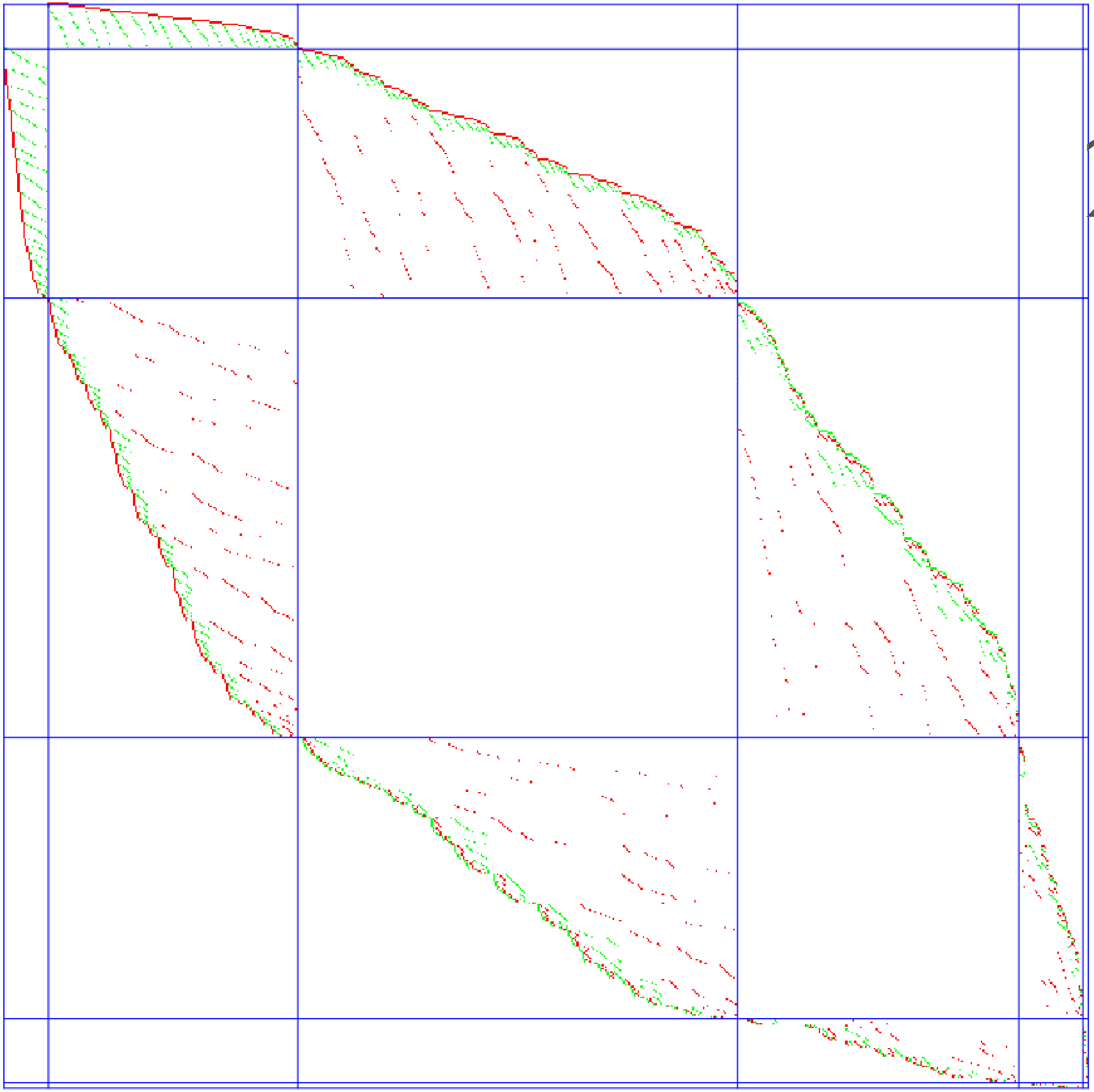


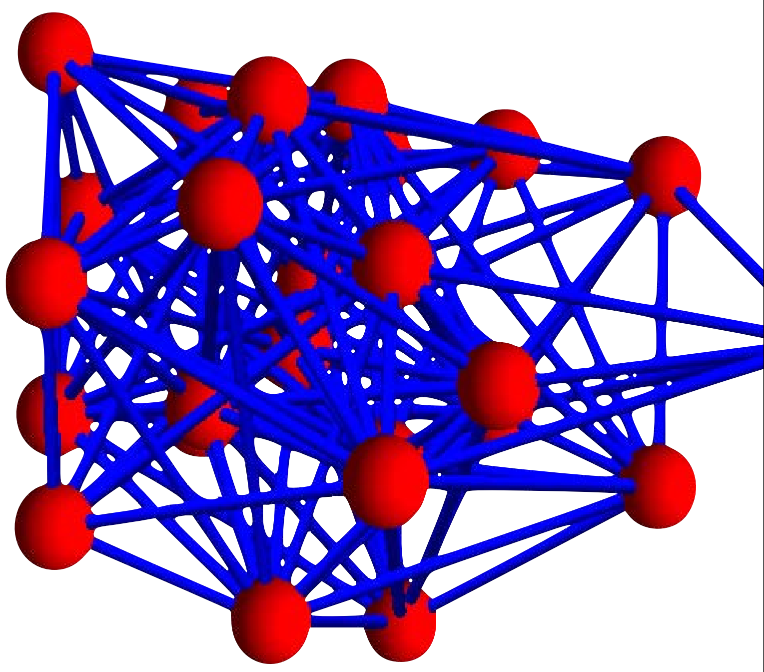
$$L = (d + d)^2 =$$

1	-1	0	0	0	0	0	0	0
-1	3	-1	-1	0	0	0	0	0
0	-1	2	-1	0	0	0	0	0
0	-1	-1	2	0	0	0	0	0
0	0	0	0	2	-1	-1	0	0
0	0	0	0	-1	3	0	0	0
0	0	0	0	-1	0	3	0	0
0	0	0	0	0	0	0	3	0
0	0	0	0	0	0	0	0	3

is block diagonal.

RANDOM 25 VERTEX GRAPH





COHOMOLOGY

$H^k(G) = \ker(d_k)/\mathrm{im}(d_{k-1})$

k'th cohomology group

$b_k(G)=\dim(H^k(G))$ Betti number

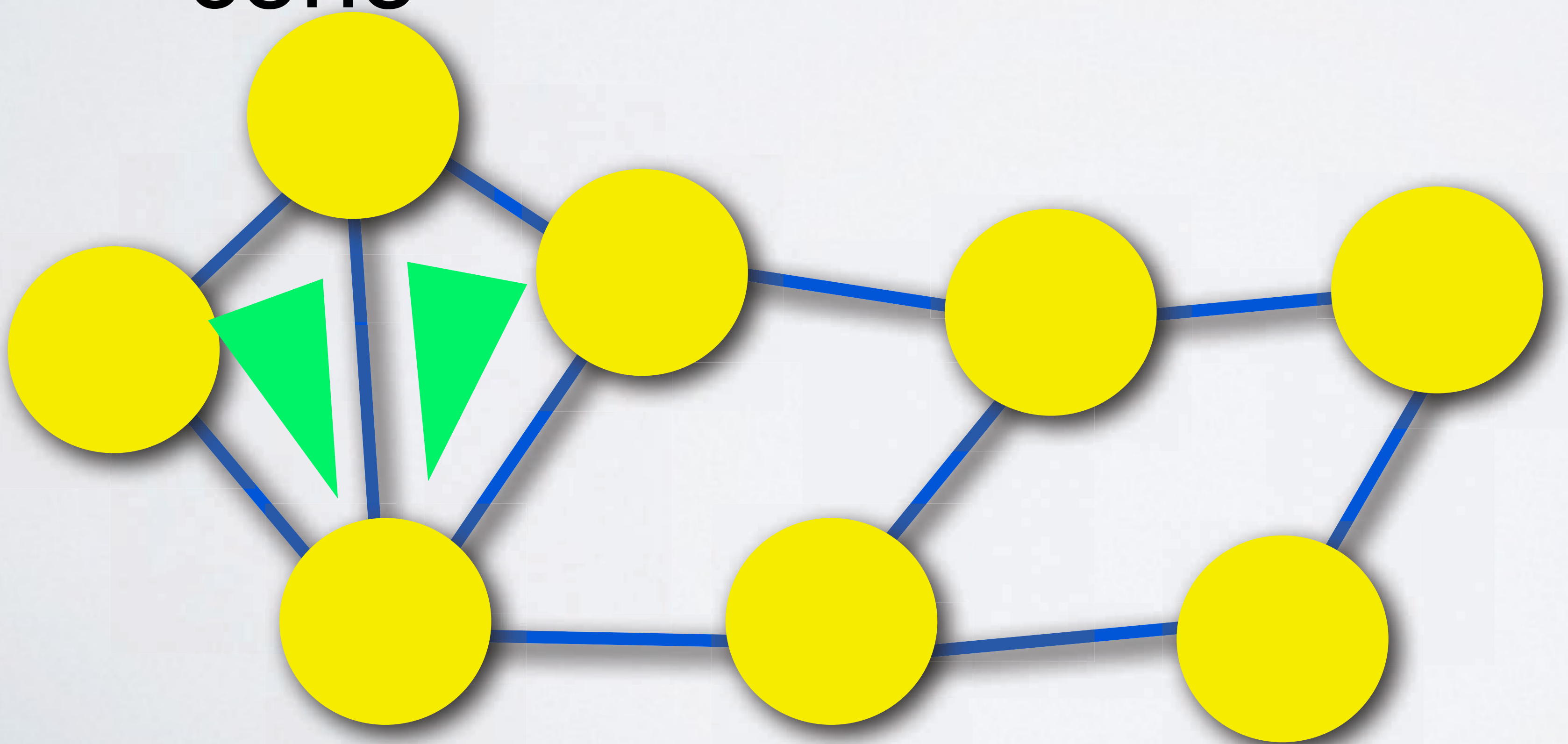
$b_0 - b_1 + b_2 \ldots = \chi(G)$

cohomological Euler characteristic

EULER CHARACTERISTIC

$$\chi_{simp}(G) = v_0 - v_1 + v_2 - \ldots = 8-11+2$$

$$\chi_{coho}(G) = b_0 - b_1 + b_2 - \ldots = 1-2$$



Example: no tetrahedra, then

$v-e+f=1-g$

EULER-POINCARE

$$\chi_{simp}(G) = \chi_{coho}(G)$$

Proof. C_m space of m forms

$Z_m = \ker(d_m)$ space of m cocycles

$R_m = \mathrm{ran}(d_m)$ space of m coboundaries

$z_m = v_m - r_m$ rank-nullety theorem

$b_m = z_m - r_{m-1}$ definition of cohomology

$$\sum_m (-1)^m (v_m - b_m) = \sum_m (-1)^m (r_m - r_{m-1}) = 0$$ QED

HODGE THEOREM

$$H^m(G) \cong \mathrm{Ker}(L_m)$$

Proof:

