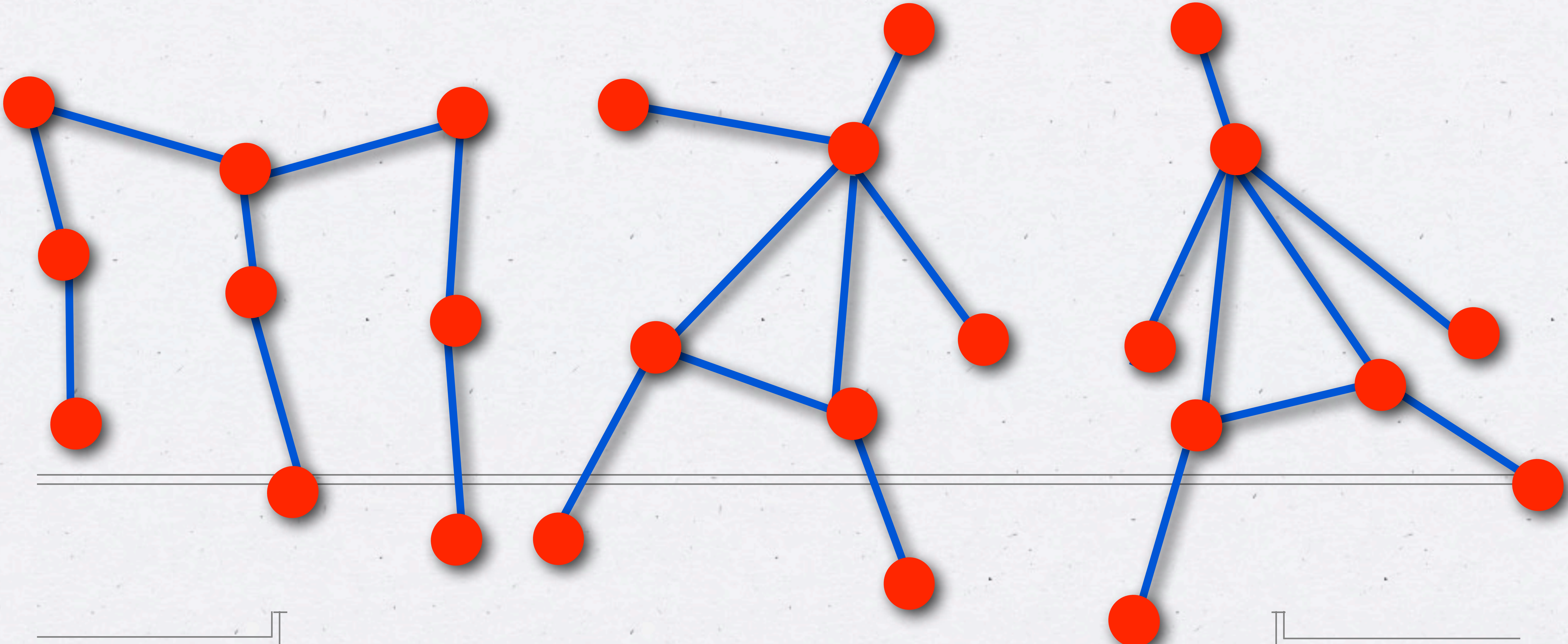
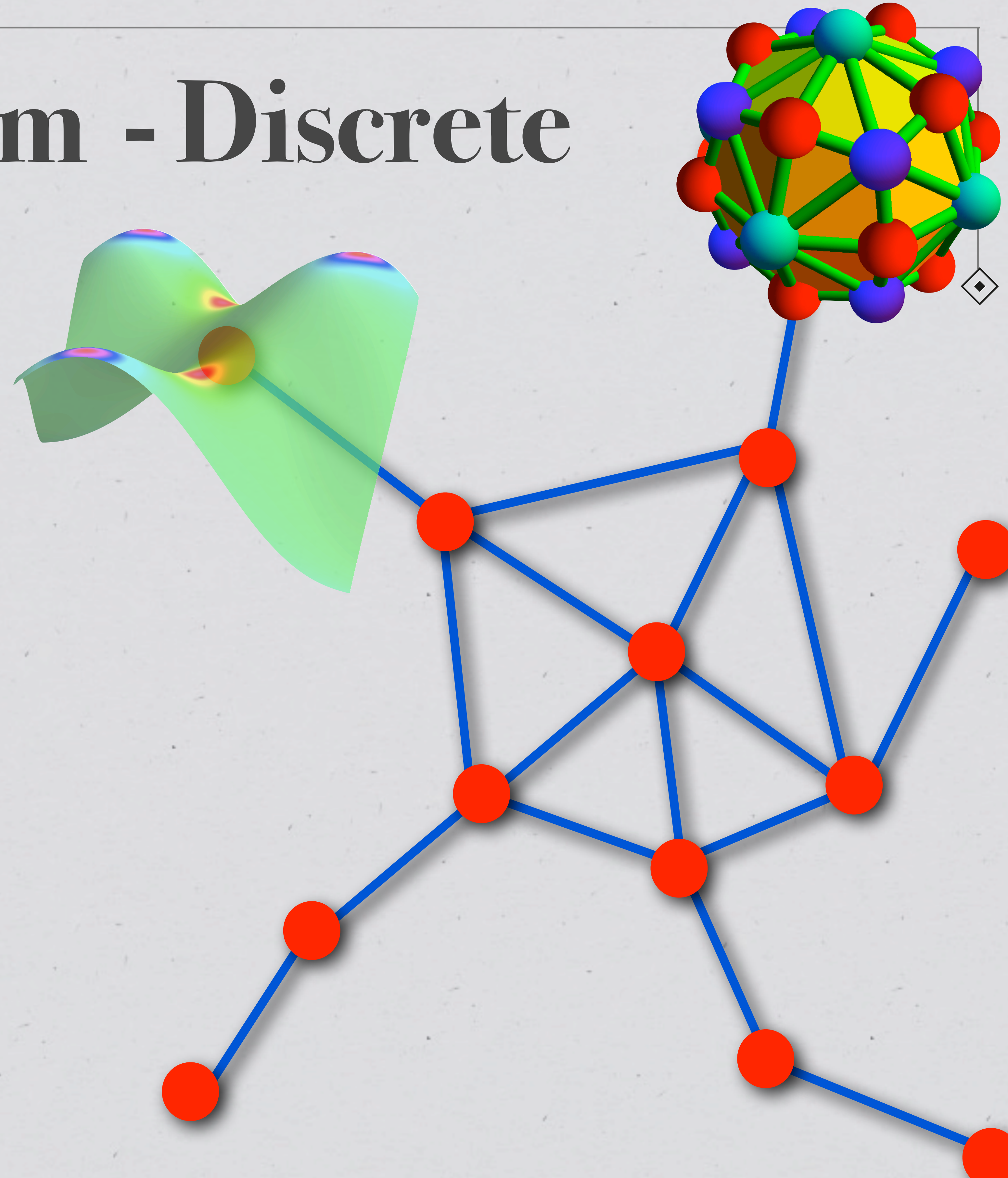


SOME DIFFERENTIAL GEOMETRIC PROBLEMS ON GRAPHS

Oliver Knill, Bridgewater, November 16,2012

Continuum - Discrete

Meta goal: carry continuum results to the discrete.



Challenges:

Notation

Complexity

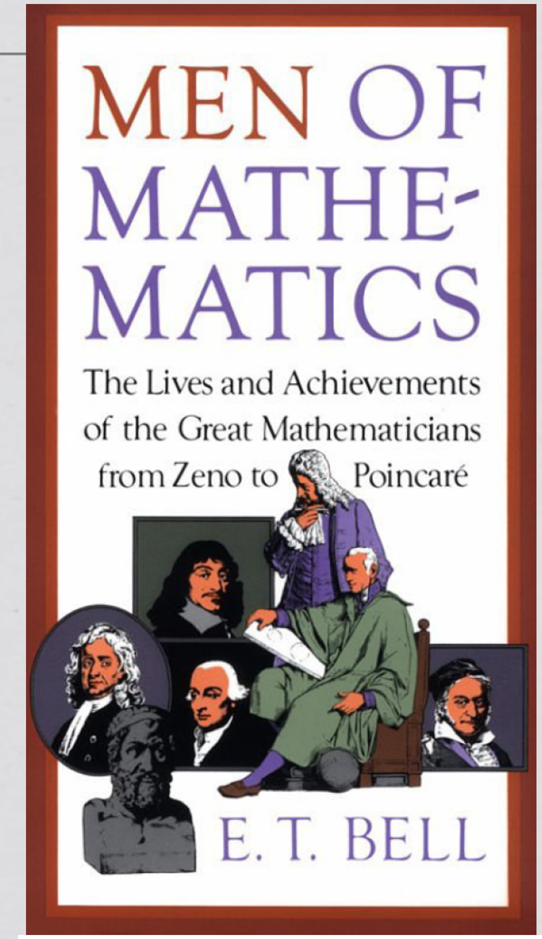
Teachability

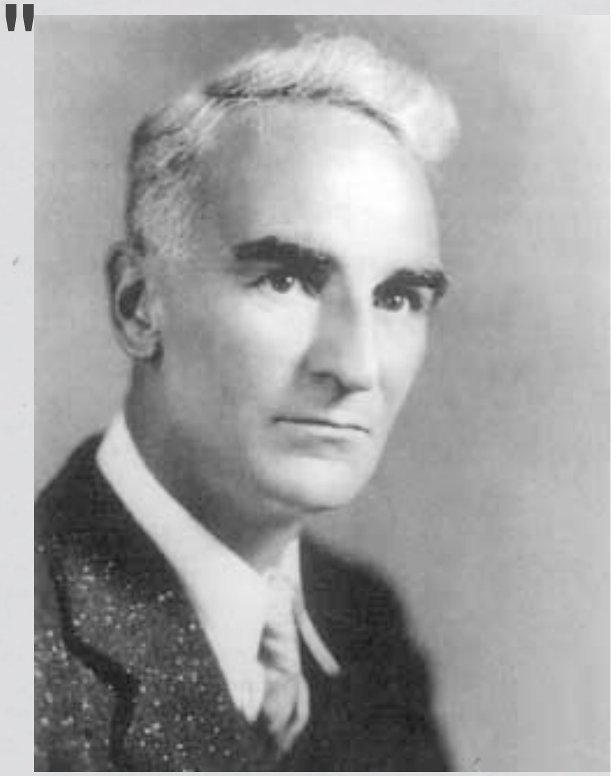
} pedagogical

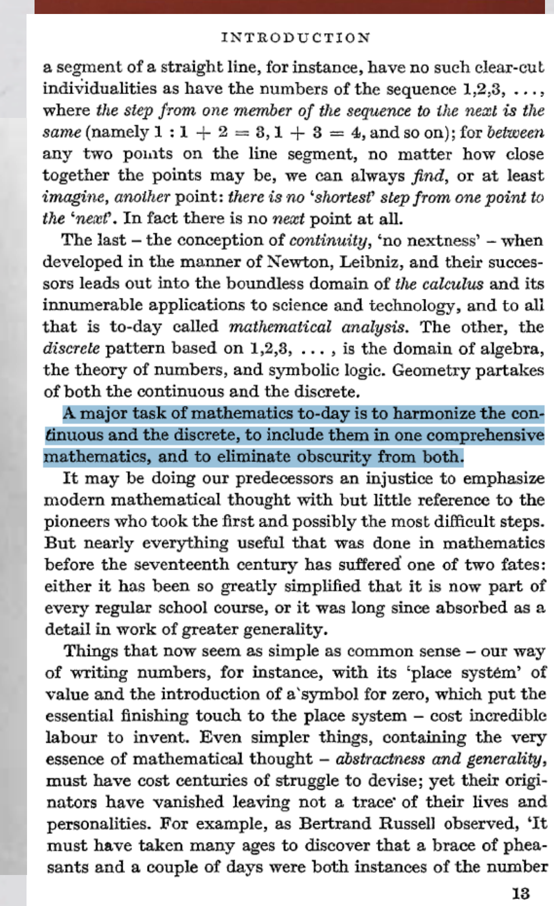
Bell Quote:

"A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both."

Eric Temple Bell, ``Men of Mathematics" 1937






INTRODUCTION

a segment of a straight line, for instance, have no such clear-cut individualities as have the numbers of the sequence 1,2,3, . . ., where *the step from one member of the sequence to the next is the same* (namely $1 : 1 + 2 = 3, 1 + 3 = 4$, and so on); for *between* any two points on the line segment, no matter how close together the points may be, we can always *find*, or at least *imagine*, *another* point: *there is no 'shortest' step from one point to the 'next'*. In fact there is no *next* point at all.

The last – the conception of *continuity*, 'no nextness' – when developed in the manner of Newton, Leibniz, and their successors leads out into the boundless domain of *the calculus* and its innumerable applications to science and technology, and to all that is to-day called *mathematical analysis*. The other, the *discrete* pattern based on 1,2,3, . . . , is the domain of algebra, the theory of numbers, and symbolic logic. Geometry partakes of both the continuous and the discrete.

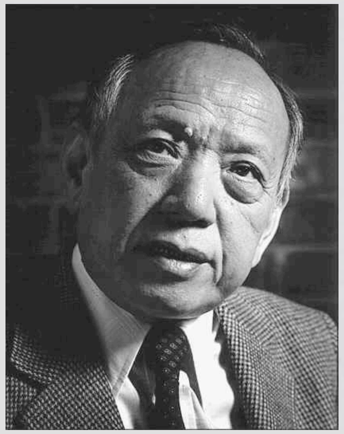
A major task of mathematics to-day is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.

It may be doing our predecessors an injustice to emphasize modern mathematical thought with but little reference to the pioneers who took the first and possibly the most difficult steps. But nearly everything useful that was done in mathematics before the seventeenth century has suffered one of two fates: either it has been so greatly simplified that it is now part of every regular school course, or it was long since absorbed as a detail in work of greater generality.

