BILLIARDS. Most of you probably have played billiards. Mathematicians like to play it too. However, they want to explore it also on more general tables, not only on the usual rectangular table. Their aim is to understand the path of a ball in such a table - not only for a few bounces - but over a very long time. While billiard balls slow down due to friction, "phonon" or "photon" balls do so much less. For phonon or photon billiards, the question of the propagation of sound in a concert hall or the problem of light reflection in an optical cable are billiard ball problems: as in usual billiards, light or sound reflects at the wall or mirror with the law that the incoming angle is equal to the outgoing angle. One can ask for example, which points are hit by a light or sound beam or where the light or sound waves focus is especially intense. In the first half of this lecture, we will look at the mathematicians billiard, with emphasis on very simple tables. Already the rectangular table produces surprisingly interesting problems and - this might surprise you - there are open problems even for triangular billiards, which nobody could answer yet.
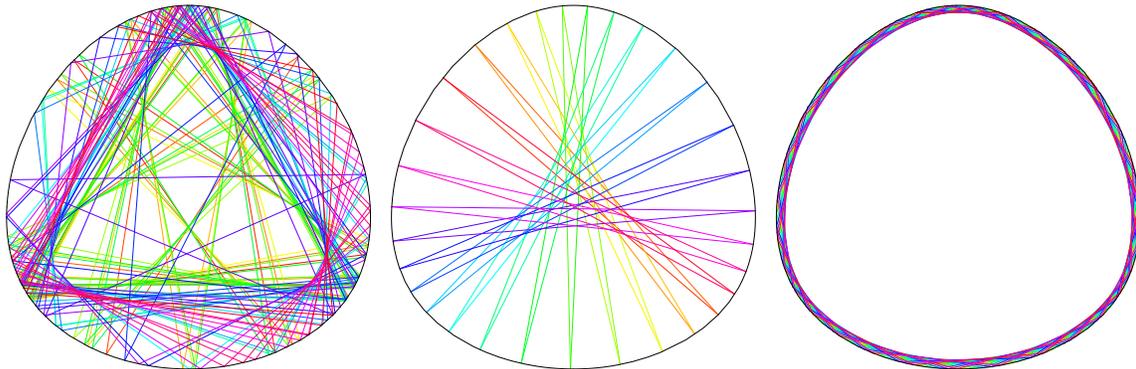

SQUARE BILLIARD. Let us start with a square billiard table. Problem: Can you shoot the ball in such a way that the trajectory comes arbitrarily close to any point of the boundary? Can you shoot a ball from a given point on the boundary in such a way that it comes back to the same point after exactly 6 bounces?
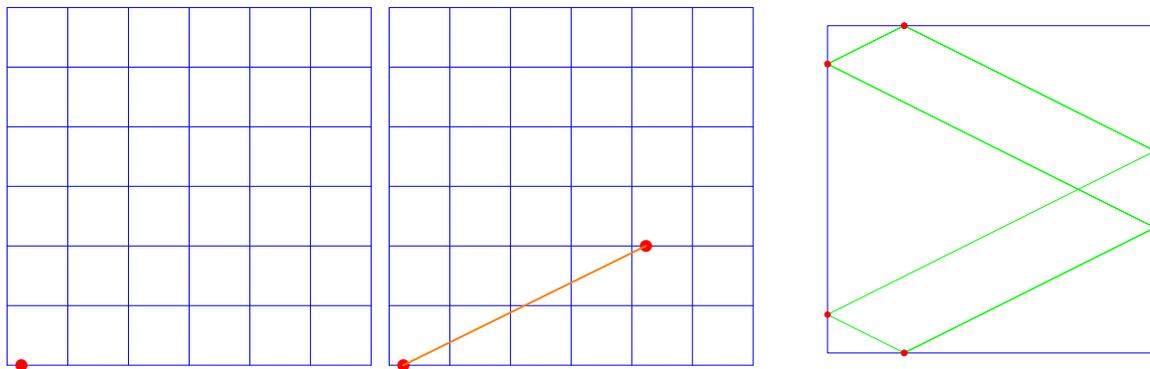

CIRCULAR BILLIARD. Can you shoot the ball in such a way that the trajectory returns back to you after hitting exactly 17 times the boundary? Can you figure out to do that in more than 2 different ways? Can you shoot in such a way that the orbit never returns back at the same spot twice?