$L f = 0$ is equivalent to $df = d^* f = 0$

$\langle f, Lf \rangle = \langle d^* f, d^* f \rangle + \langle df, df \rangle$

$\mathrm{im}(d) + \mathrm{im}(d^*) + \ker(L) = R^n$

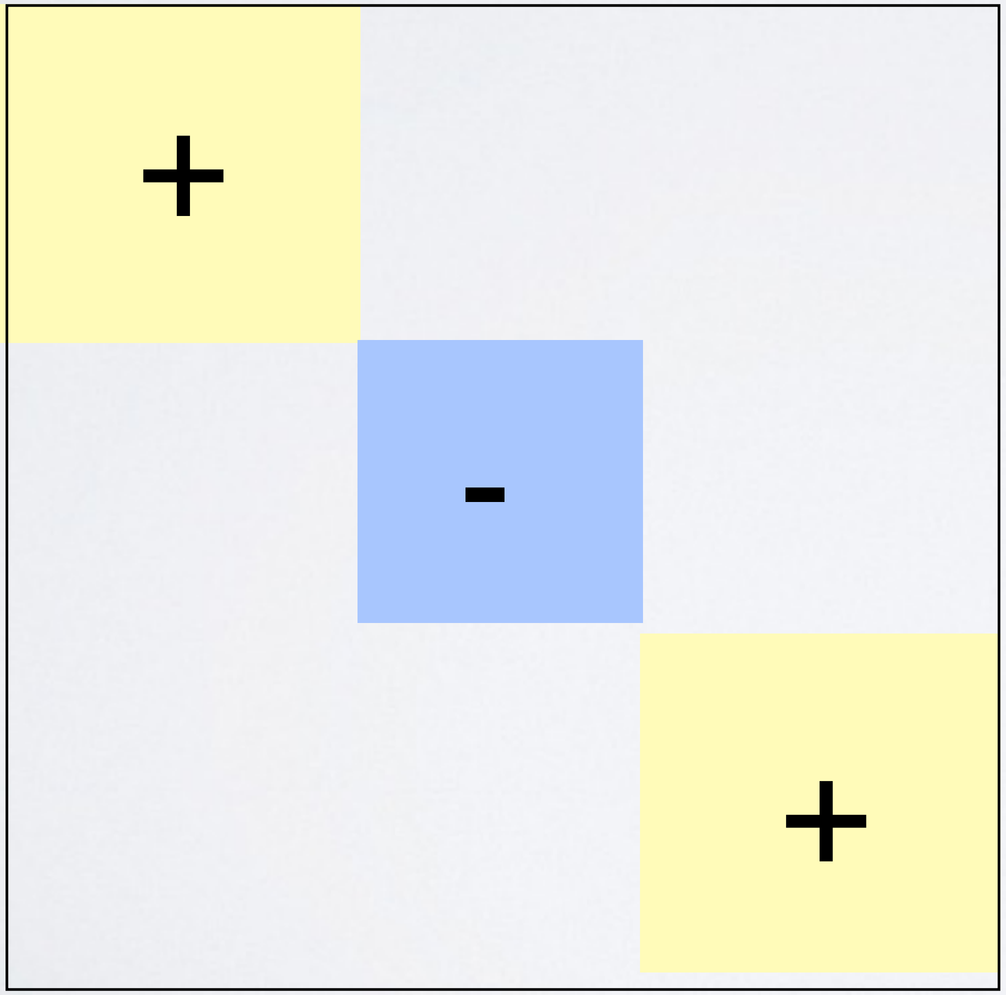
$\langle dg, d^* h \rangle = 0$. Also ker(L) and im(L) are perp.

$dg=0$ implies $g = d f + h$

can match cohomology class [g] - harmonic form

SUPER TRACE

$$str(A) = tr(P\,A)$$



Example:

$str(1)=tr(P)$

$=\chi(G)$

MCKEAN SINGER

$$\mathrm{str}(\exp(-t\,L)) = \chi(G)$$

Proof. Each Boson f matches a Fermion Df

$$Lf = \lambda\, f$$

$$LDf = D\,Lf = D\lambda\, f = \lambda\, Df$$

$$\mathrm{str}(L\ ) = 0 \quad \text{for } n > 0$$

$$\mathrm{str}(\exp(-t\,L)) = \mathrm{str}(1) = \chi(G)$$

COMBINATORICS

GUSTAV KIRCHHOFF

1824-1887

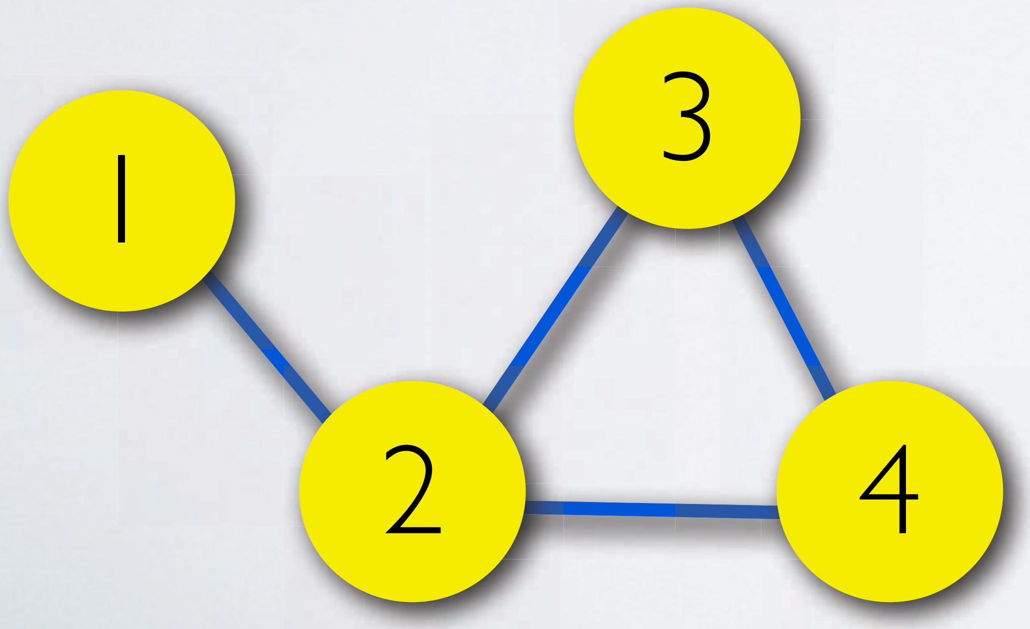
MATRIX TREE THEOREM

L = B-A =

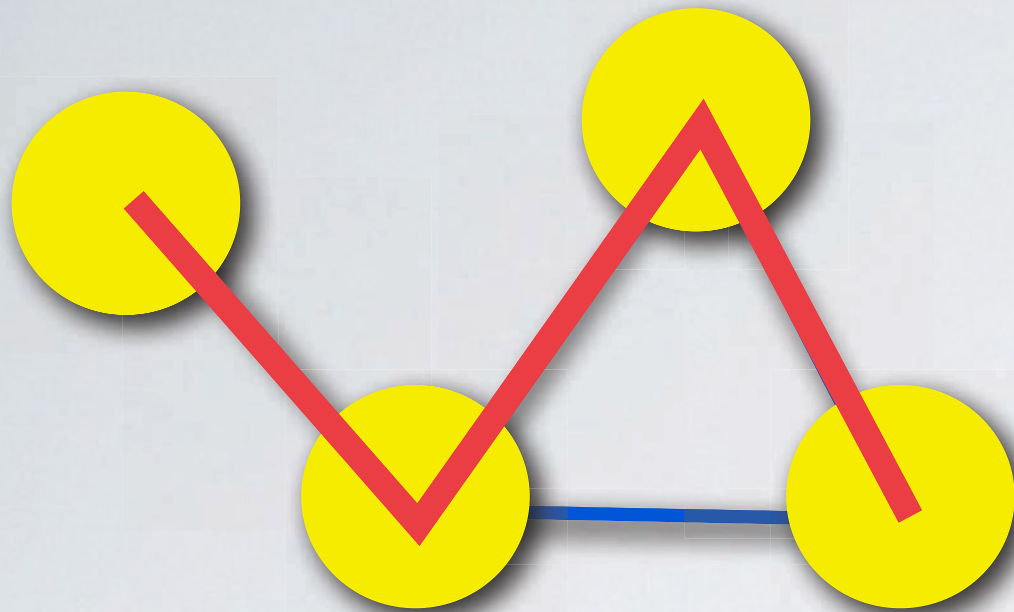
B = Degree matrix

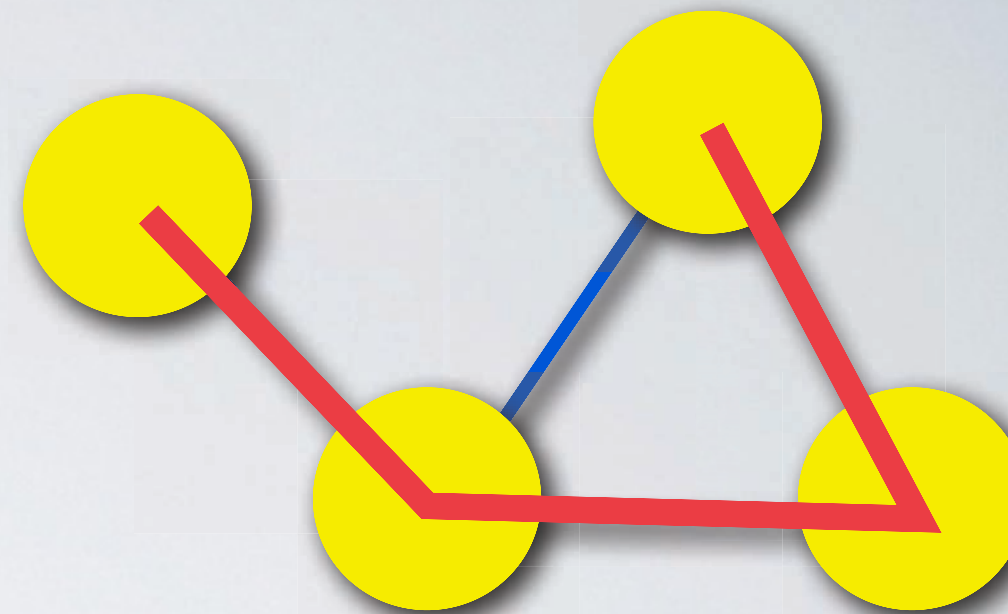
L=

1	-1	0	0
-1	3	-1	-1
0	-1	2	-1
0	-1	-1	2

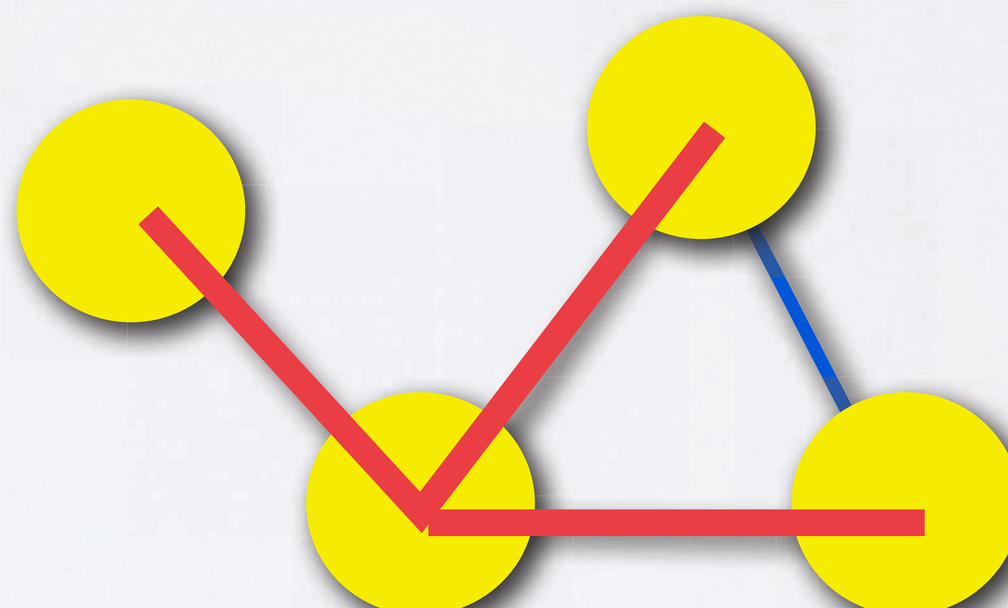


Eigenvalues: 4,3,1,0





$$\det(E+L)/4=12/4 = 3$$



$$E_{ij} = 1/n$$

FOREST THEOREM

Chebotarev-Shamis

det(L+1) is the number of rooted spanning forests

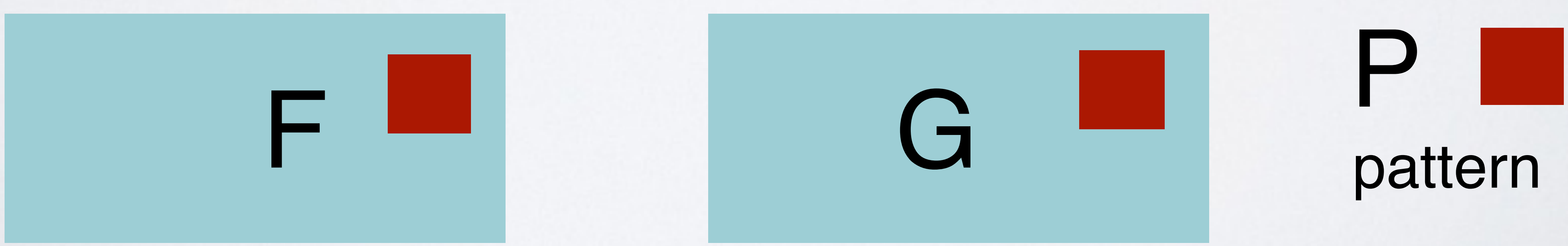
We have a completely new proof, using a new identity in linear algebra:

CAUCHY-BINET

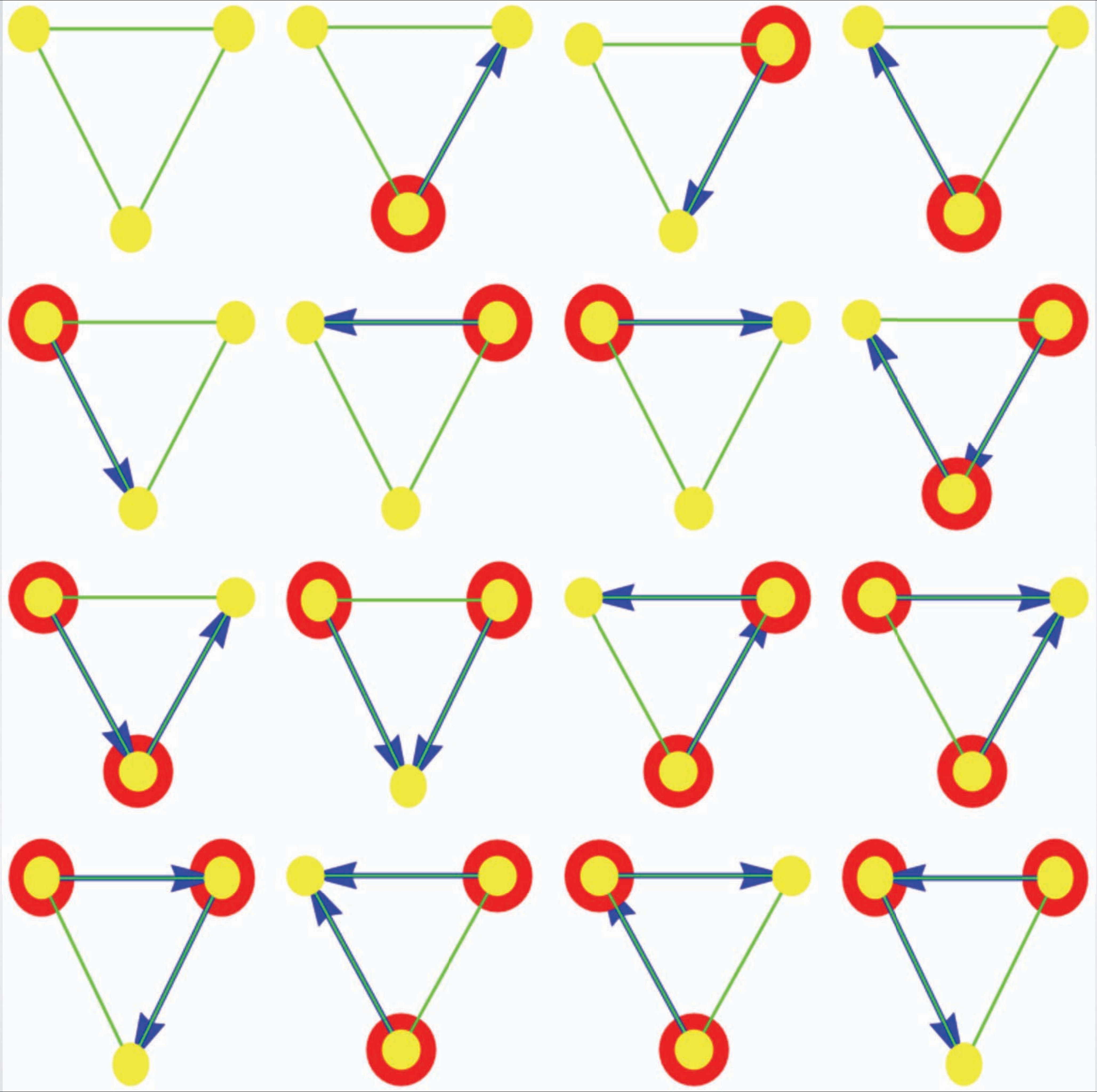
F, G arbitrary $n \times m$ matrices.

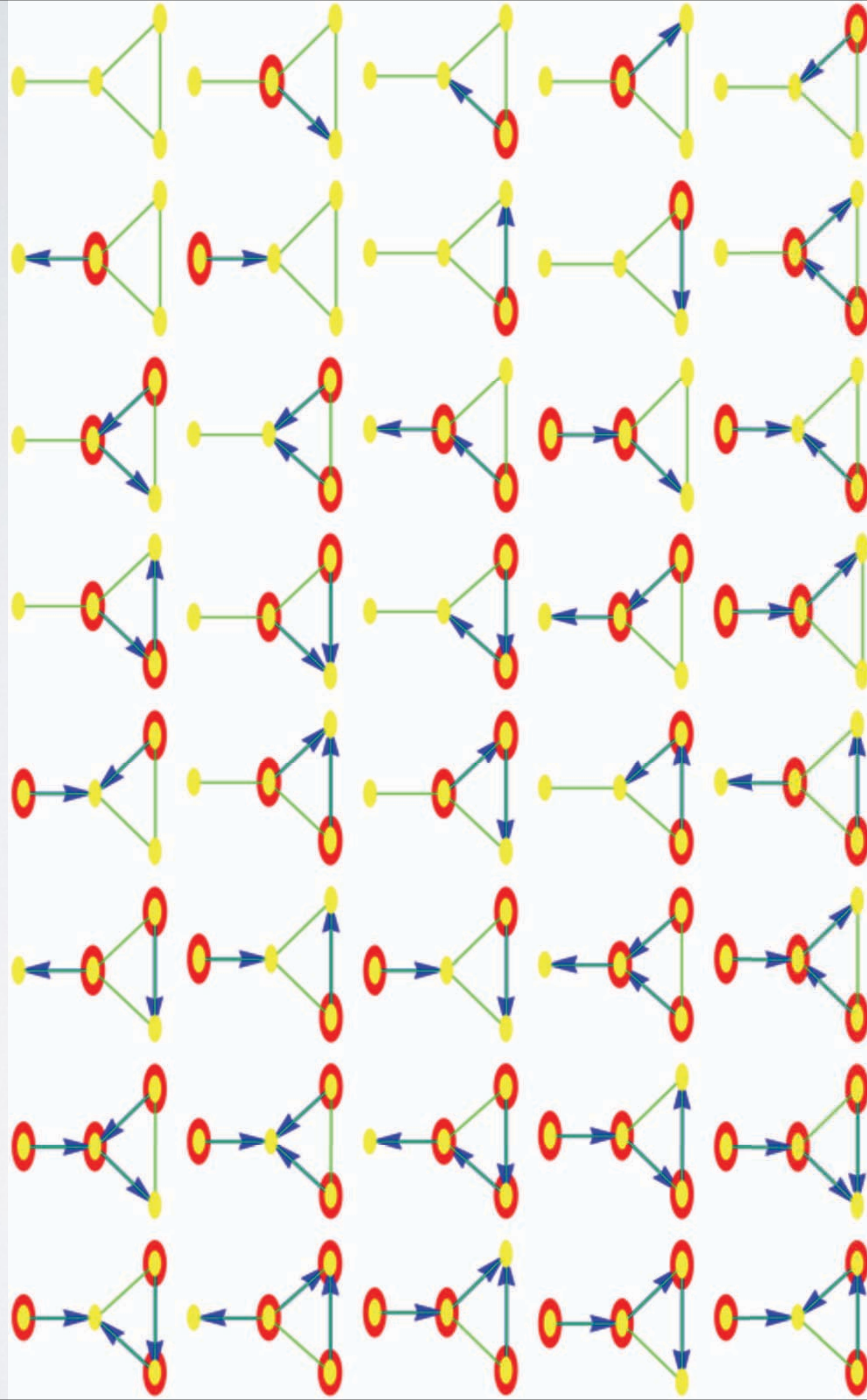
$$\det(1+x F^T G) = \sum_P x^{|P|} \det(F^T) \det(G)$$

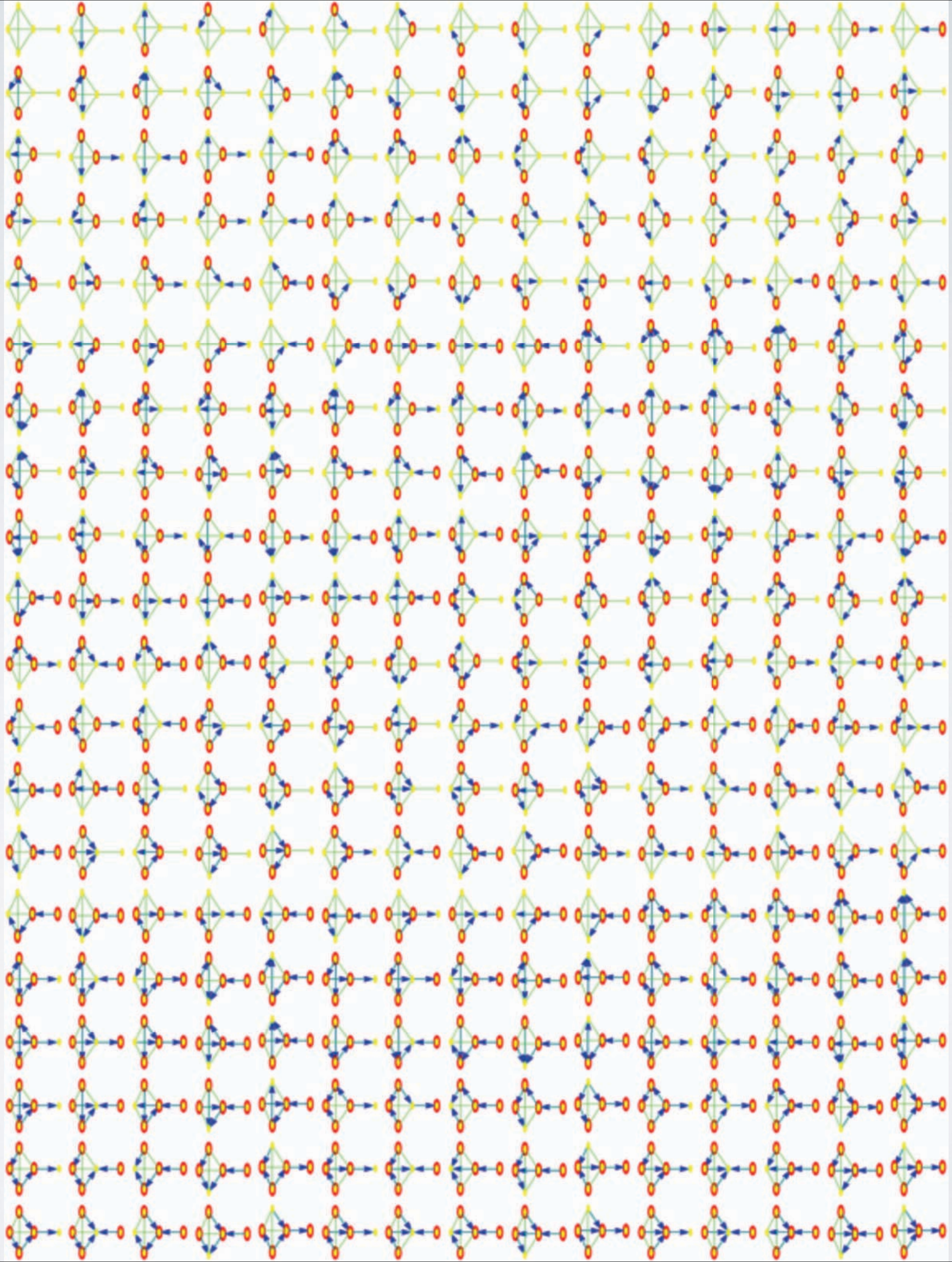
We are not aware that this has appeared anywhere.