Things that now seem as simple as common sense – our way of writing numbers, for instance, with its 'place system' of value and the introduction of a symbol for zero, which put the essential finishing touch to the place system – cost incredible labour to invent. Even simpler things, containing the very essence of mathematical thought – *abstractness and generality*, must have cost centuries of struggle to devise; yet their originators have vanished leaving not a trace of their lives and personalities. For example, as Bertrand Russell observed, 'It must have taken many ages to discover that a brace of pheasants and a couple of days were both instances of the number

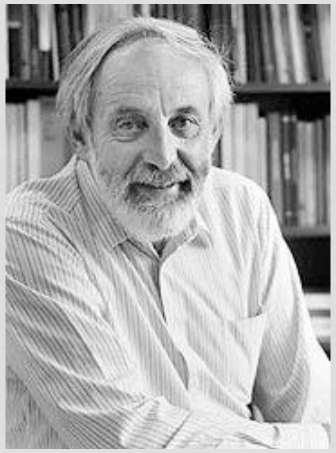
13



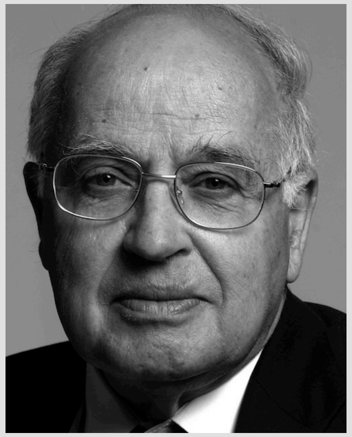
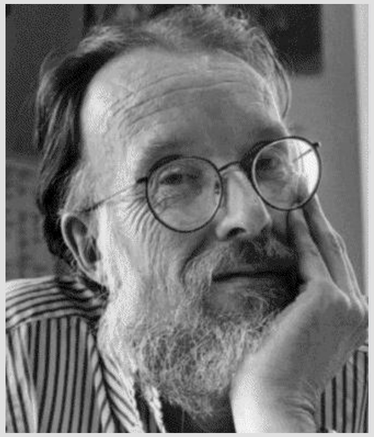
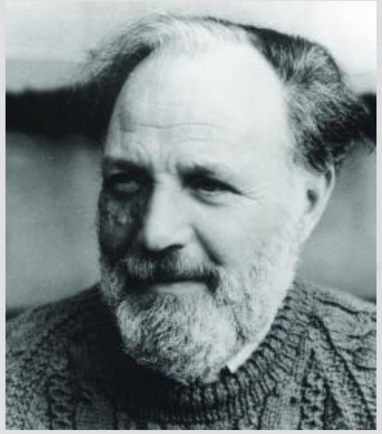