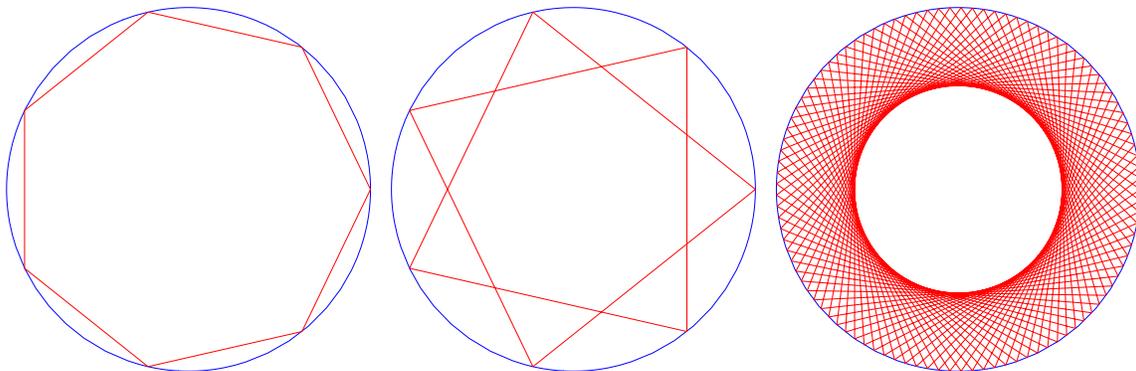
POLYGONAL BILLIARDS. For which polygons can one easily answer all the questions about a path by just drawing one line?
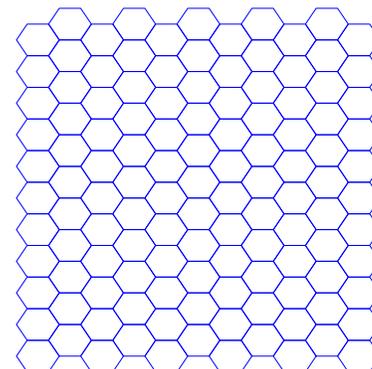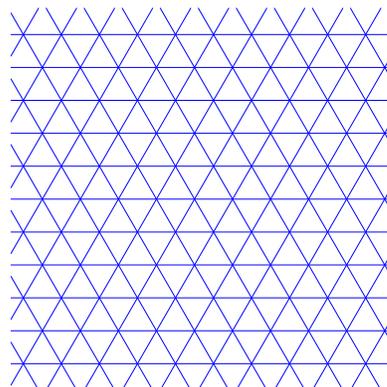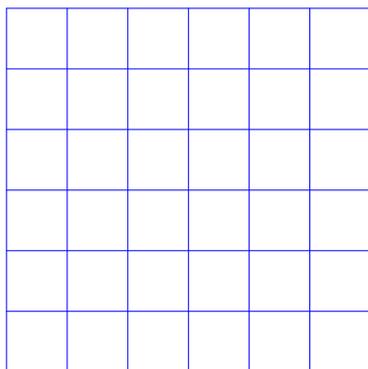
CAUSTICS. Places where light rays focus after some time are called caustics. You all probably have seen the coffee cup caustic in a cup, a round billiard table
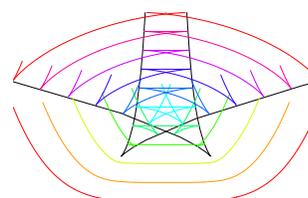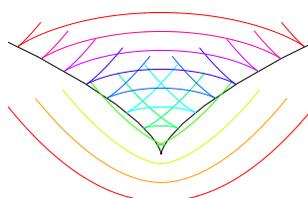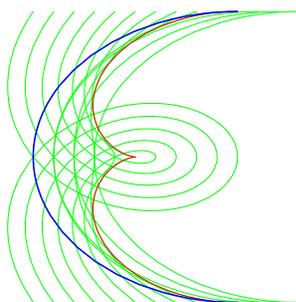
AN UNSOLVED PROBLEM FOR TRIANGULAR BILLIARDS. Does every triangular billiard have a billiard path which is closed in the sense that the ball traces again and again the same path? This problem is even unsolved for billiards in triangles. Can you solve the problem for triangles with a right angle? Can you solve the problem for acute triangles?