EXAMPLES OF FORESTS







THE END

DIFFERENTIAL EQUATIONS

DIFFERENTIAL EQUATIONS

Also some of the Dynamics stuff had been mentioned in the ILAS talk. Linear differential equations are of course tightly mingled with linear algebra.

PDE ANALOGUES ON GRAPHS

$Lu=j$ **Poisson**

$u' = -Lu$ **Heat**

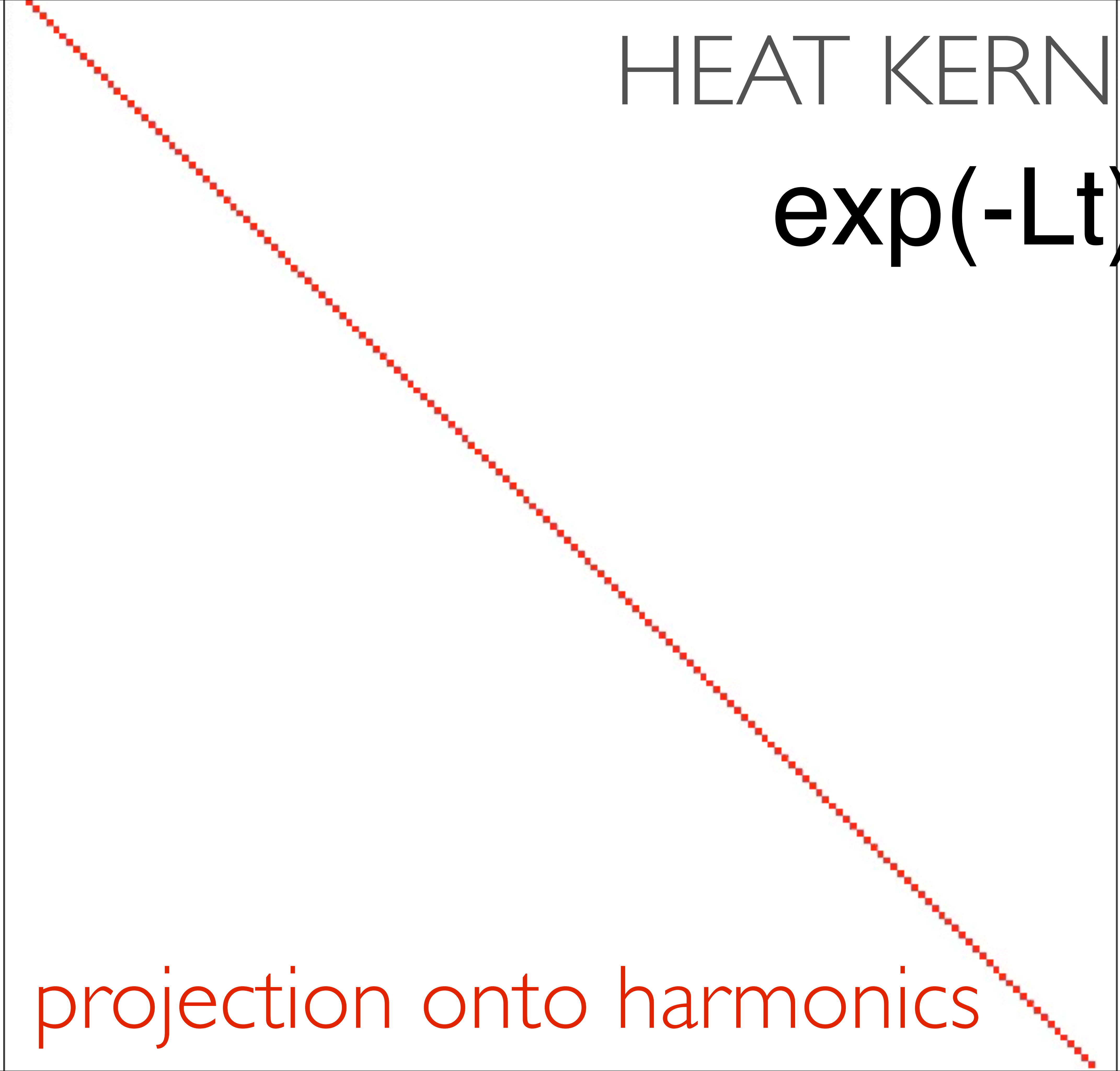
$u' = iLu$ **Schrödinger**

$u'' = -Lu$ **Wave**

$dF=0, d^*F=j$ **Maxwell**

HEAT KERNEL

$$\exp(-Lt)$$



projection onto harmonics

WAVE EQUATION SOLUTION

$$u'' = - L u$$

$$u'' = - D^2 u$$

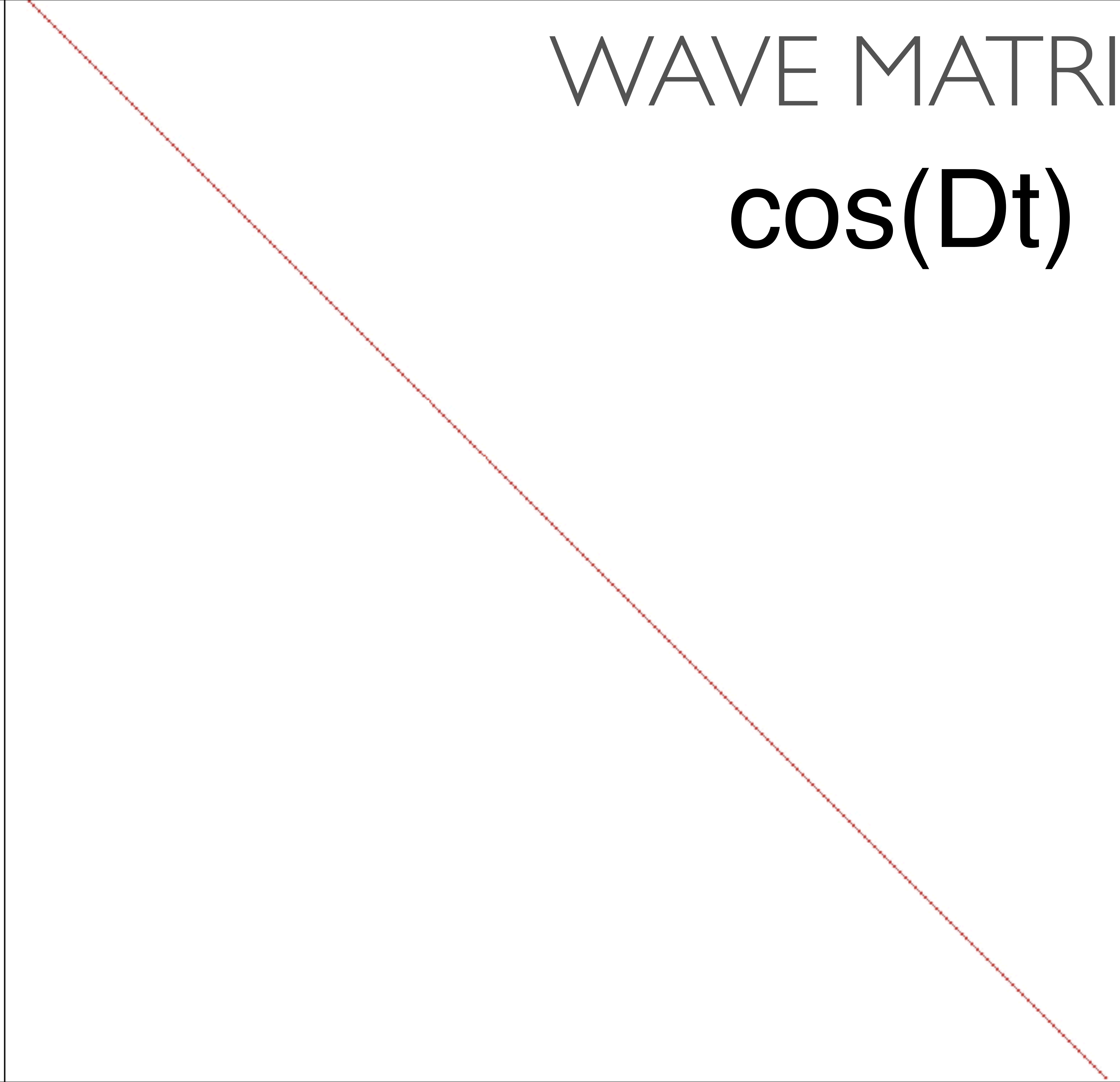
$$(\partial + i D)(\partial - i D) u = 0$$