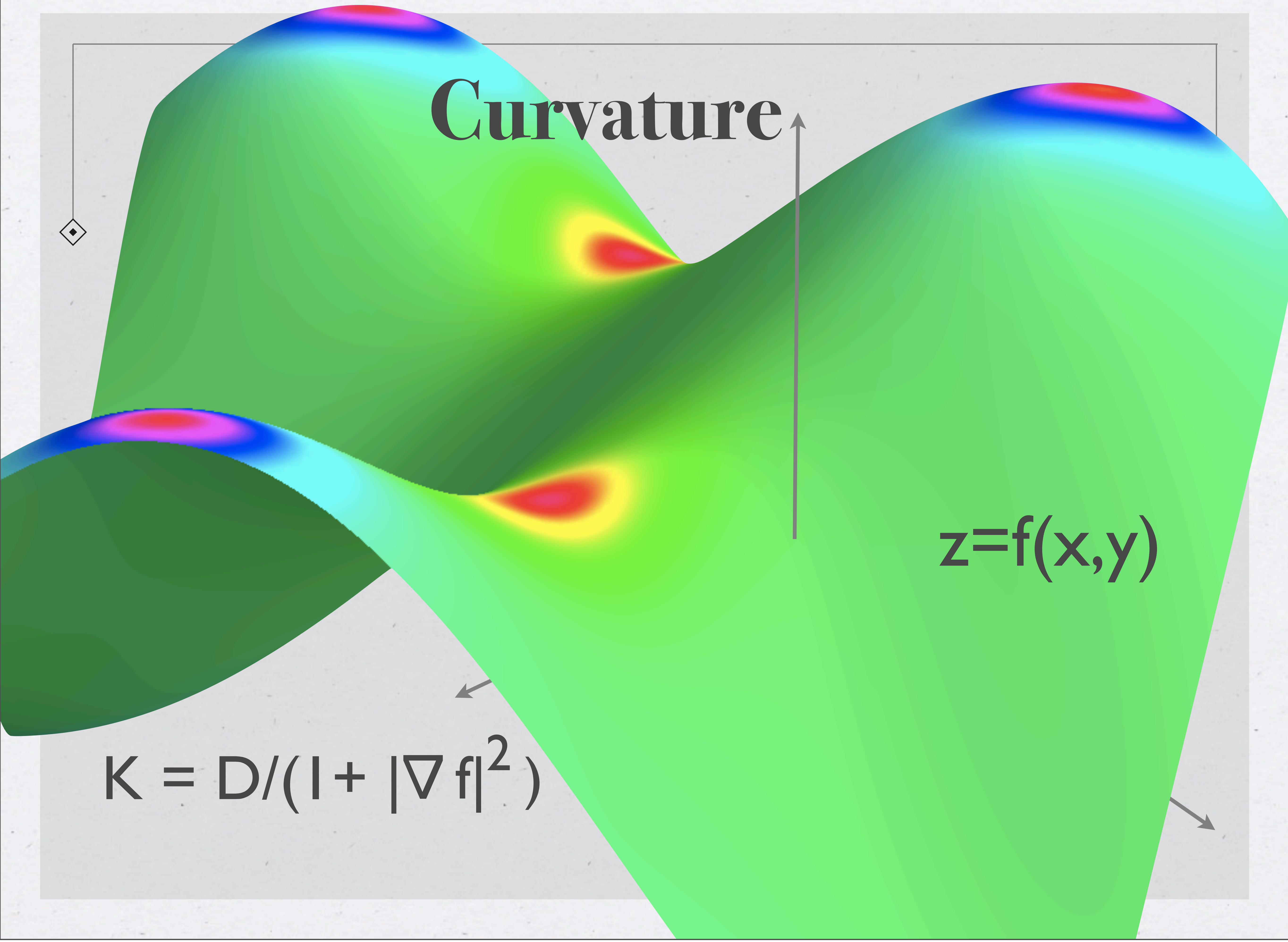




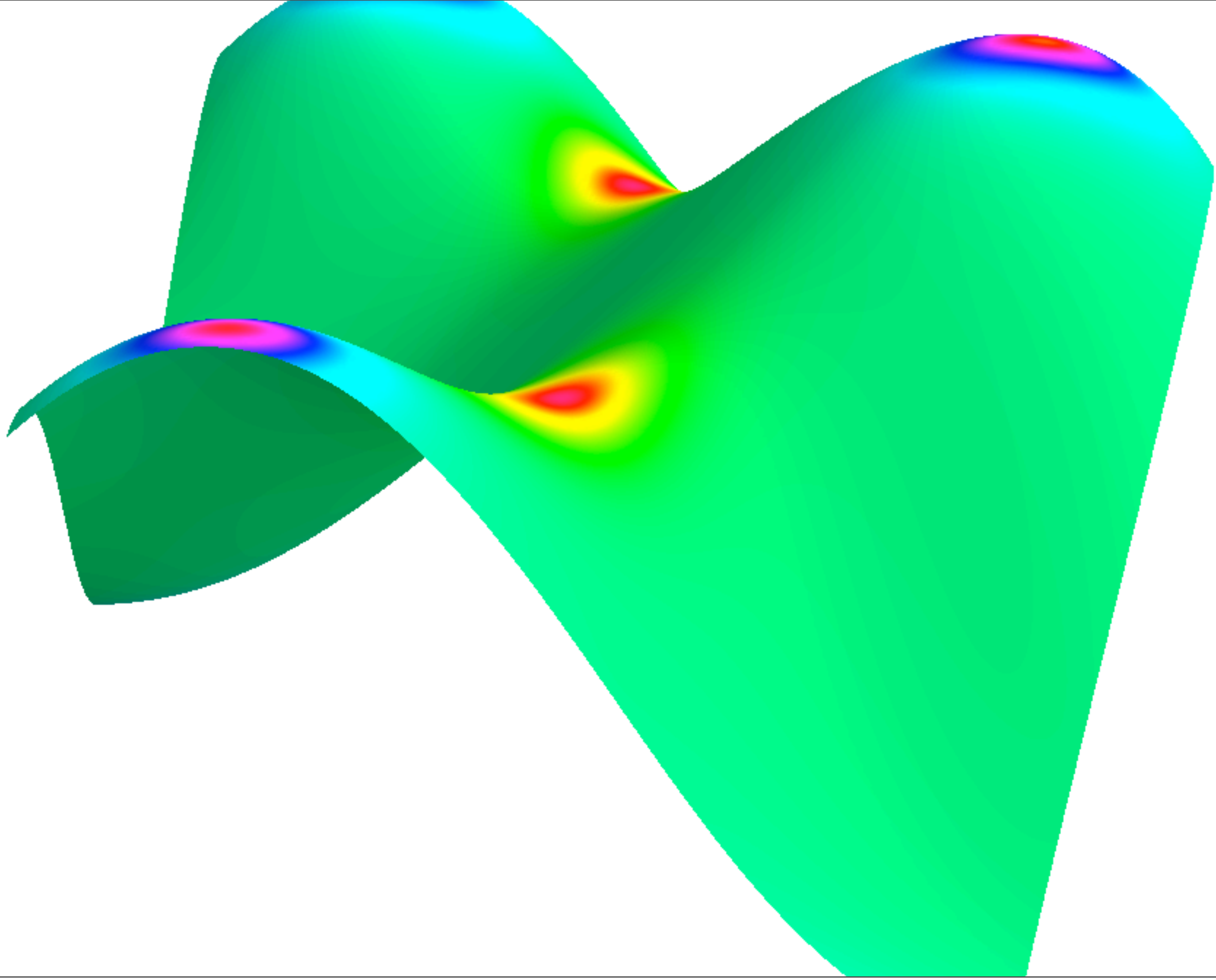




Part I: Graphs and Graphs

Curvature



$$K = D/(1 + |\nabla f|^2)$$

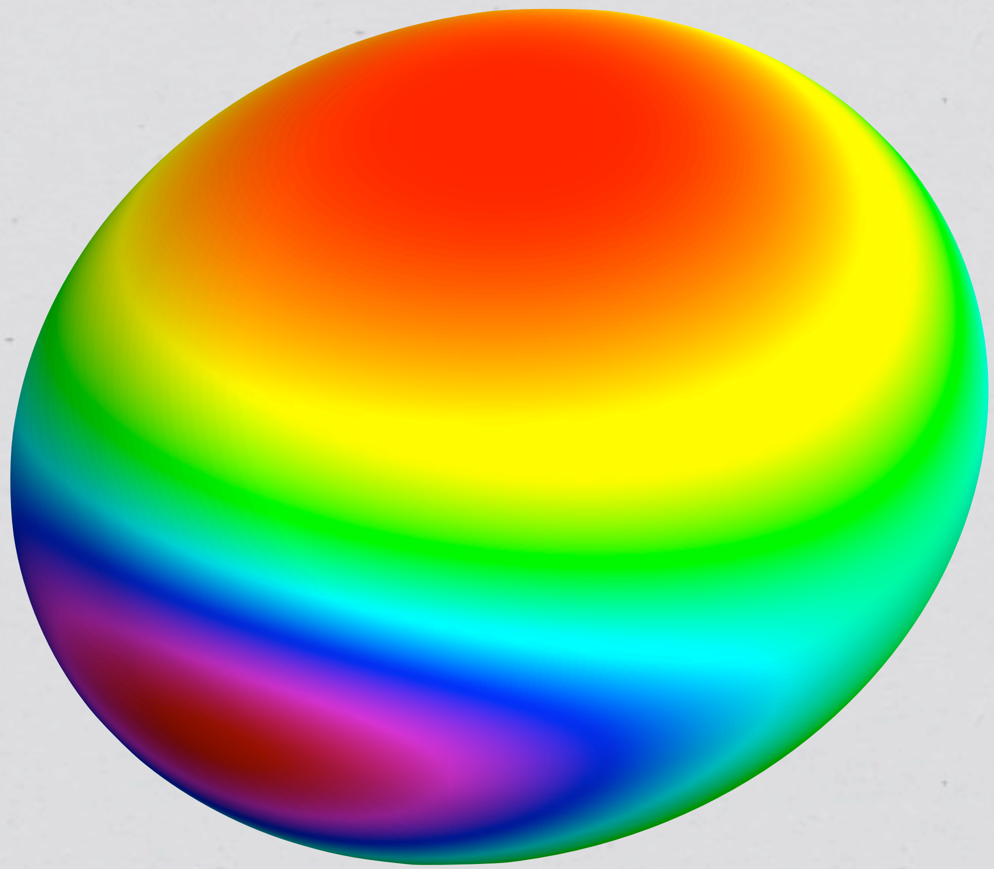


A beautiful Theorem

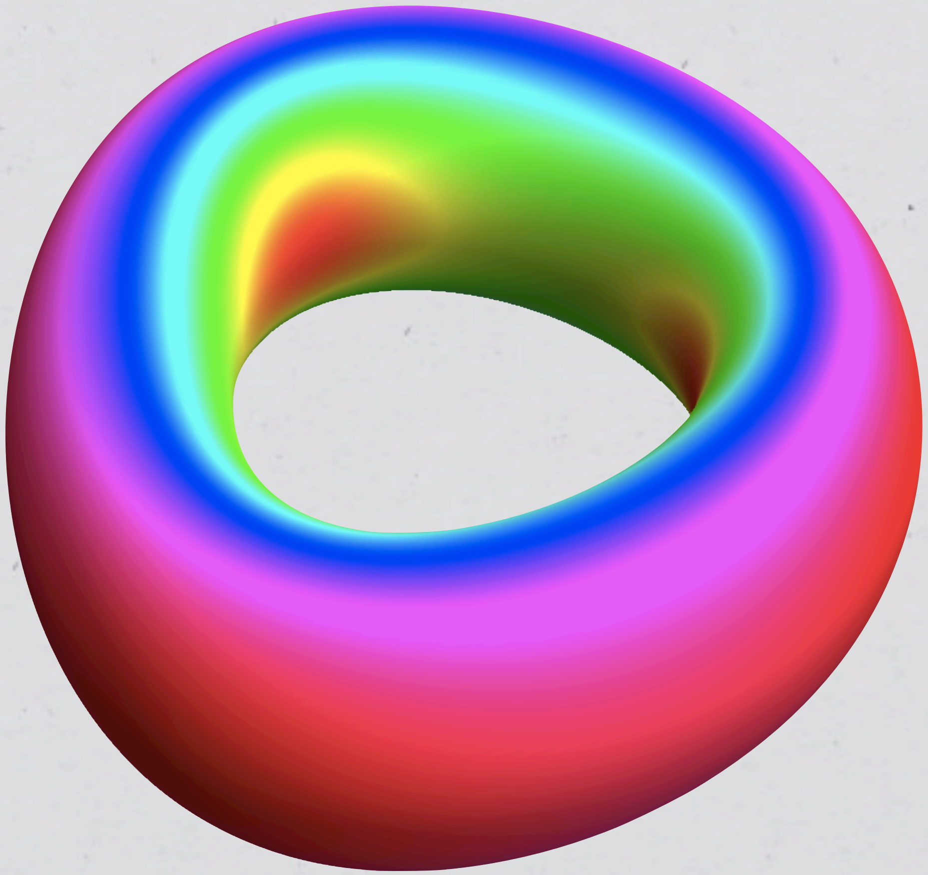
Gauss Bonnet

The integral of curvature is the

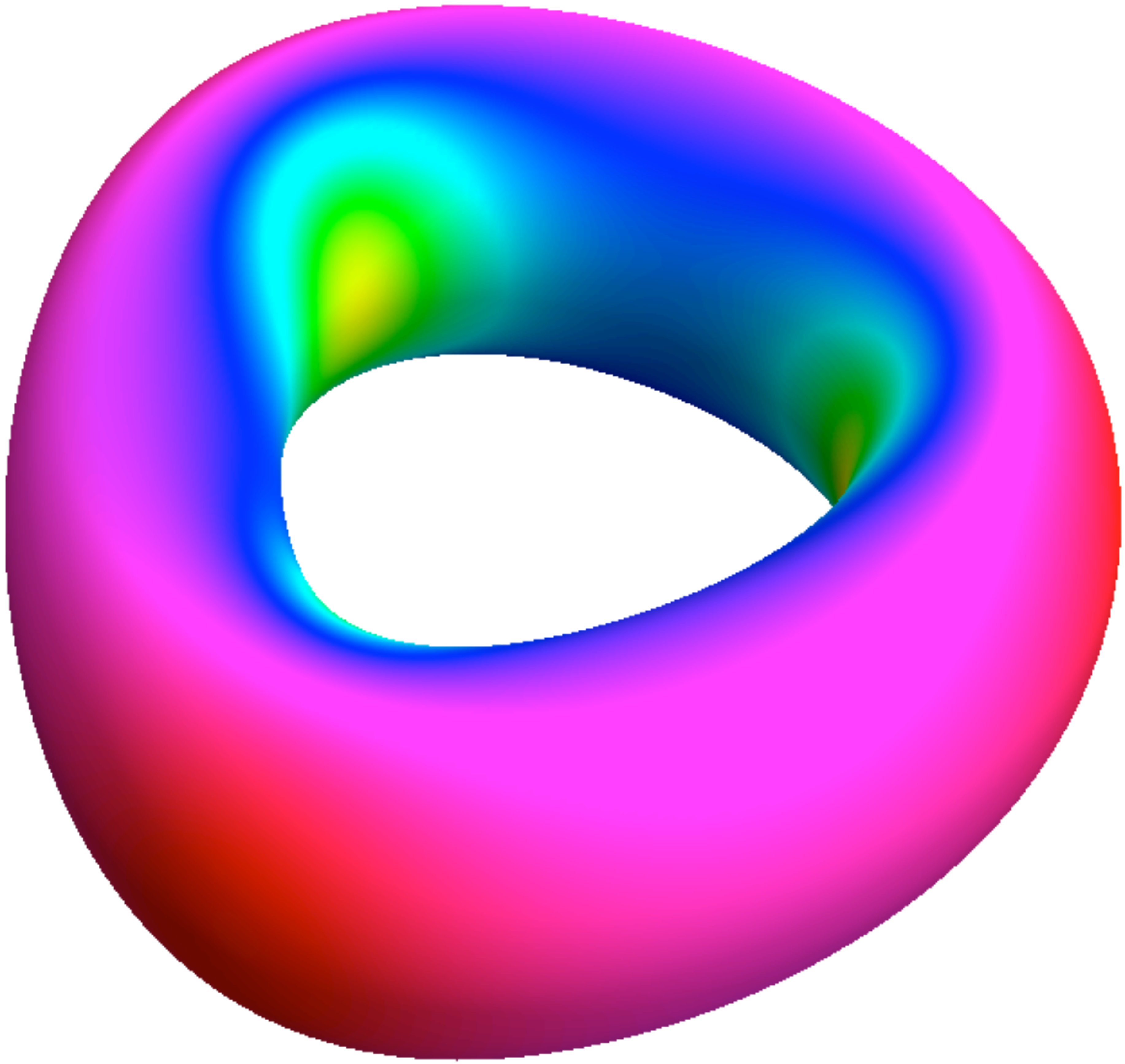
2 π times Euler characteristic



$\chi = 2$



$\chi = 0$



Generalization: Gauss-Bonnet-Chern

The integral of the curvature K over a compact Riemannian manifold M is the Euler characteristic of M.

The formula for K

Complicated!

$$\langle \nabla_i X_j, X_k \rangle = \frac{1}{2}\left\{\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij}\right\}$$

(M,g) Riemannian of dimension n=2k

so

$$\nabla_i X_j = \sum \Gamma_{ij}^k X_k$$

$$\Gamma_{ij}^k = \sum_l g^{kl} \langle \nabla_i X_j, X_l \rangle$$

$$R_k{}^l{}_{ij} = \frac{\partial}{\partial x_i} \Gamma_{jk}^l - \frac{\partial}{\partial x_j} \Gamma_{ik}^l + \sum_r (\Gamma_{jk}^r \Gamma_{ir}^l - \Gamma_{ik}^r \Gamma_{jr}^l) .$$

$$K(x) = \frac{(-1)^k}{(4\pi)^k k! 2^k} \sum_{\pi, \sigma} (-1)^\pi (-1)^\sigma R^{\pi(1)\pi(2)}{}_{\sigma(1)\sigma(2)} \cdots R^{\pi(n-1)\pi(n)}{}_{\sigma(n-1)\sigma(n)}$$

is a scalar (independent of coordinate systems).

formulas from Cycon-Froese-Kirsch-Simon, 1987

Gauss-Bonnet-Chern

Karl-Friedrich Gauss | Pierre Ossian Bonnet | Shiing-Shen Chern

1777-1855 | 1819-1892 | 1911-2004

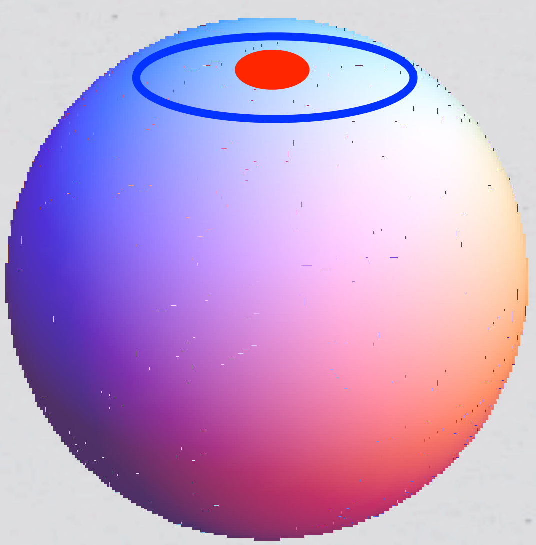
1800 | 1900 | 2000

What about Curvature for Graphs?