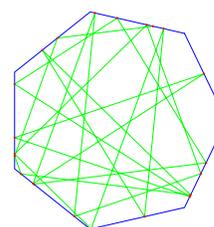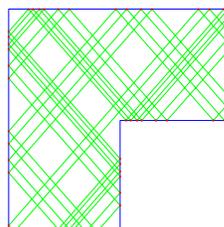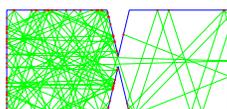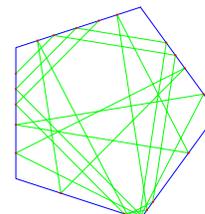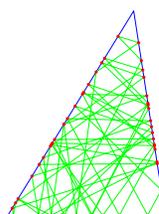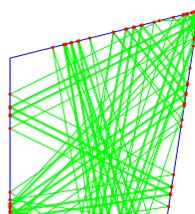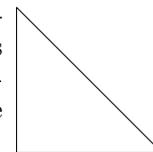
EXTERIOR BILLIARDS. There is a dual brother of billiards which is played outside a billiard table. Many questions for this game are of very similar nature then the interior billiard, but there are new ones. In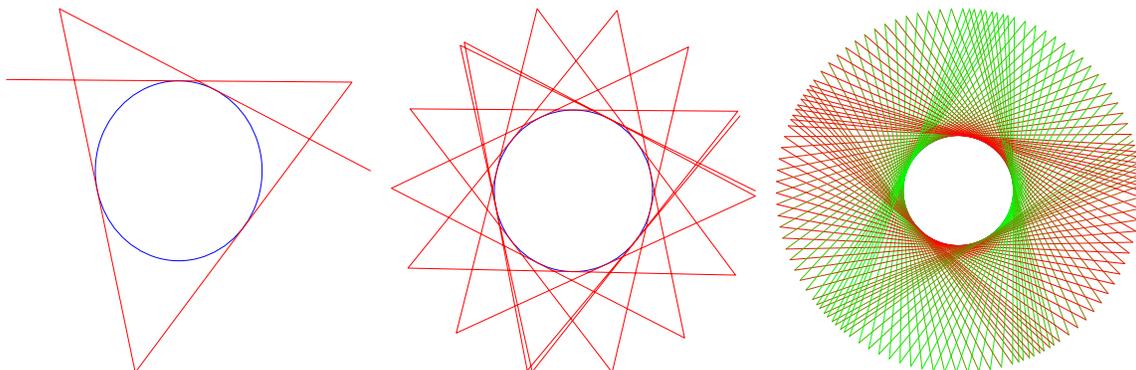 exterior billiard, we shoot from a point outside the table tangent at the table and reflect at the tangent point. Repeating this slingshot construction again and again produces a sequence of points outside the table. Where does this sequence go? Can we estimate, where the point is after millions of such sling-shots? This problem is motivated partly by an unsolved problem in celestial mechanics: is our solar system stable? While one can understand the motion of one planet around the sun since the time of Galileo and Newton, the presence of other planets make the picture much more complicated. Nobody knows whether it is possible that in a a trillion years - this is a rather academic question because the sun will die much before that - a planet like our earth could get expelled from the solar system.
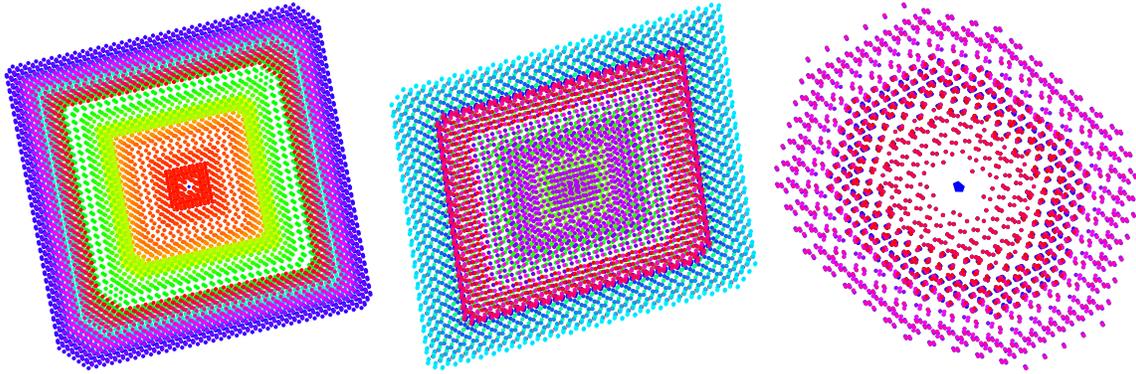

SQUARE EXTERIOR BILLIARD. Experiment with a square. What do you observe, when following trajectories? Can you understand all orbits? Can you extend this to rectangles or parallelograms?