$$u(t) = \cos(Dt)u(0) + \sin(Dt)D^{-1}u'(0)$$

d'Alembert type solution

WAVE MATRIX

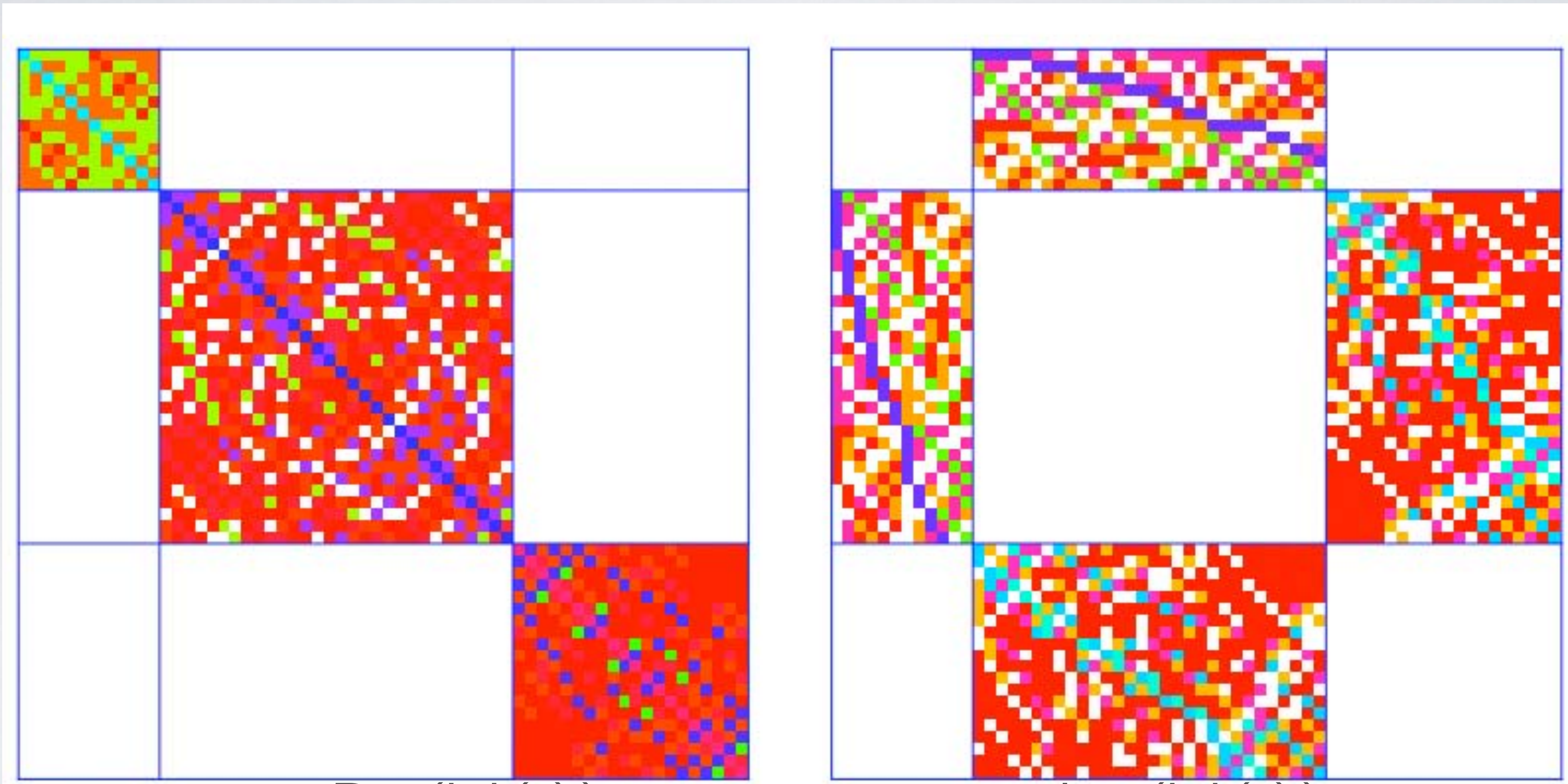
$\cos(Dt)$



PETER LAX

1926-

ISOSPECTRAL EVOLUTIONS

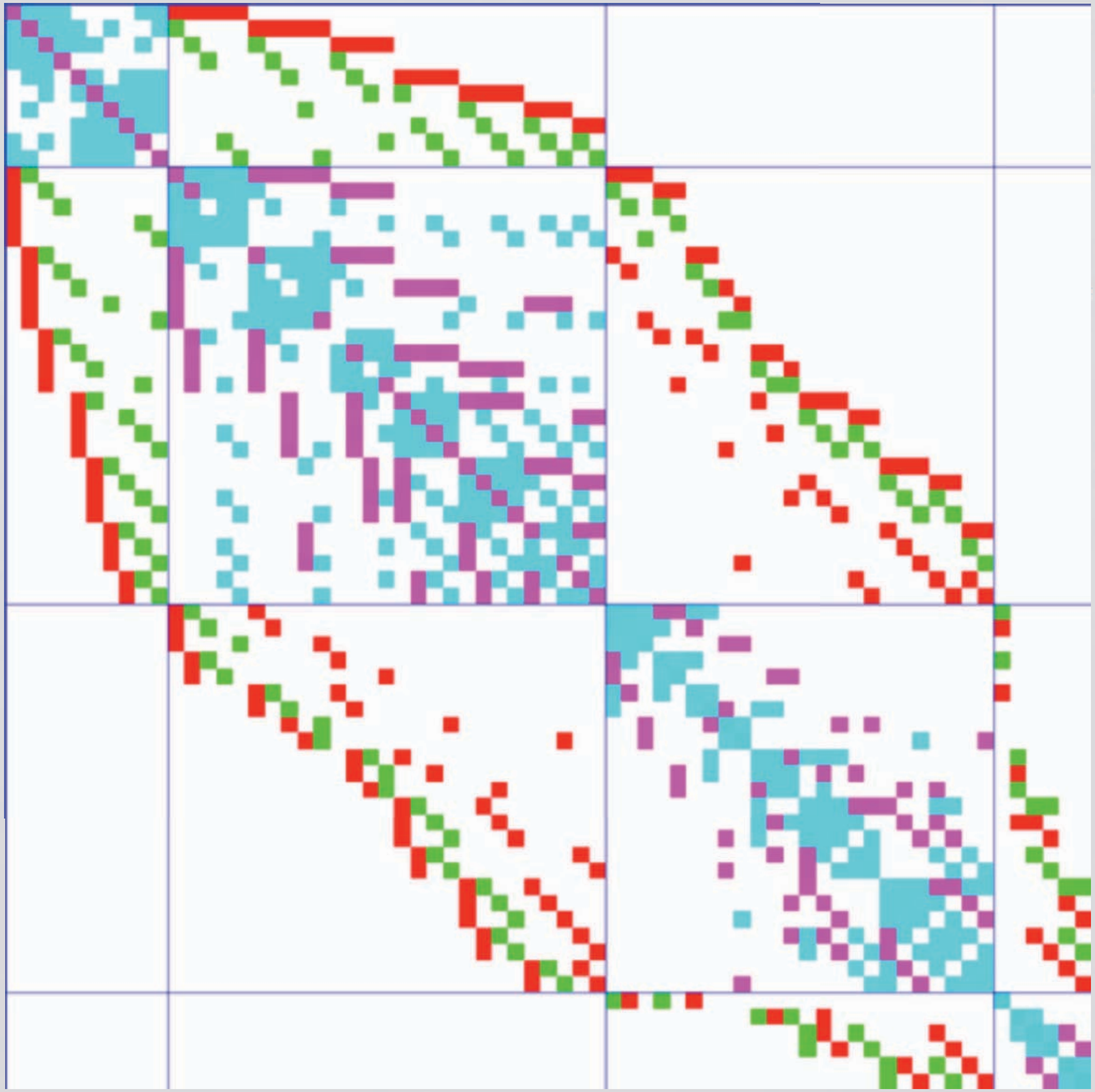


Re(U(t))   Im(U(t))

$$D' = [B, D] \quad , \quad B = d - d^*$$

$$D(t) = d^* + d + b$$

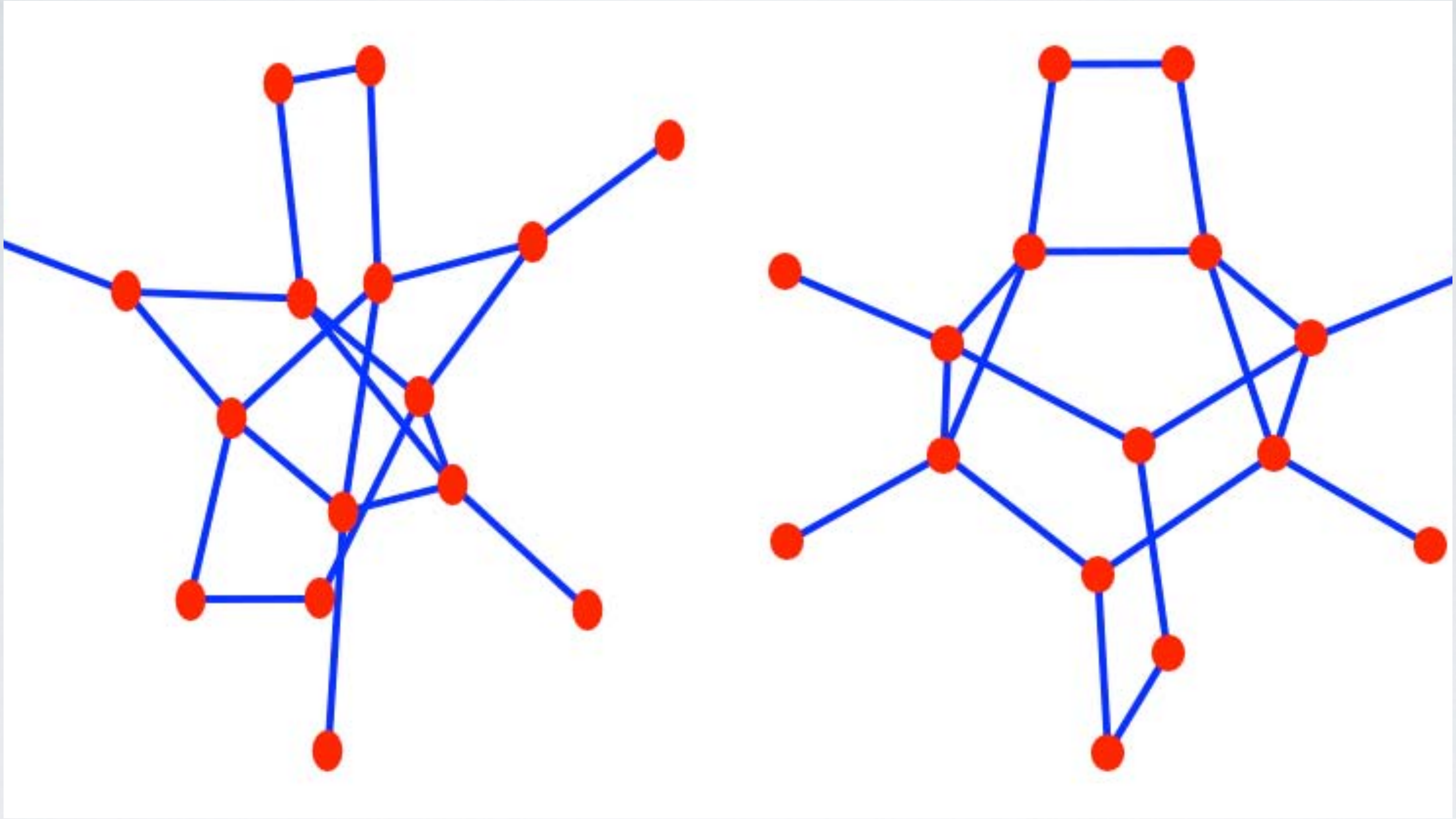
isospectral deformation



$$D(t) = U(t)^{*} D(0)\, U(t)$$

SPECTRAL PROBLEMS

ISOSPECTRAL GRAPHS



H-H Graphs are Dirac Isospectral!

proof: L_0, L_2, McKean Singer: L_1

ZETA FUNCTION

Having been mentioned last spring, this research has made progress in the summer.

ZETA FUNCTION

Let A be an operator for which the spectrum is discrete and satisfies $\sigma(A)=-\sigma(A)$. Define

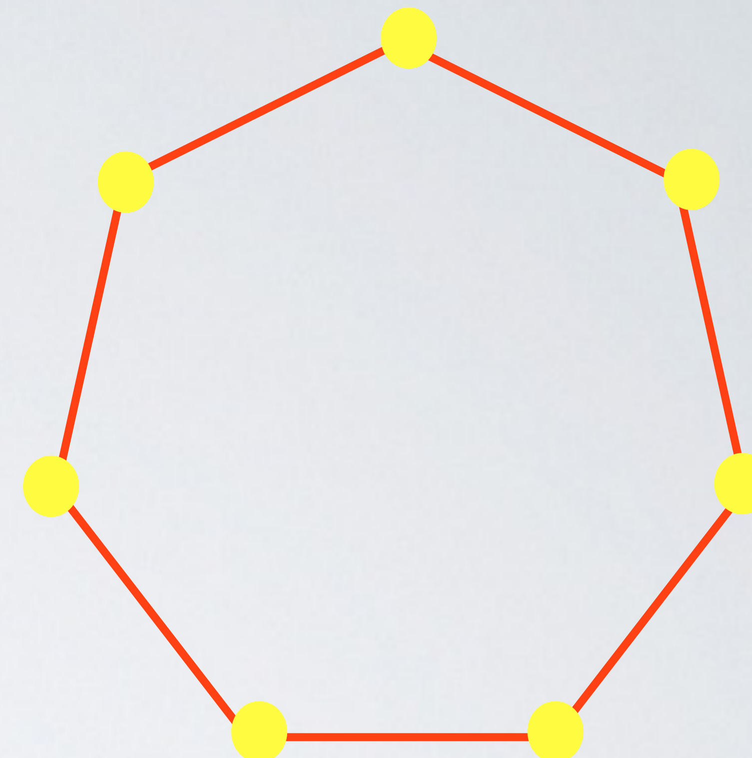
$$\zeta(s) = \sum_{\lambda>0} \lambda^{-s}$$

where λ runs over the set $\sigma(A)$ of eigenvalues of A.