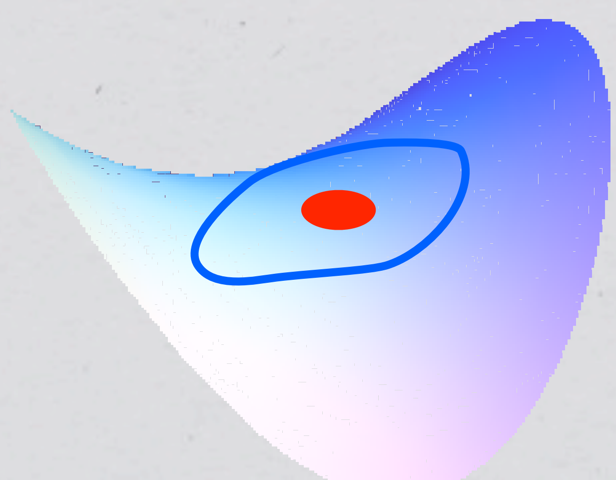
Bertrand-Diquet-Puiseux

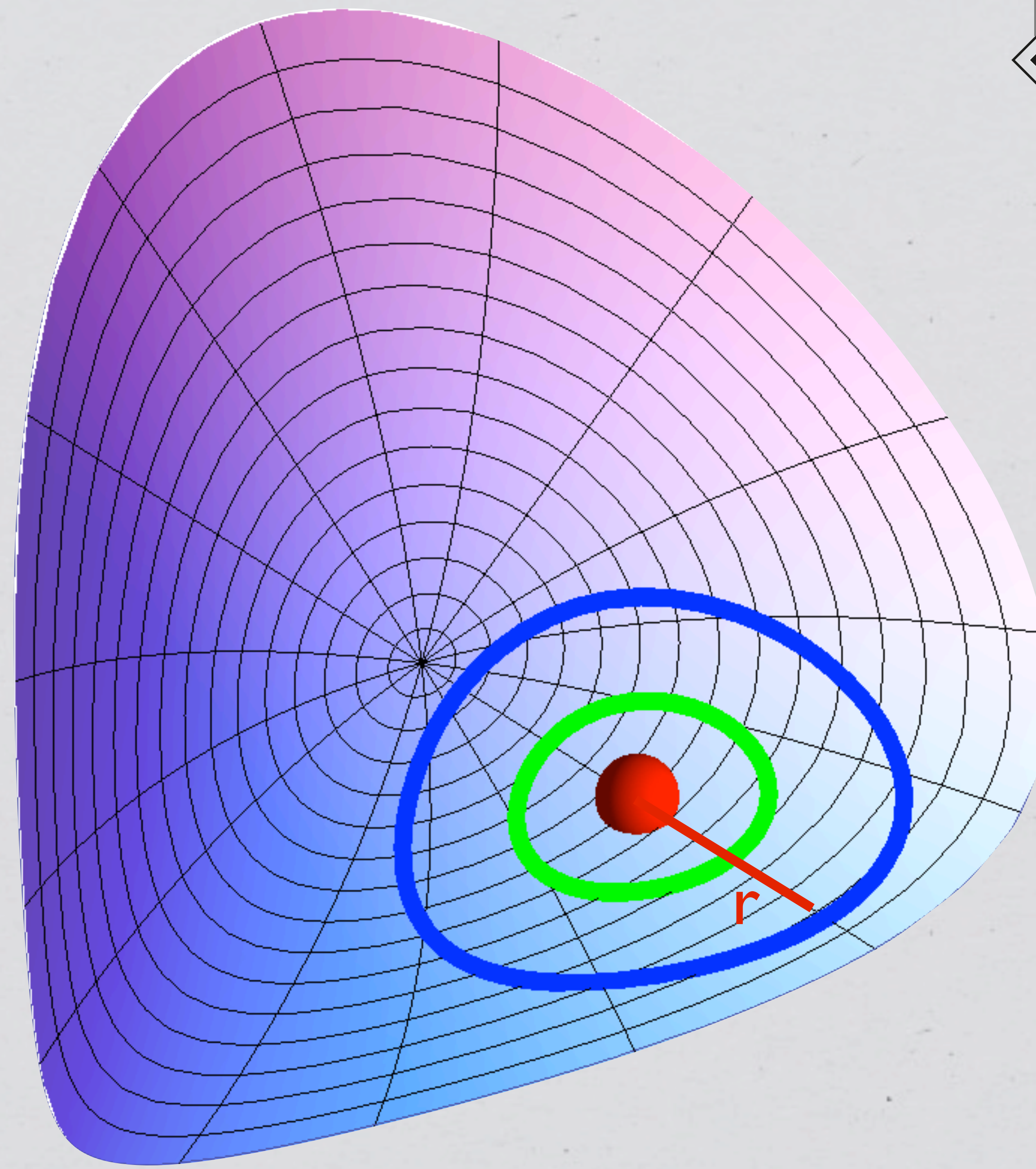
$K = (2\pi r - |S_r|) \quad 3\pi / r^3$

K>0



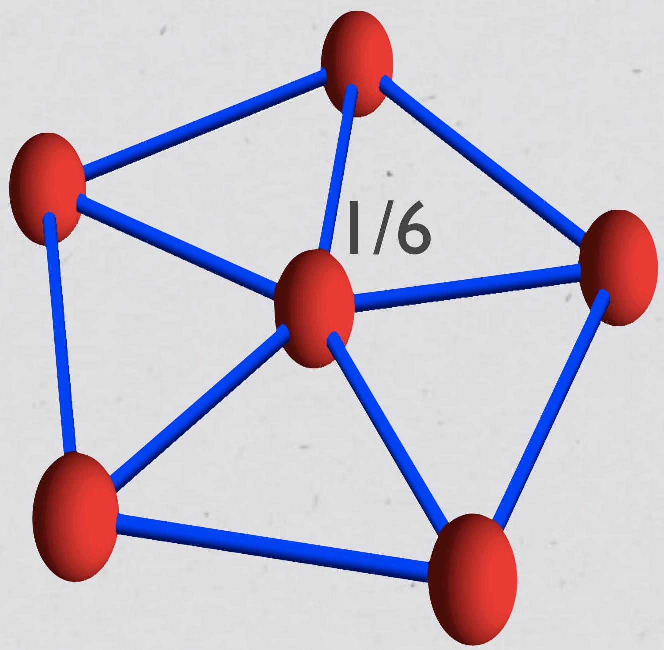
K<0

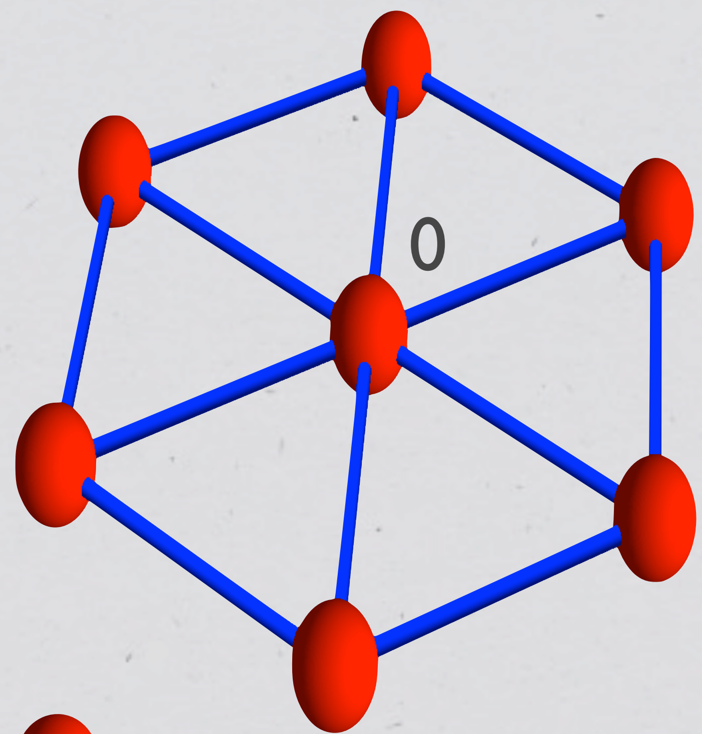




Curvature for Graphs

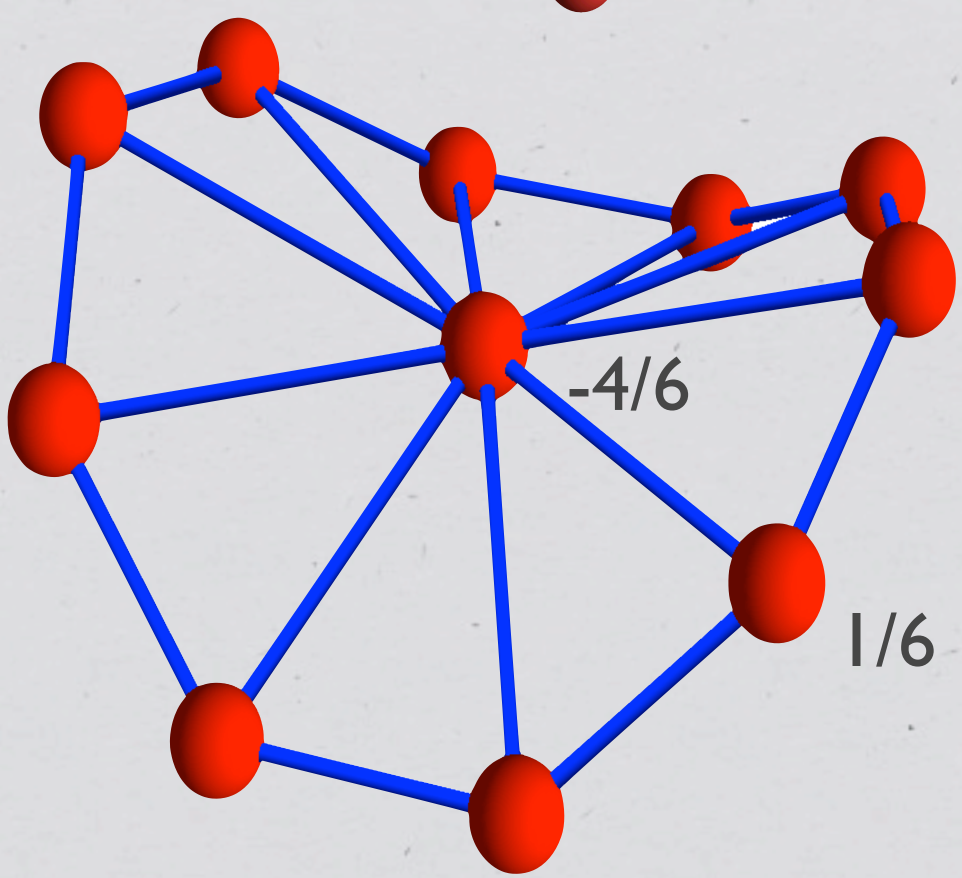
$$K = 1 - V/2 + E/3$$

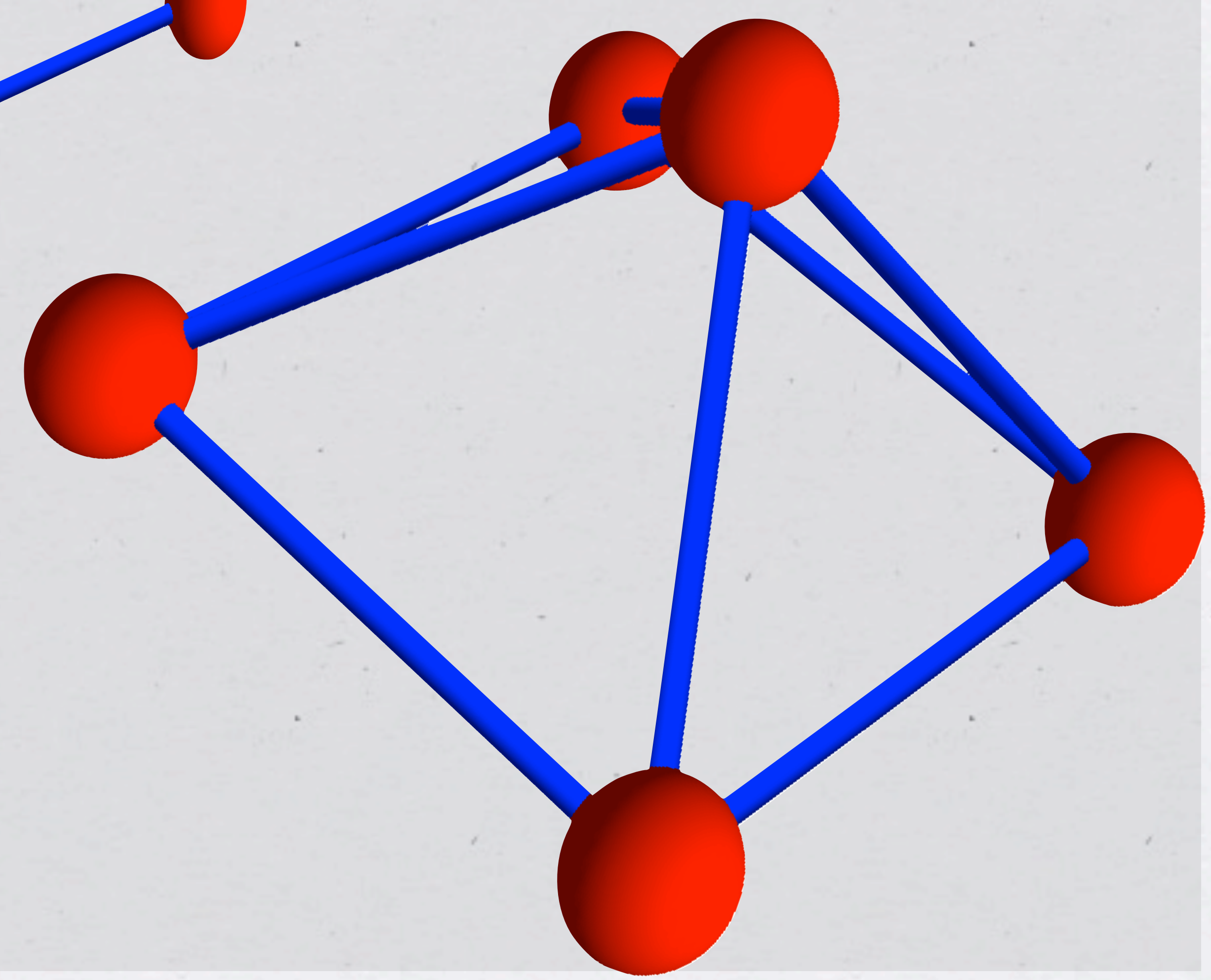


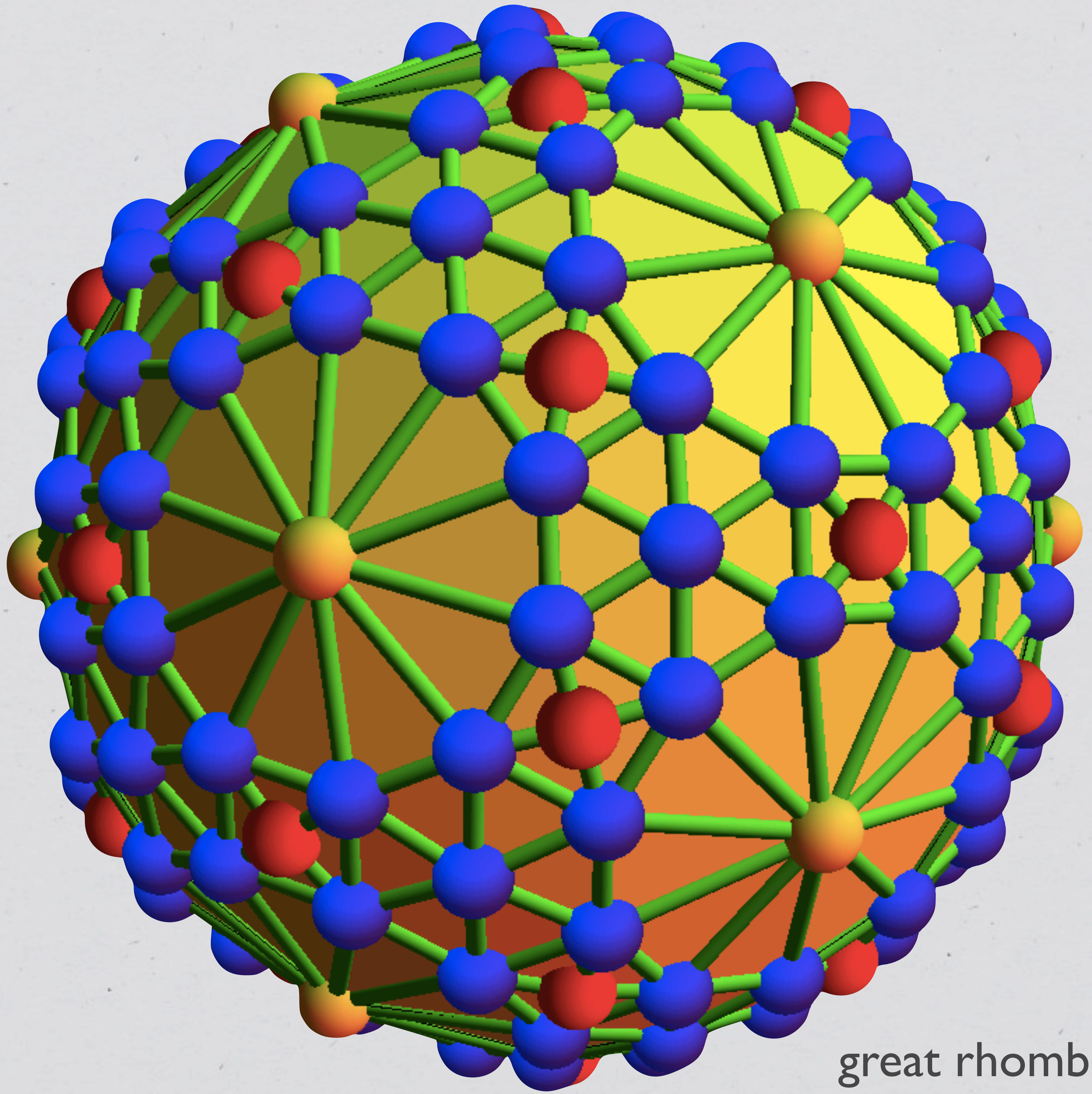


1-3/2+2/3=1/6

1-4/2-4/3=1/3







12 times -4/6

30 times +2/6