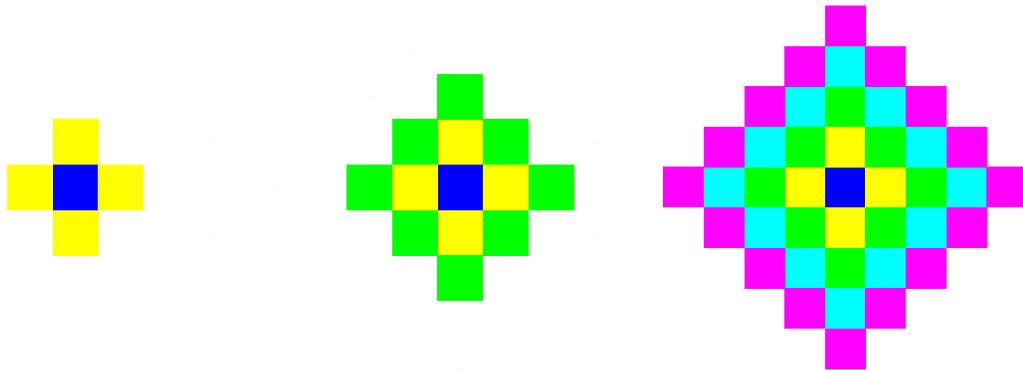
CIRCULAR EXTERIOR BILLIARD. What do you observe for a circle? Can you find points for which you come back to the same spot in exactly 17 sling-shots? Can you say something about exterior billiards on ellipses?
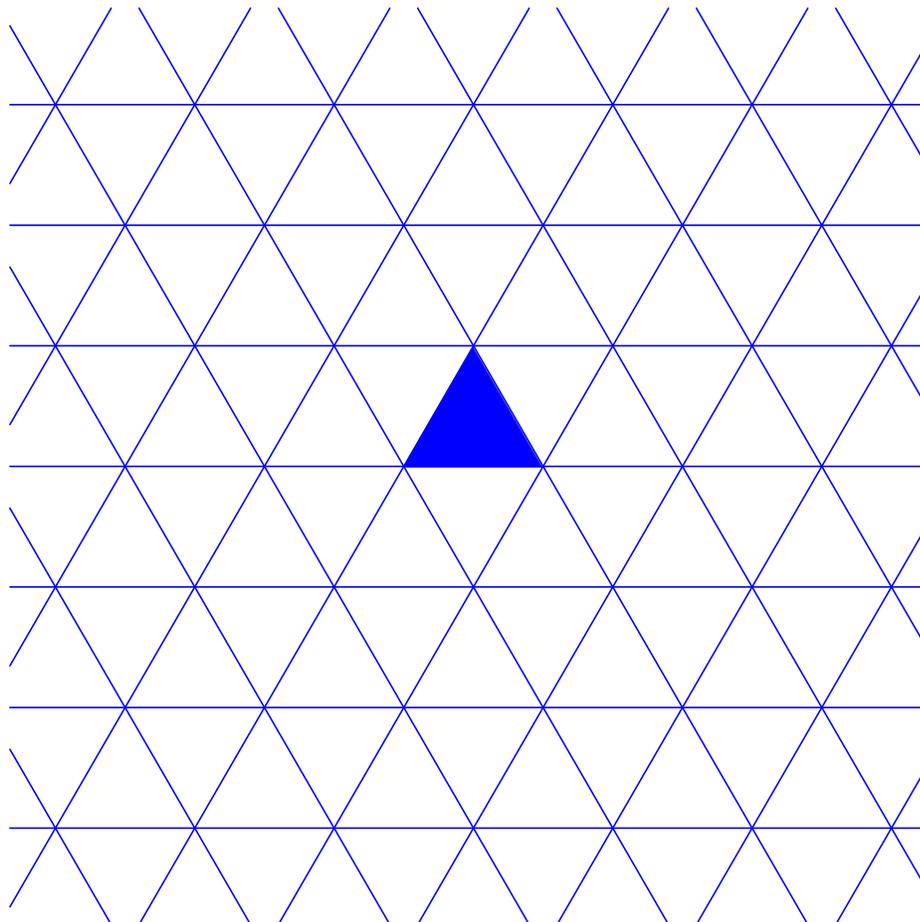
AN UNSOLVED PROBLEM. Is there a table and a point, such that the orbit of that point does not stay in any bounded region?

REMARKS TO THE PROBLEM. A simple example of where one does not know the answer is a half-circle. It would probably be easier to find an unstable table (if it exists), then to prove that every table is stable (if that is the case). There is a third possibility as with any unsolved mathematical problem. But this is very unlikely. It could be that our mathematical axioms do not cover the answer and that one could fork two axiom systems, one in which there is stability, and one in which there is no.