CIRCLE CASE

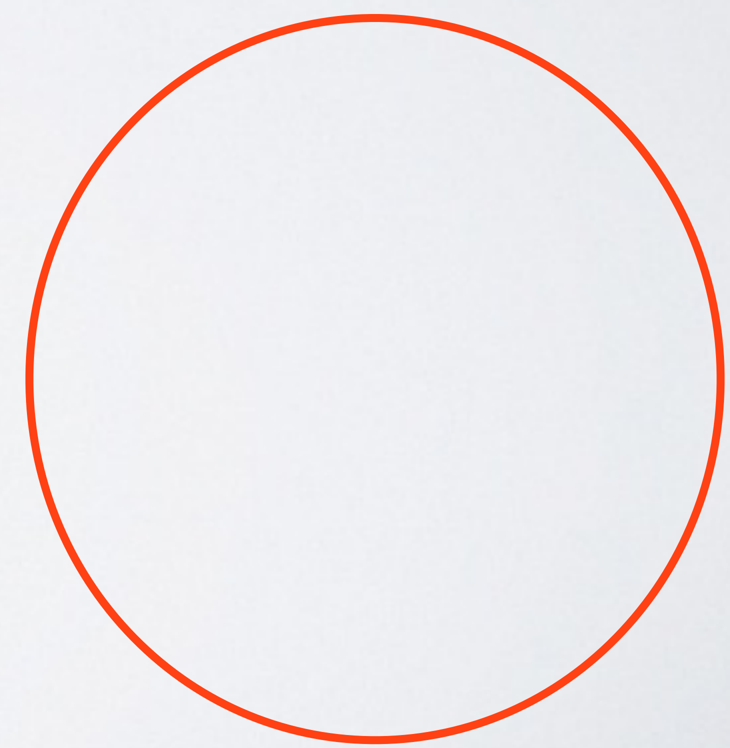
A=Dirac matrix of graph C_n

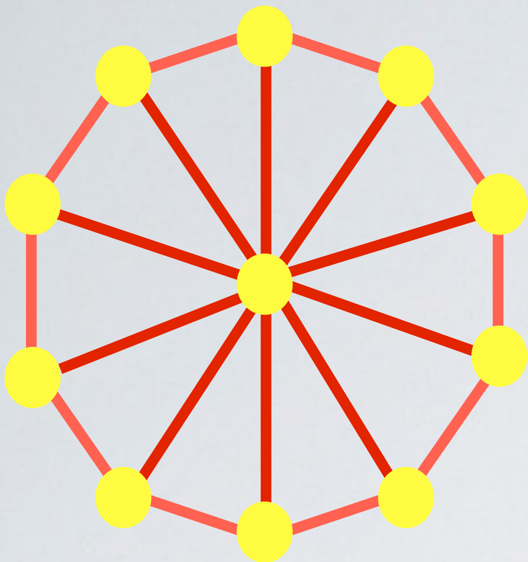
$$\zeta_n(s) = \sum_{k=1}^{n} \sin(\pi k/n)^{-s}$$



A= $i\partial/\partial x$

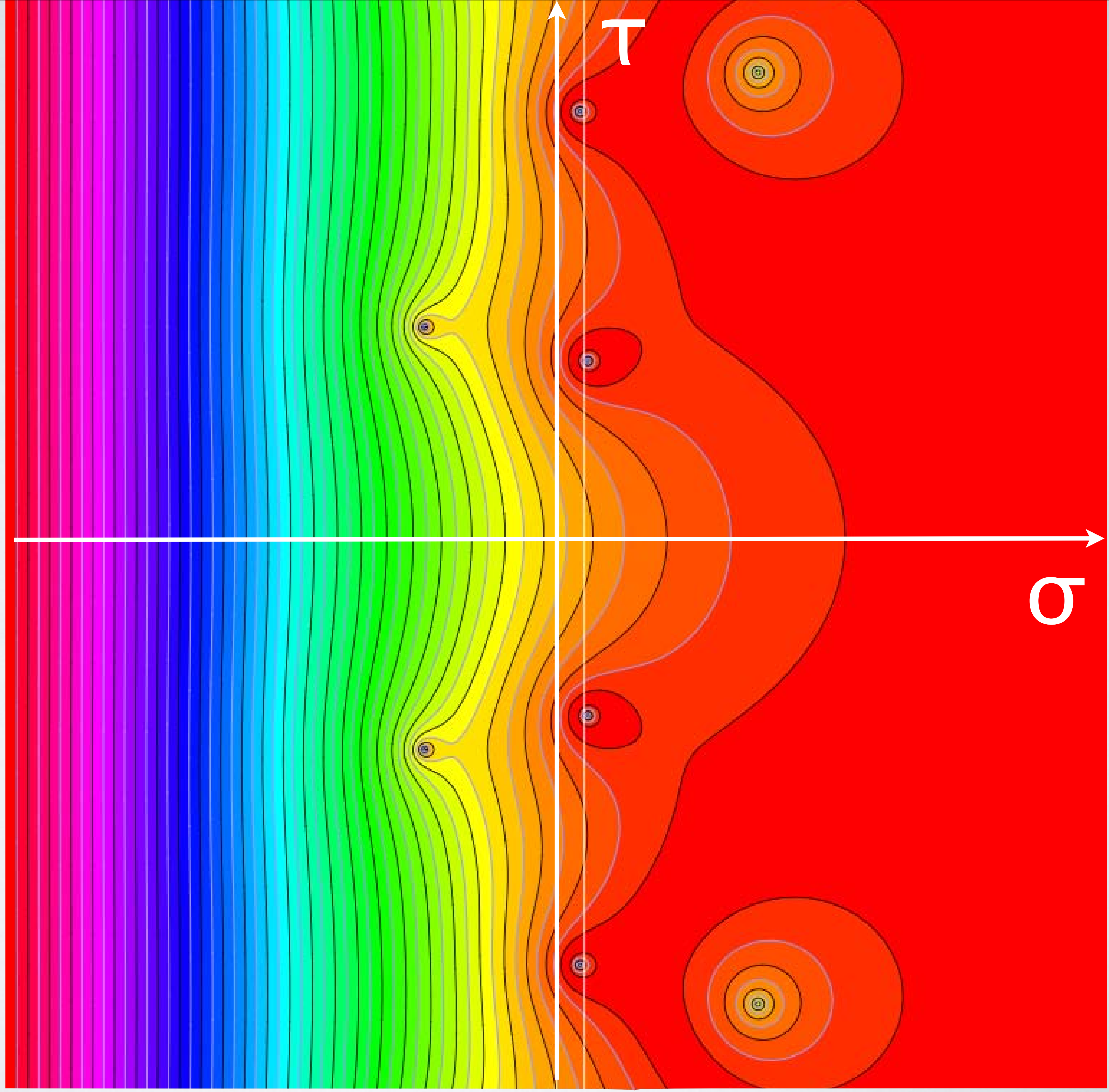
$$\zeta(s) = \sum_{k>0} k^{-s}$$



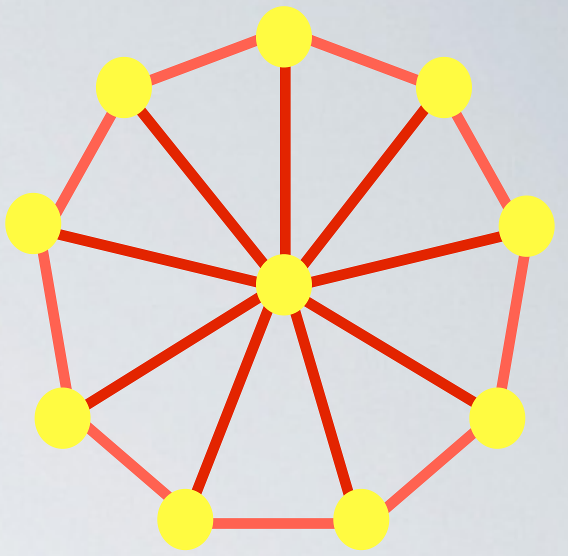


W_{10}

A = Dirac operator of the graph



$\zeta_n(2s)$ is the Zeta function of the Laplacian L



$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Theorem: (Baby Riemann)

The roots of the Laplace Zeta function $\zeta_n(2s)$ converge to the line $\sigma=1/2$.

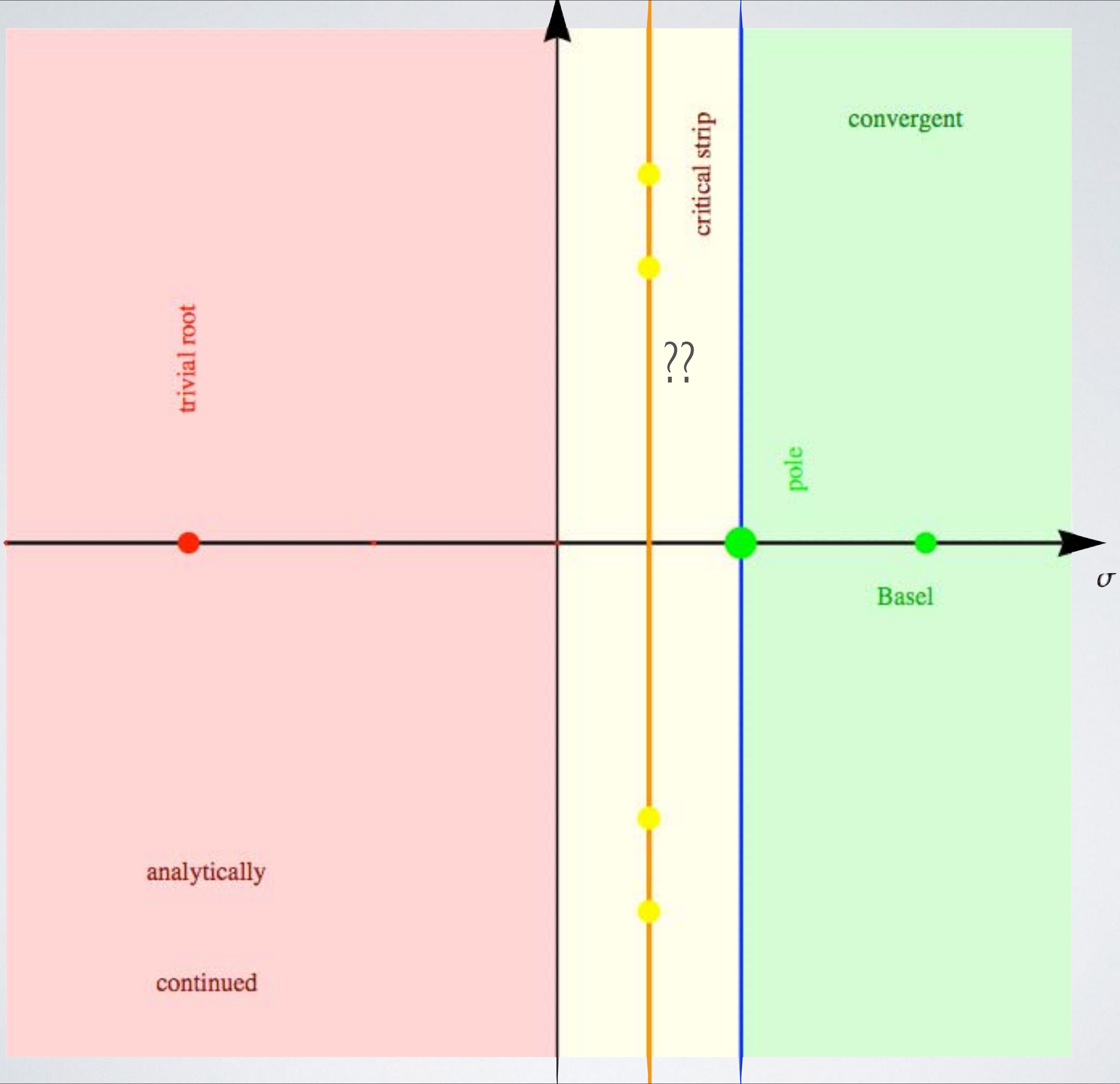
Proof: three parts. Exclude roots of $\zeta_n(s)$