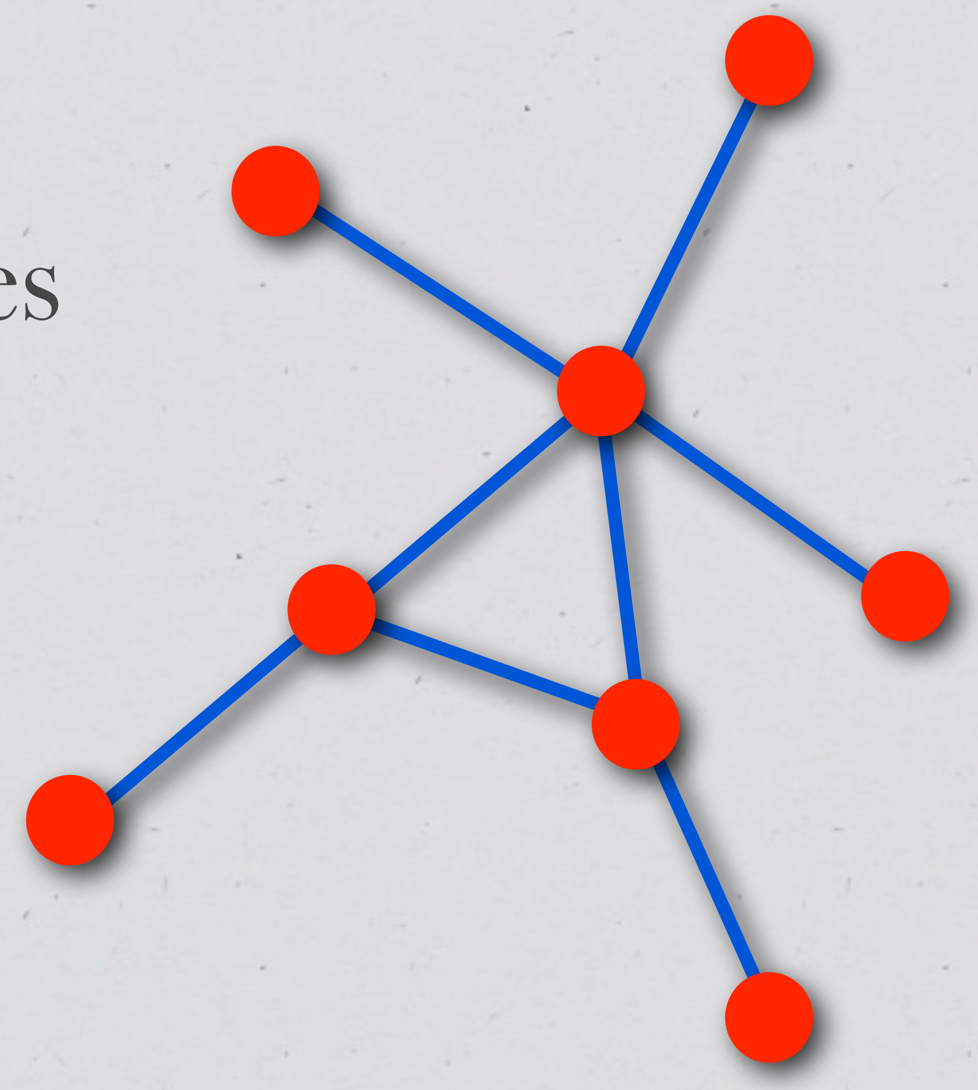
total

curvature = 2

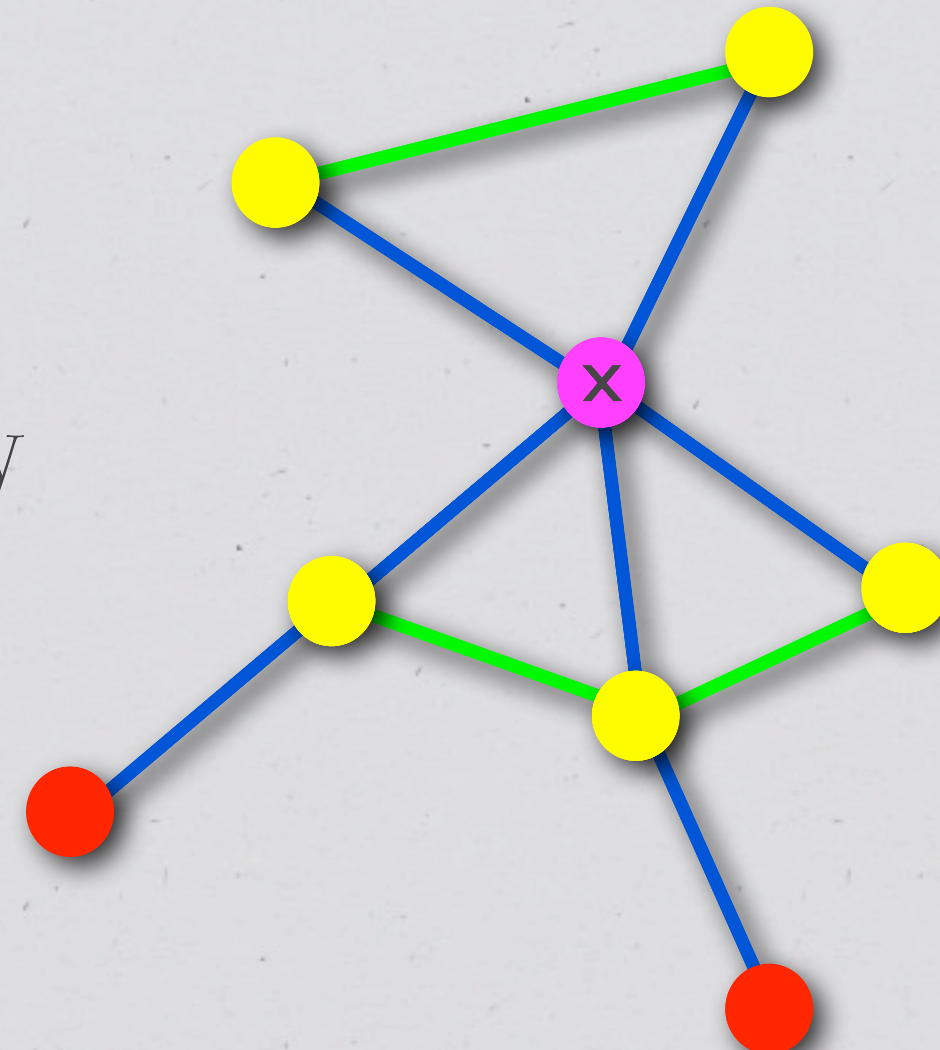
great rhomb icosi dodecahedron

Simple Graph

- G=(V,E)   Vertices and Edges
- V finite
- Undirected
- No self loops
- No multiple connections

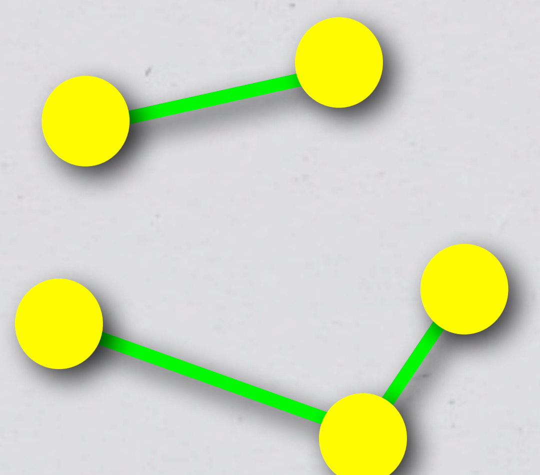


Sphere



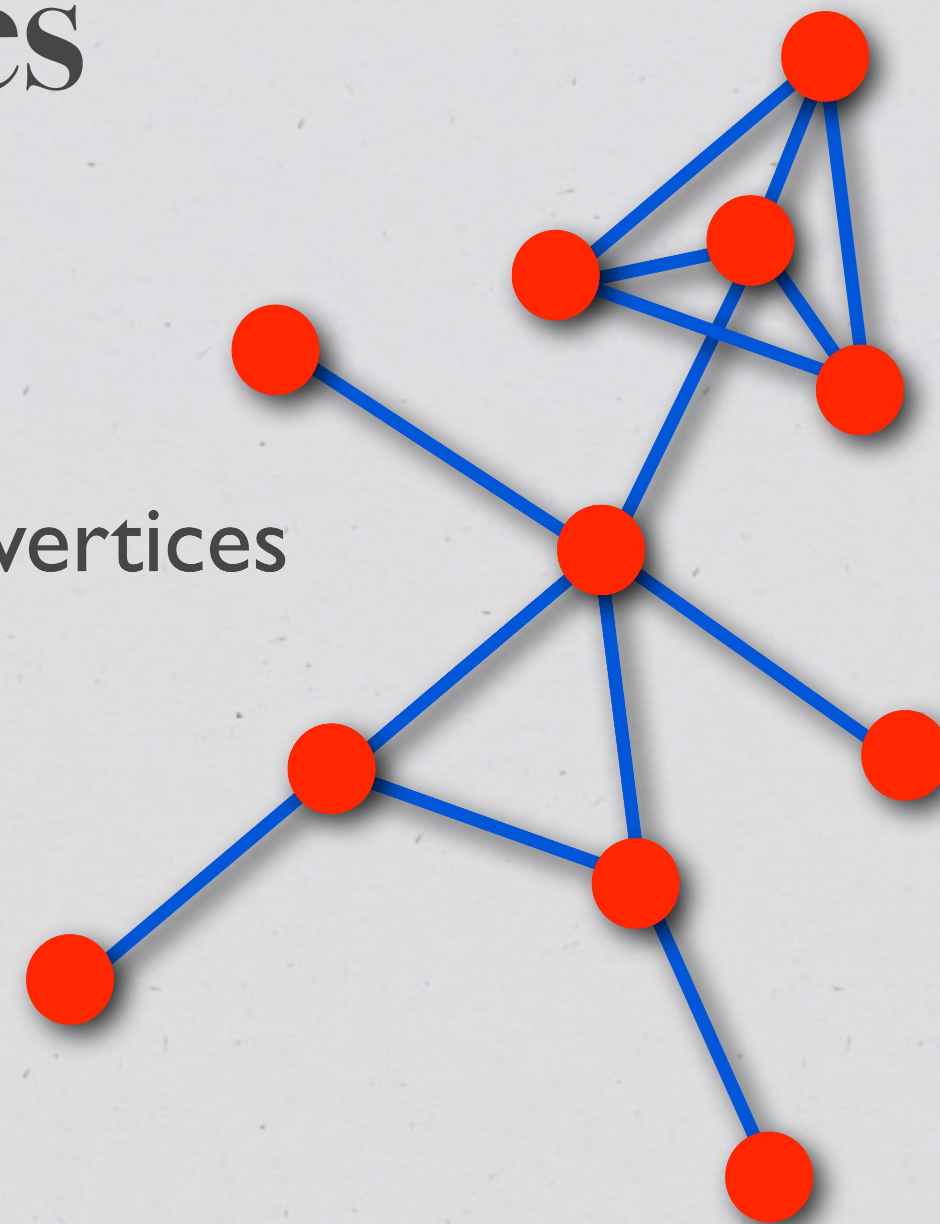
- x is a vertex
- S(x) subgraph generated by all vertices adjacent to x

S(x) =



Cliques

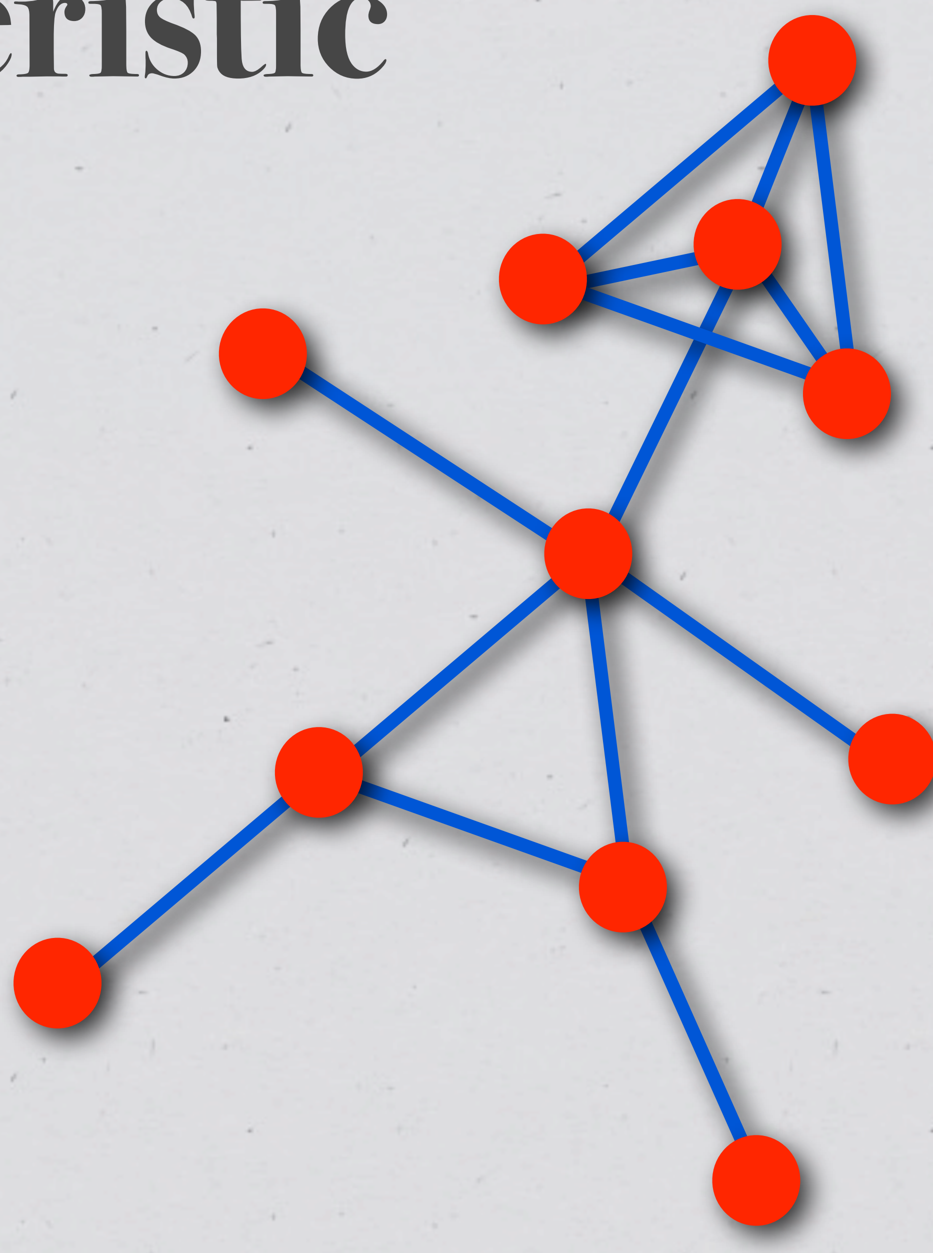
K_{n+1} complete graph with n+1 vertices