$\sigma>1$: Approximate with $\zeta(s)$
$0<\sigma<1$: "Central limit"
$\sigma<0$: Riemann sum

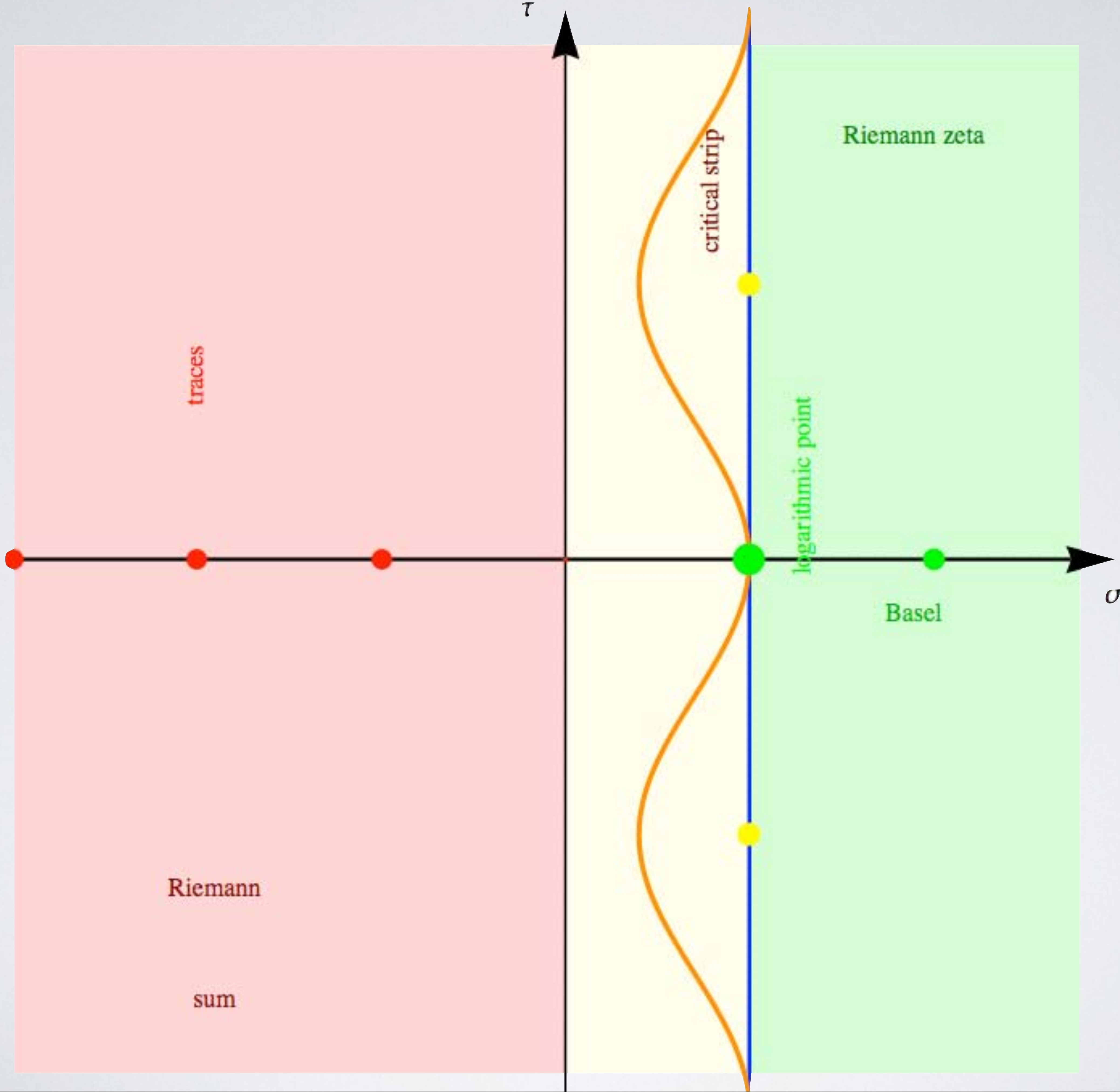
There is no relation with the actual Riemann hypothesis!

BERNHARD RIEMANN

1826-1866

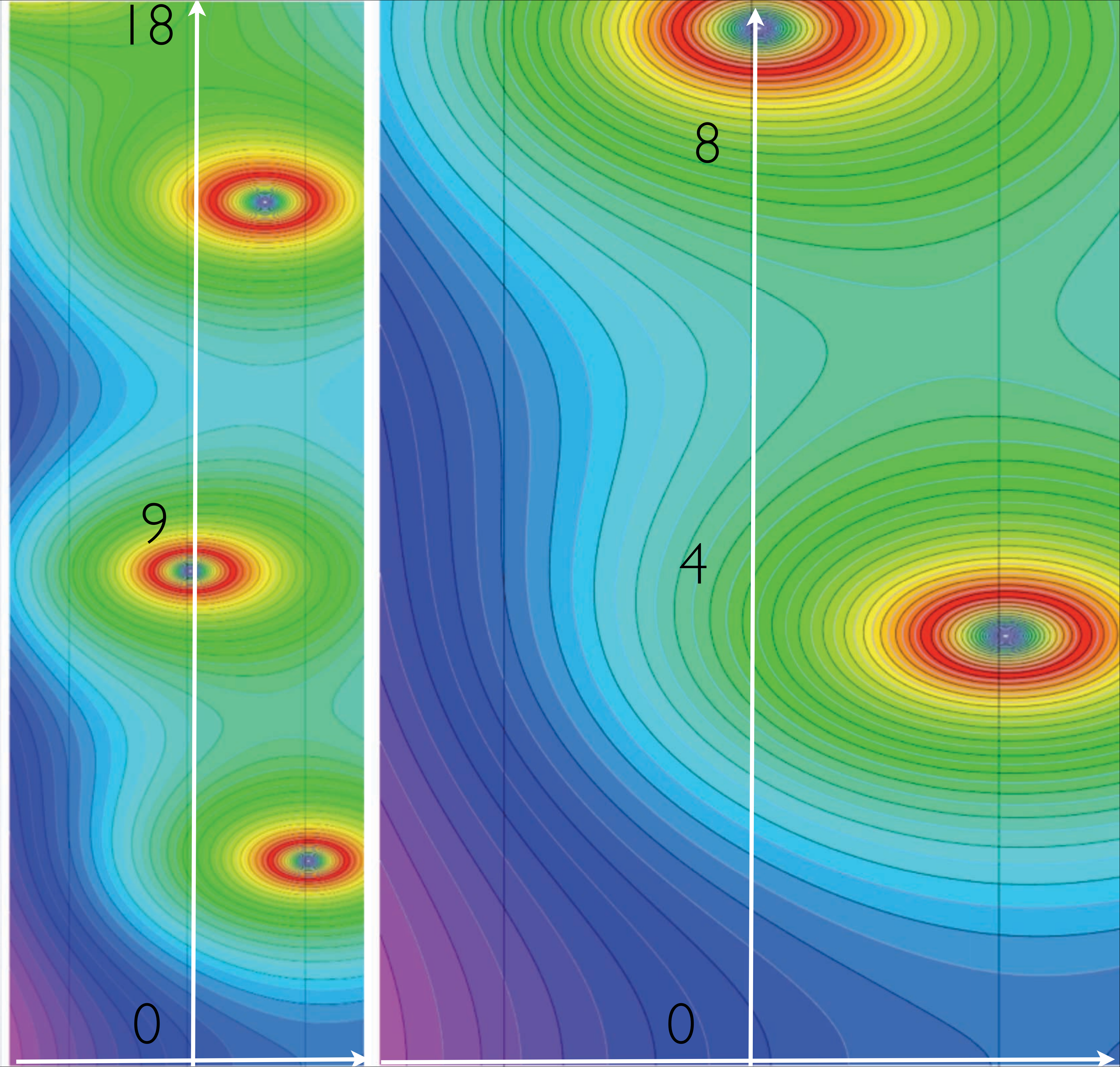


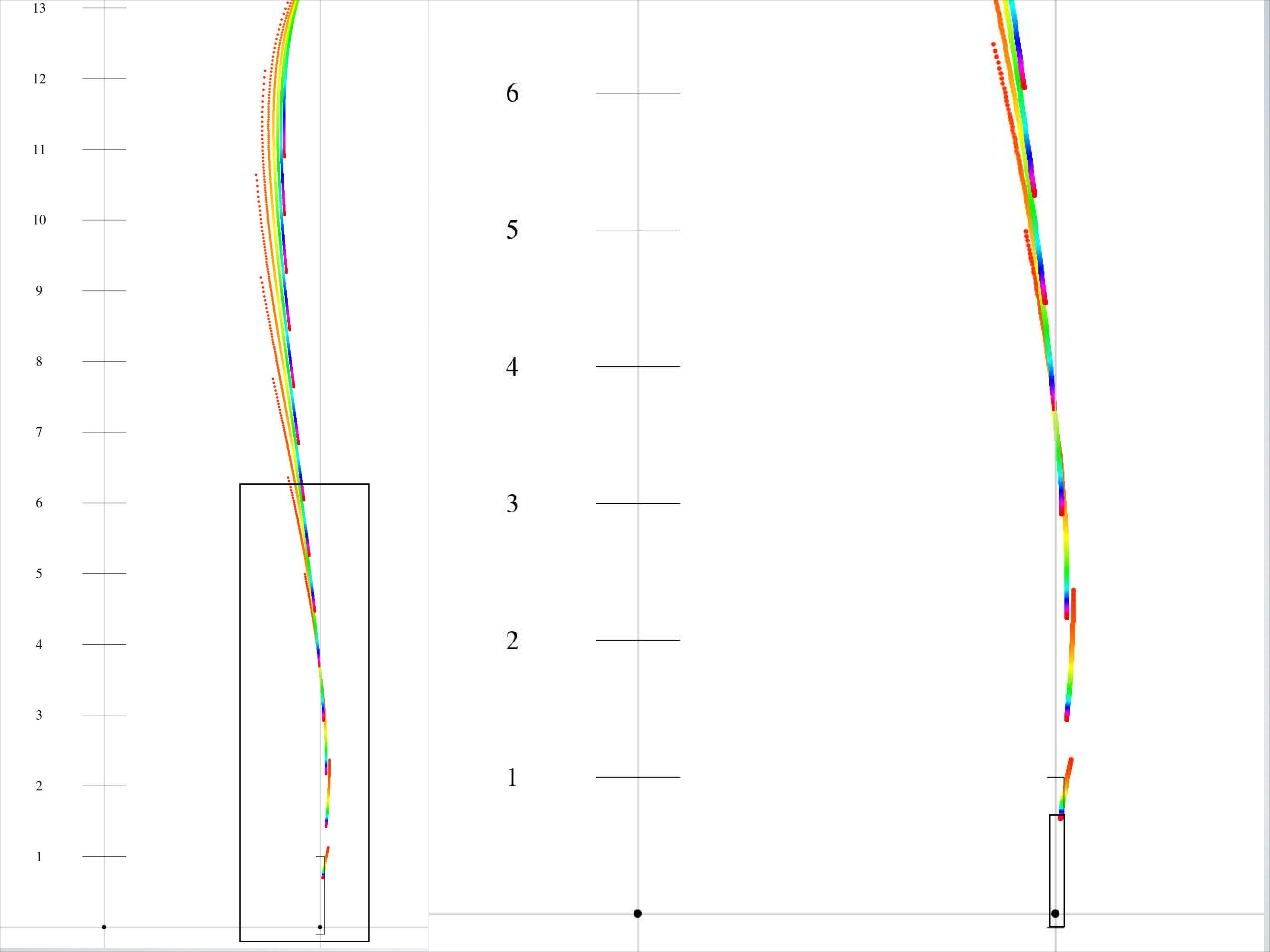
convergent
critical strip
trivial root
??
pole
σ
Basel
analytically
continued