- n=0: Vertices v_0 11
- n=1: Edges v_1 14
- n=2: Triangles v_2 5
- n=3: Tetrahedra v_3 1

Euler Characteristic

$$\chi(G) = v_0 - v_1 + v_2 - v_3$$
$$= 11 - 14 + 5 - 1 = 1$$

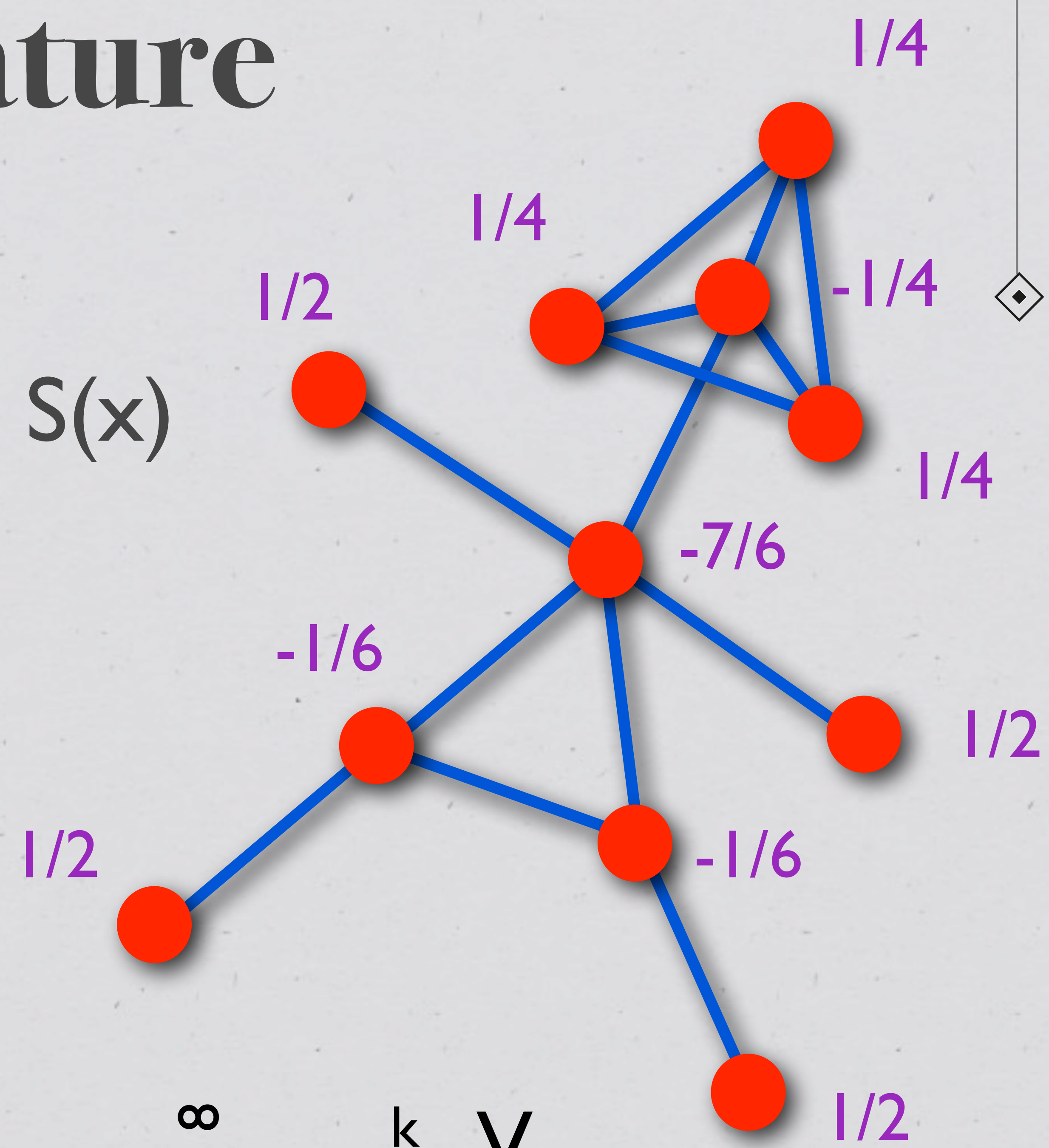


* n=0: Vertices v_0 11
* n=1: Edges v_1 14
* n=2: Triangles v_2 5
* n=3: Tetrahedra v_3 1

Curvature

$V_k(x)$ number of K_{k+1} graphs in $S(x)$

$V_{-1}(x)=1$ assumption



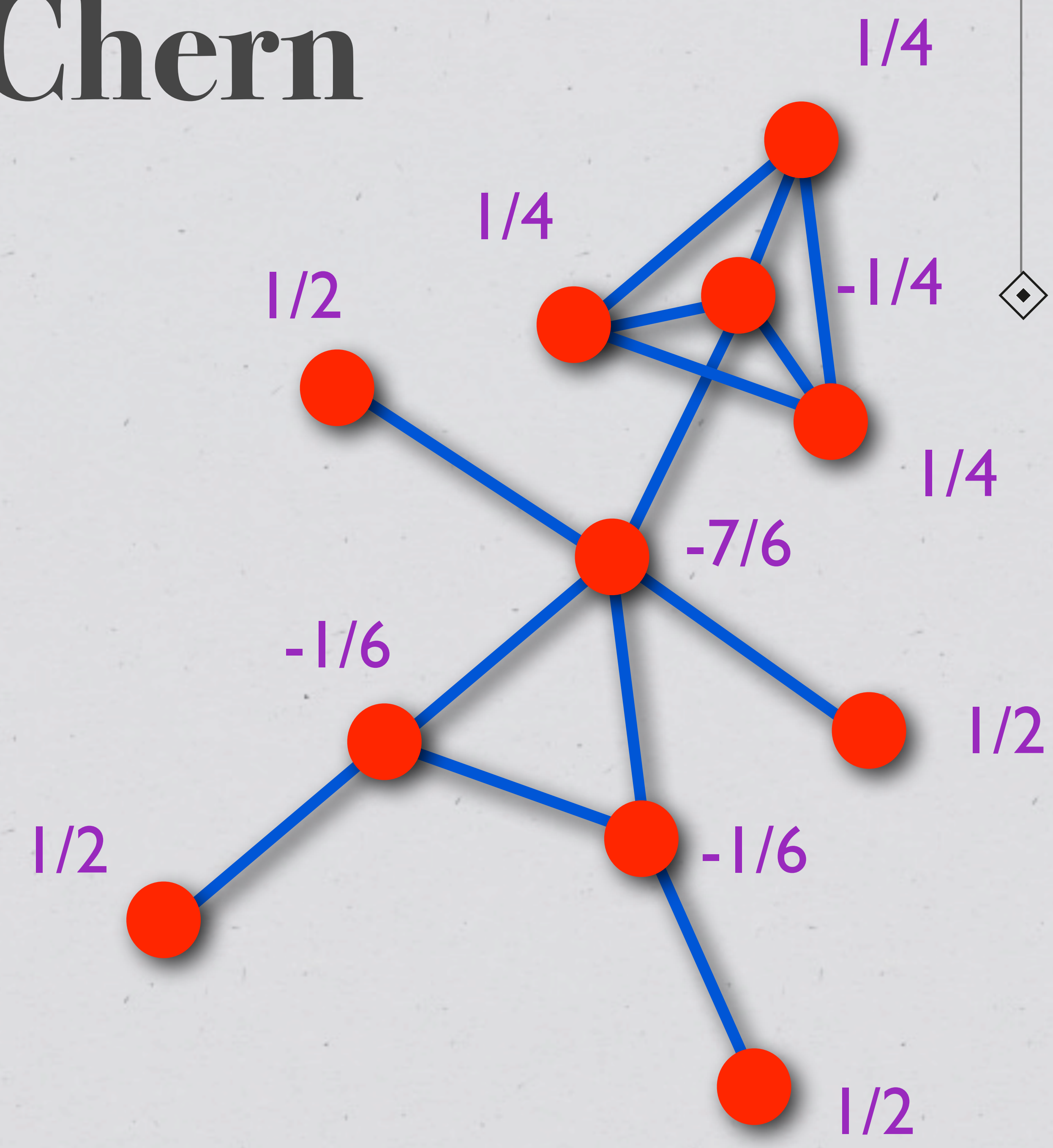
$$K(x) = \frac{V_{-1}}{1} - \frac{V_0}{2} + \frac{V_1}{3} \quad \ldots \quad = \sum_{k=0}^{\infty} (-1)^k \frac{V_{k-1}}{k+1}$$

$$= 1 - V/2 + E/3 - \ldots$$

Gauss-Bonnet-Chern

Theorem:

$$\sum_{x \in V} K(x) = \chi(G)$$

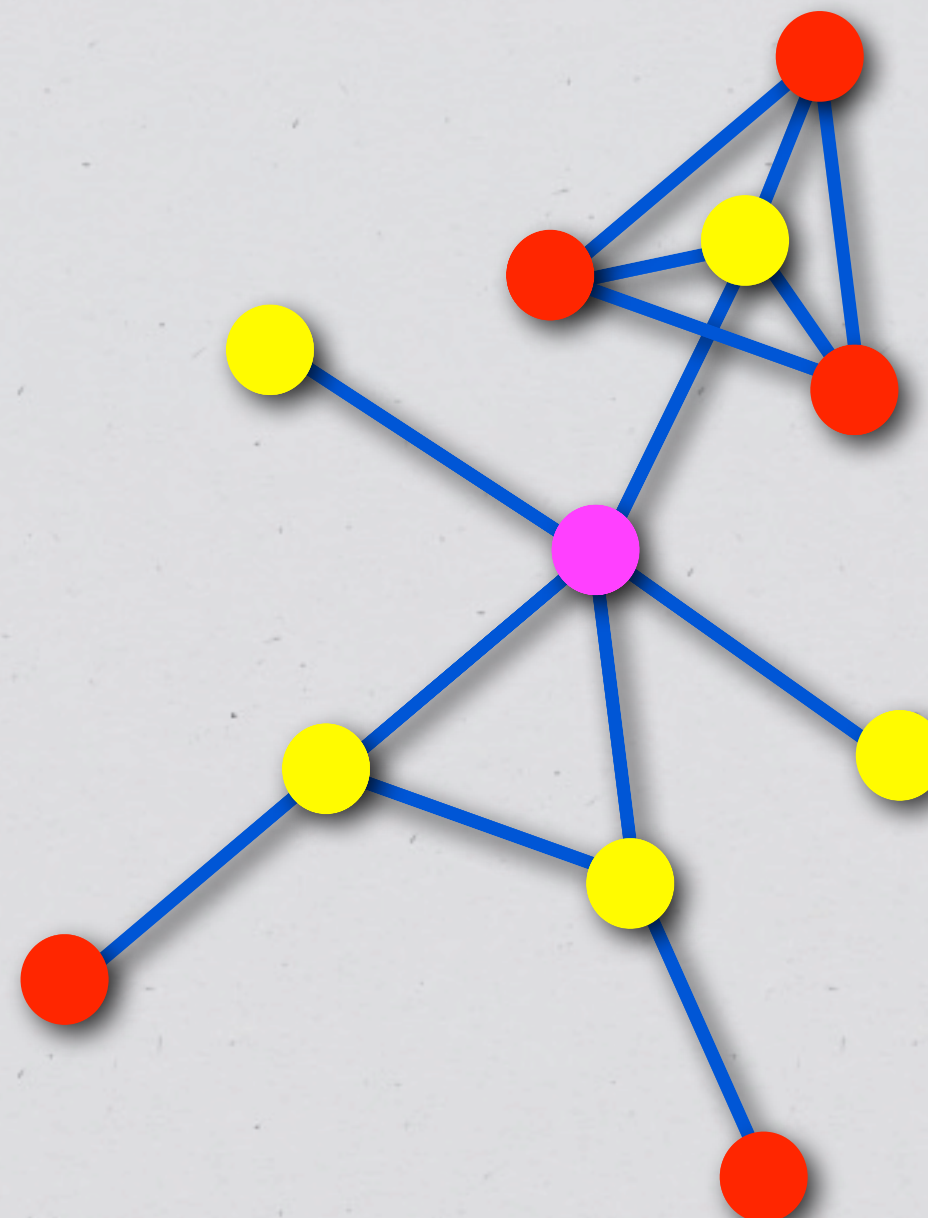


$$\chi(G) = v_0 - v_1 + v_2 - v_3$$
$$= 11 - 14 + 5 - 1 = 1$$

2/2-2/6+2/2-7/6+3/4-1/4 = 1

Handshaking

$$\sum_{x \varepsilon V} V_{k-1}(x) = (k+1)\ v_k$$



k=0: trivial

k=1: Euler:

every edge
couns twice

k=2: every triangle
gets counted three
times

Proof of G-B-C

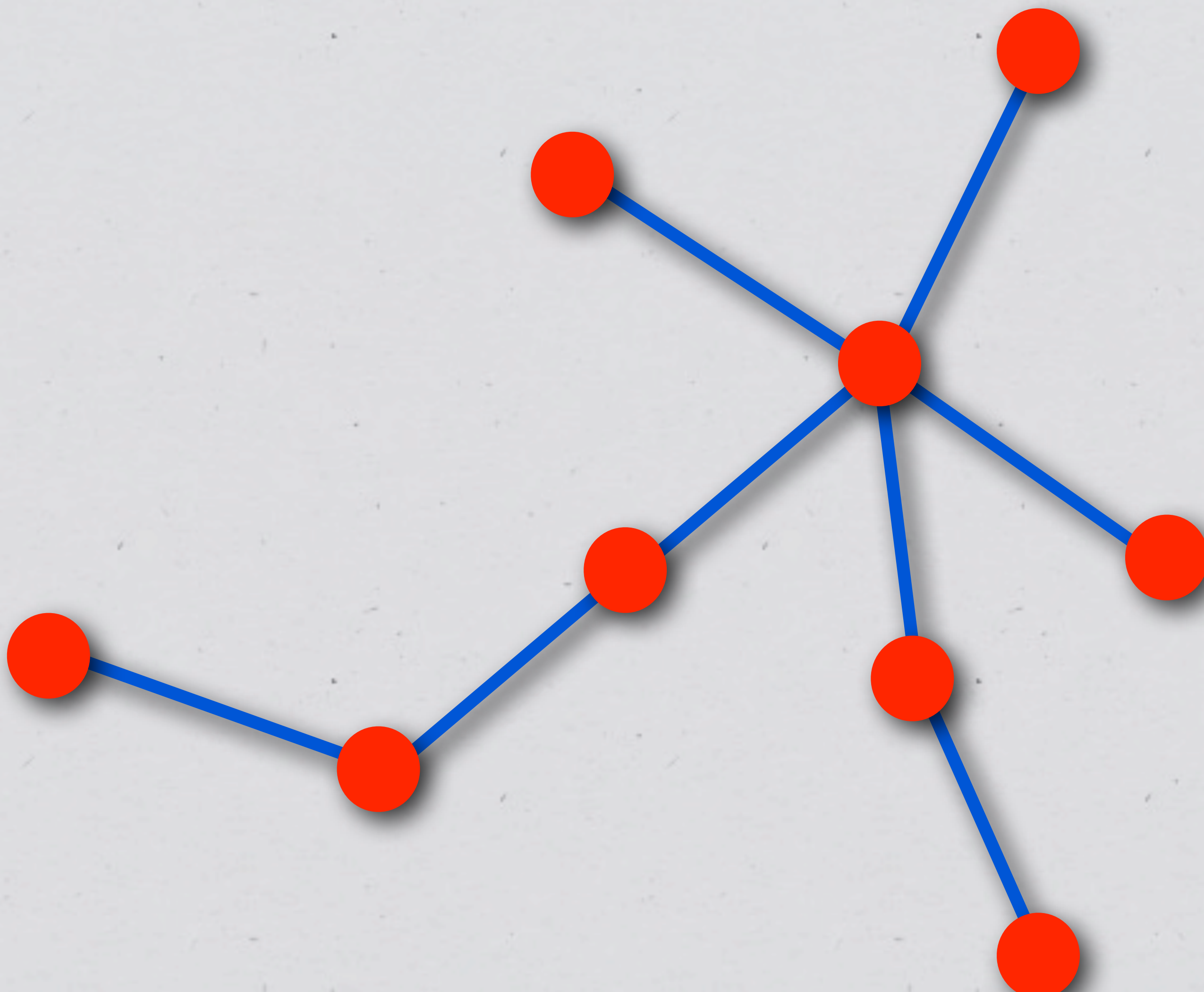
$$\sum_{x \in V} K(x) = \sum_{x \in V} \sum_{k=0}^{\infty} (-1)^k \frac{V_{k-1}(x)}{k+1}$$ definition

$$= \sum_{k=0}^{\infty} (-1)^k \sum_{x \in V} \frac{V_{k-1}(x)}{k+1}$$ order of summation

$$= \sum_{k=0}^{\infty} (-1)^k v_k$$ handshake

$$= \chi(G)$$ definition

Example: Trees

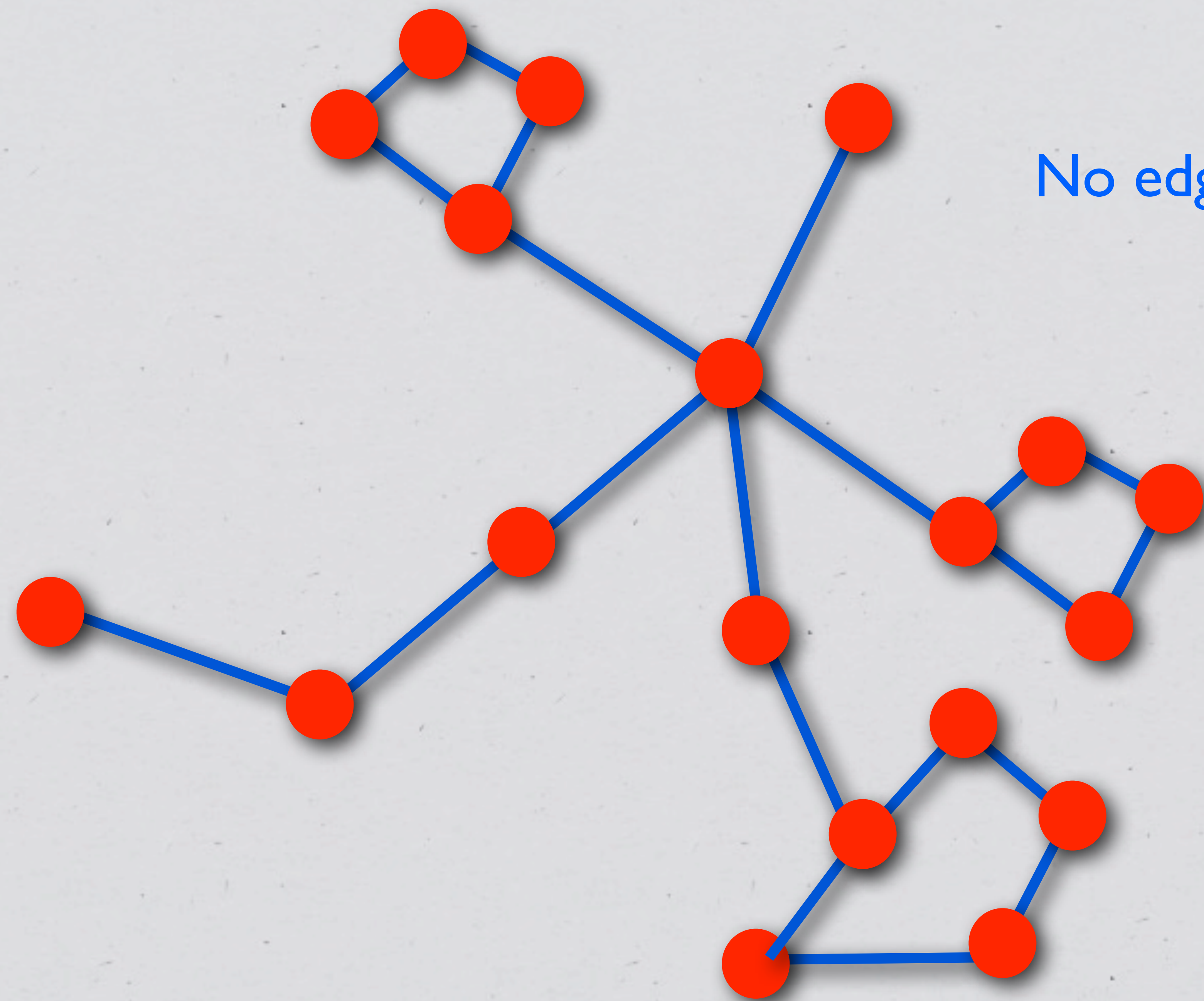


K(x)=1-s/2

s = number of neighbors

χ(G) = Number of Trees

Example: Cacti



No edge belongs to two cycles

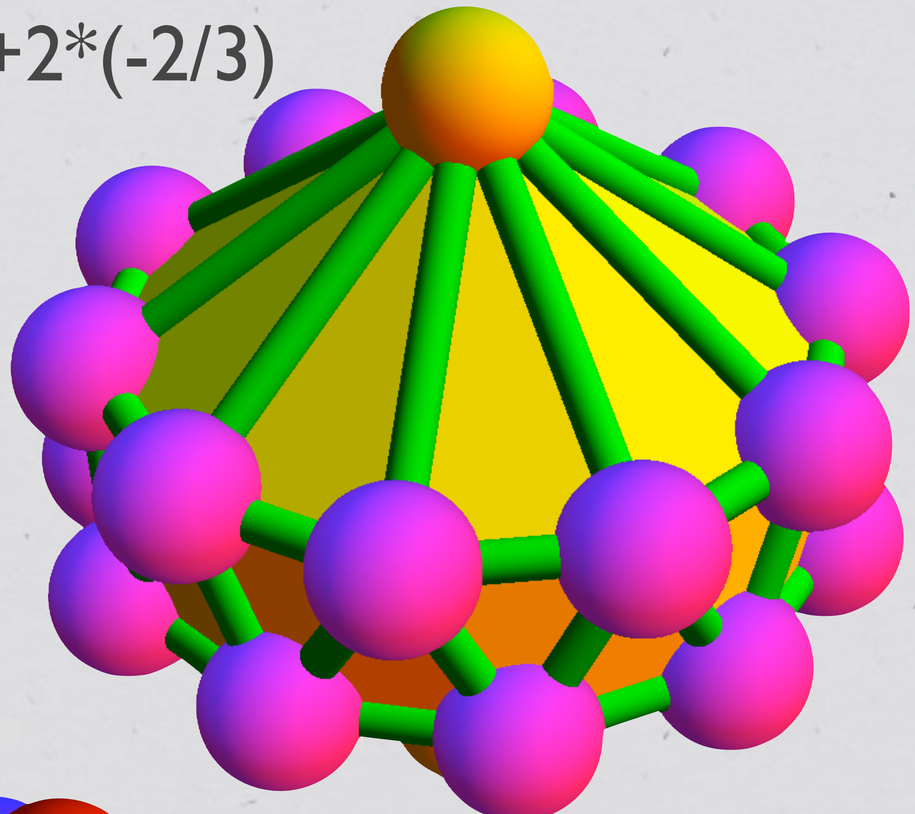
$K(x)=1-s/2$

s = number of neighbors

χ(G) = Number of Cacti - Number of Fruits

Example: Polyhedra

20*1/3+2*(-2/3)