τ
σ
critical strip
Riemann zeta
traces
logarithmic point
Basel
Riemann
sum

$\zeta(C_n)$

n=10 to 5000

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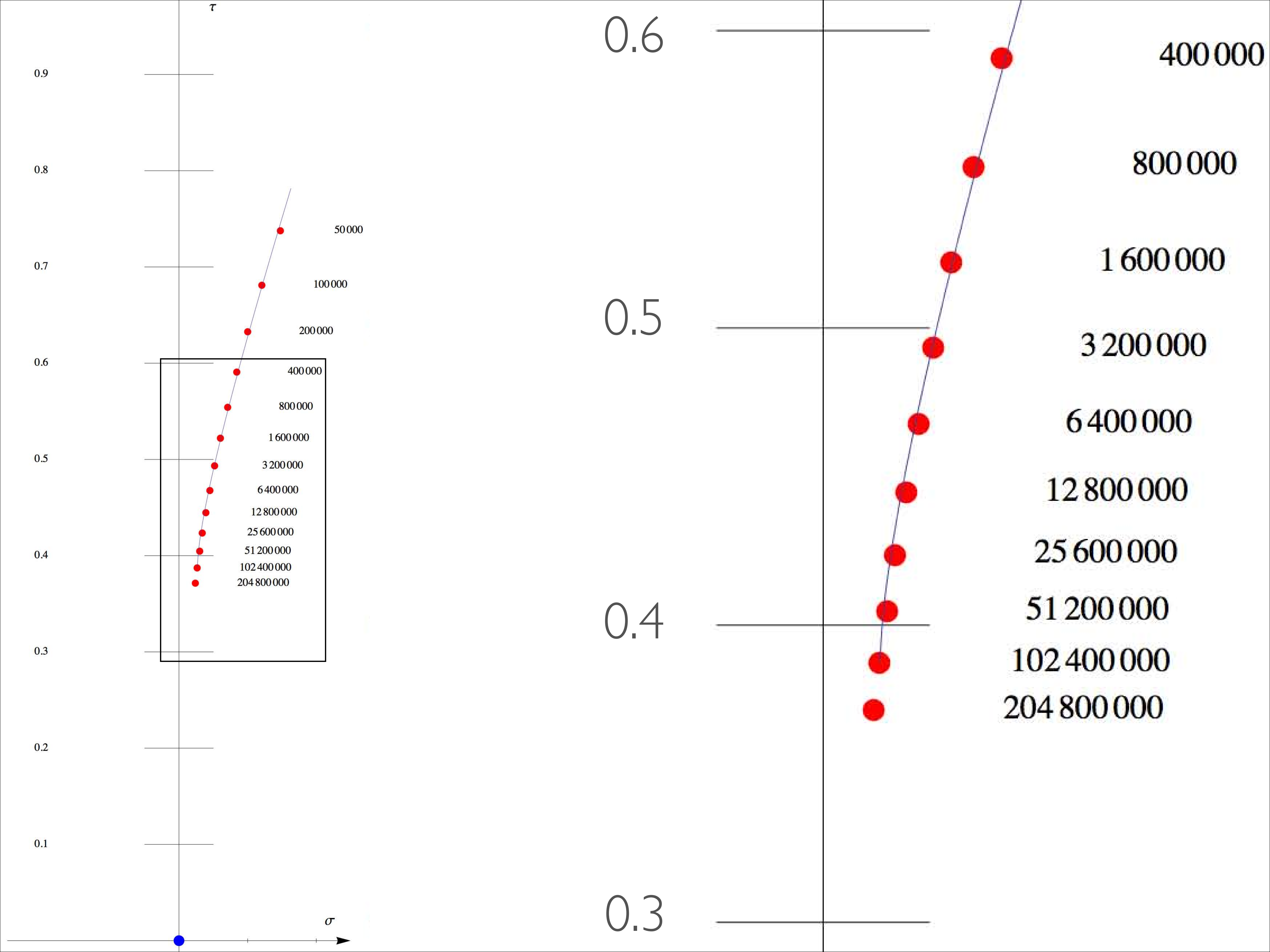
τ
0.9
0.8
0.7
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0.3
0.2
0.1
σ
50 000
100 000
200 000
400 000
800 000
1 600 000
3 200 000
6 400 000
12 800 000
25 600 000
51 200 000
102 400 000
204 800 000
0.6
0.5
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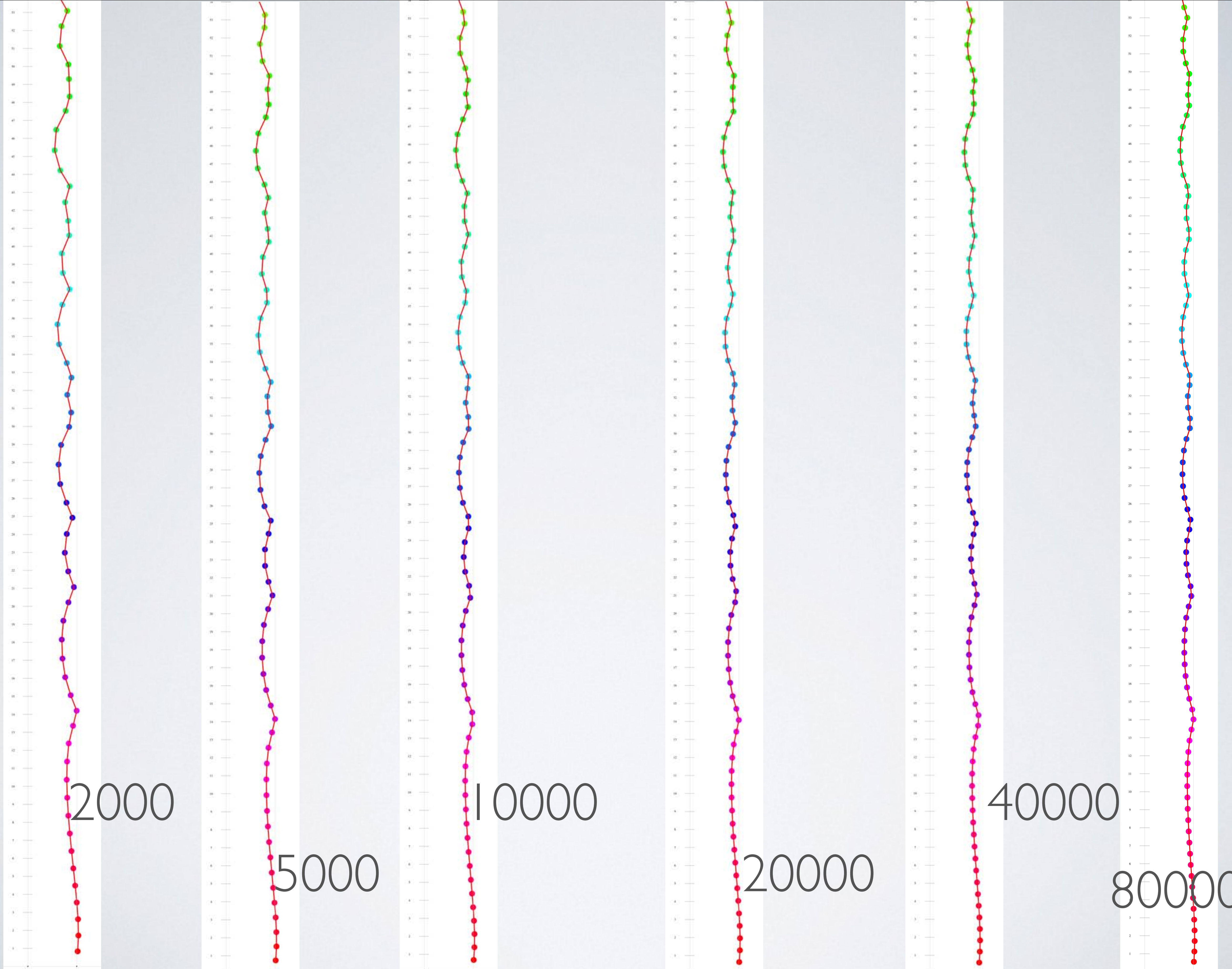
102 400 000
204 800 000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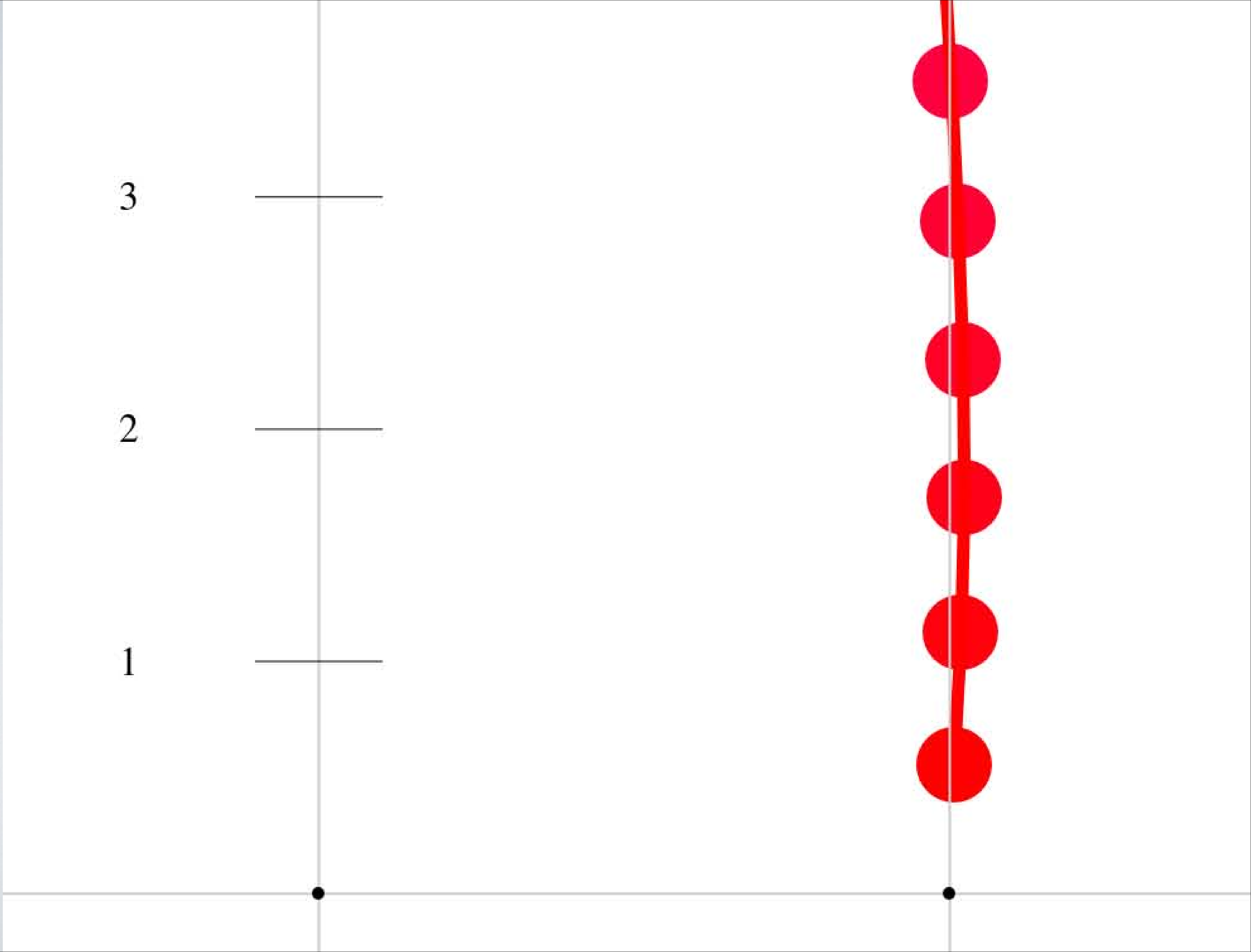
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2
1

THE END