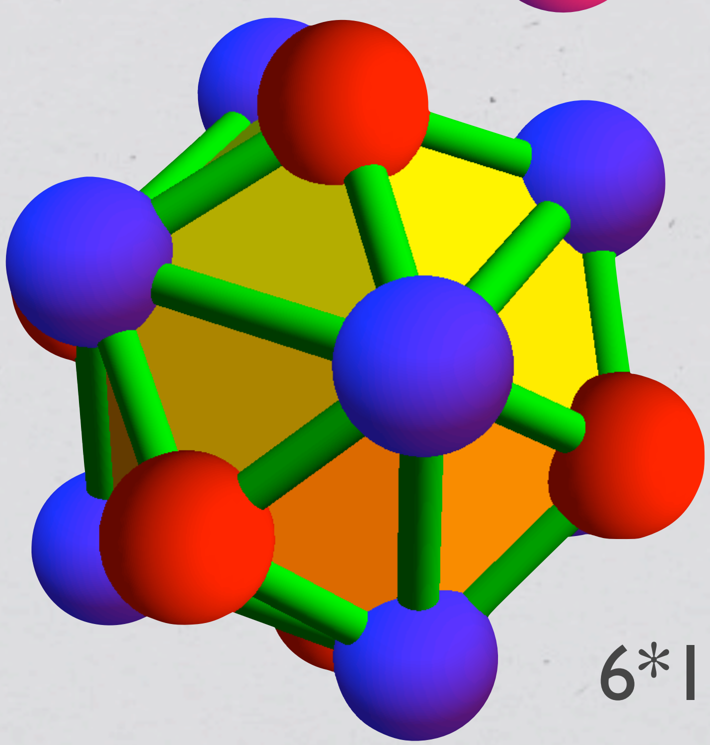
In two dimensions

K(x)=1-s/6

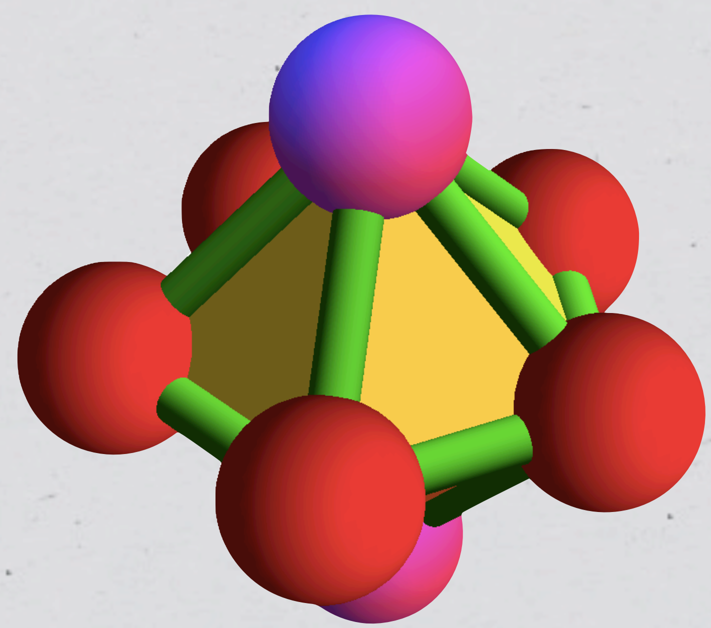
s = number of neighbors

X(G) = v-e+f=2

Euler

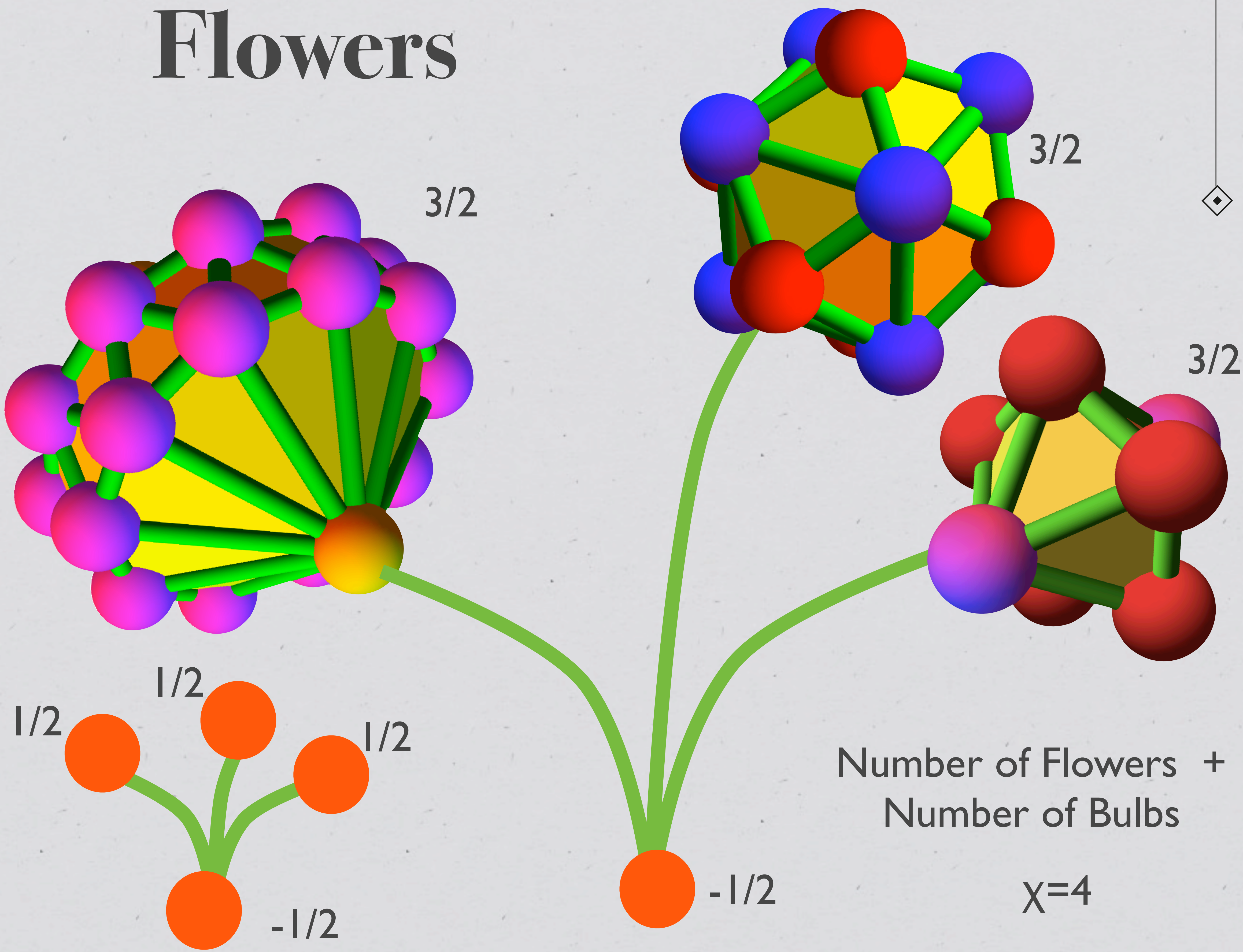


6*1/3



5*1/3+2*1/6

Flowers

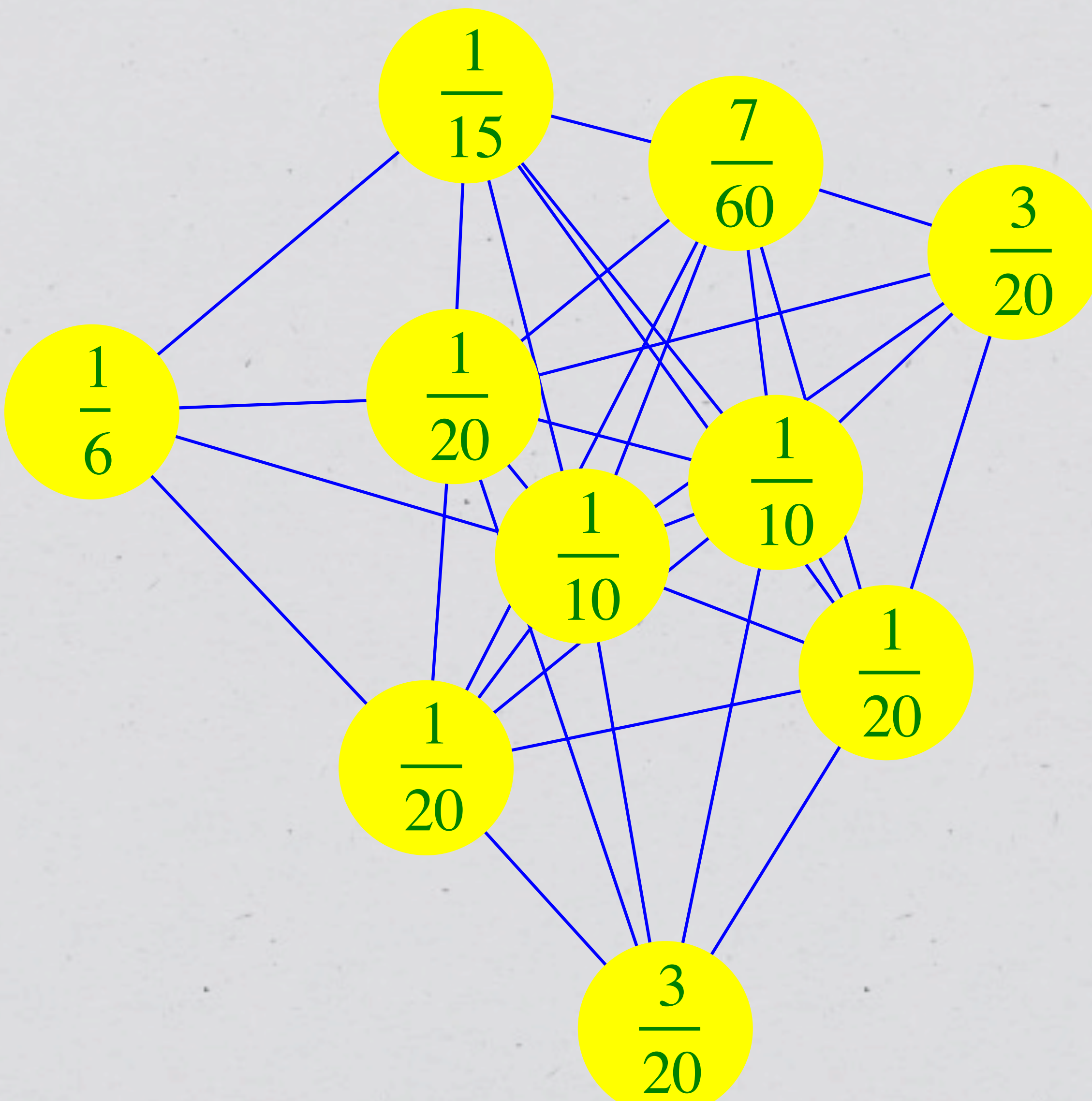


Number of Flowers +
Number of Bulbs

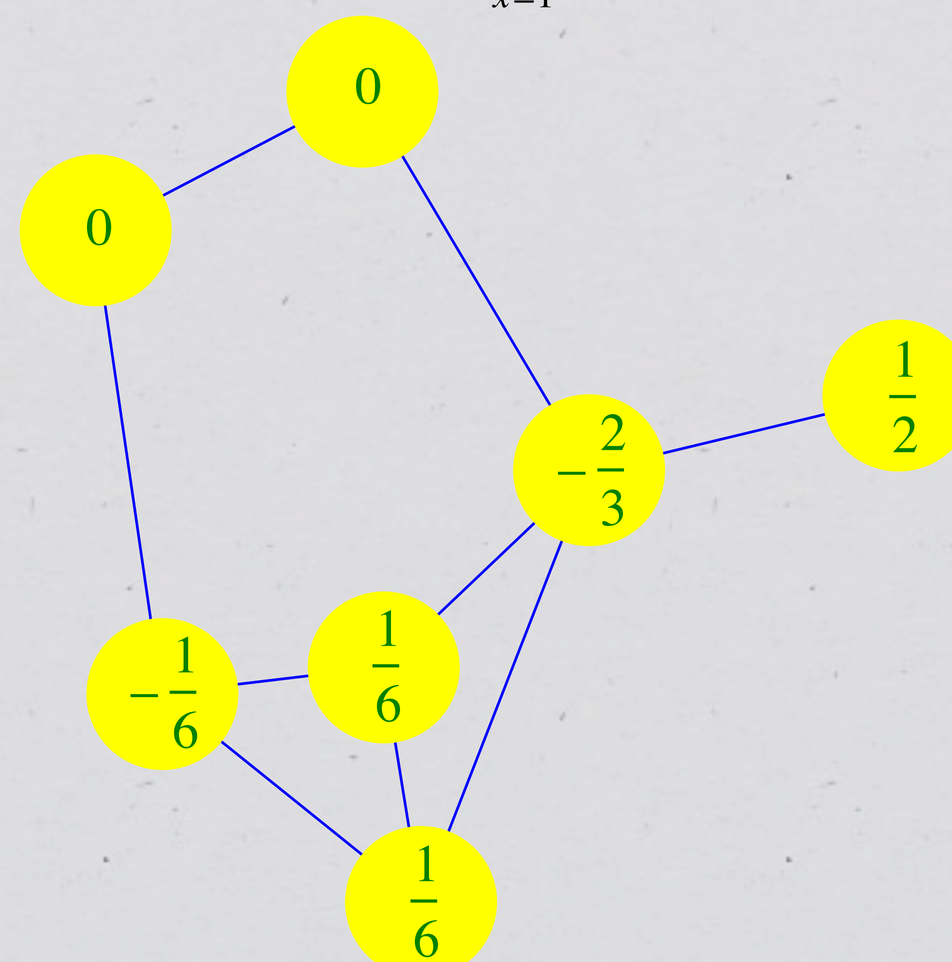
X=4

General Graphs

$$1 = \chi = \sum_{x=1}^{|V|} K[x]$$

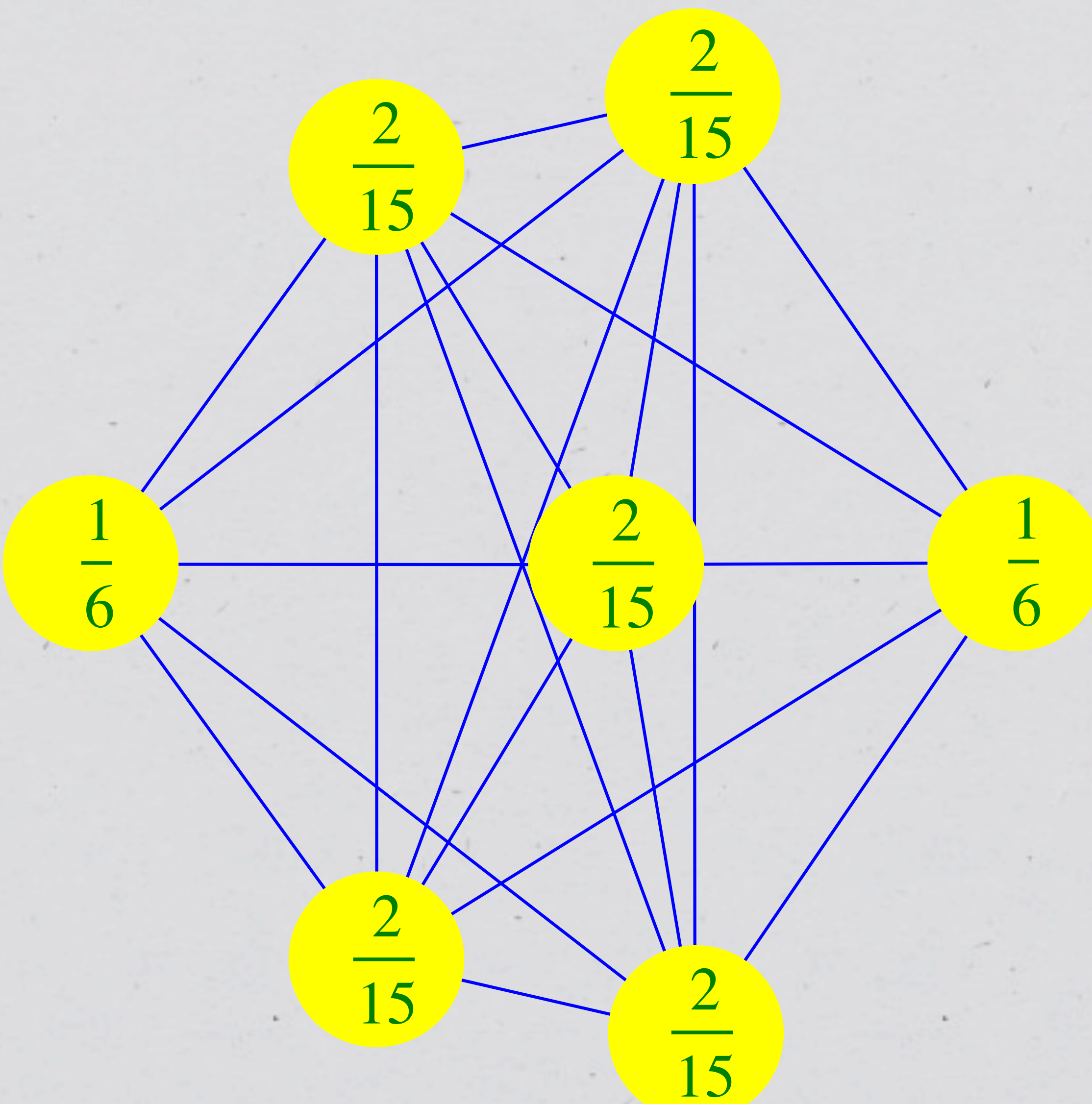


$$0 = \chi = \sum_{x=1}^{|V|} K[x]$$

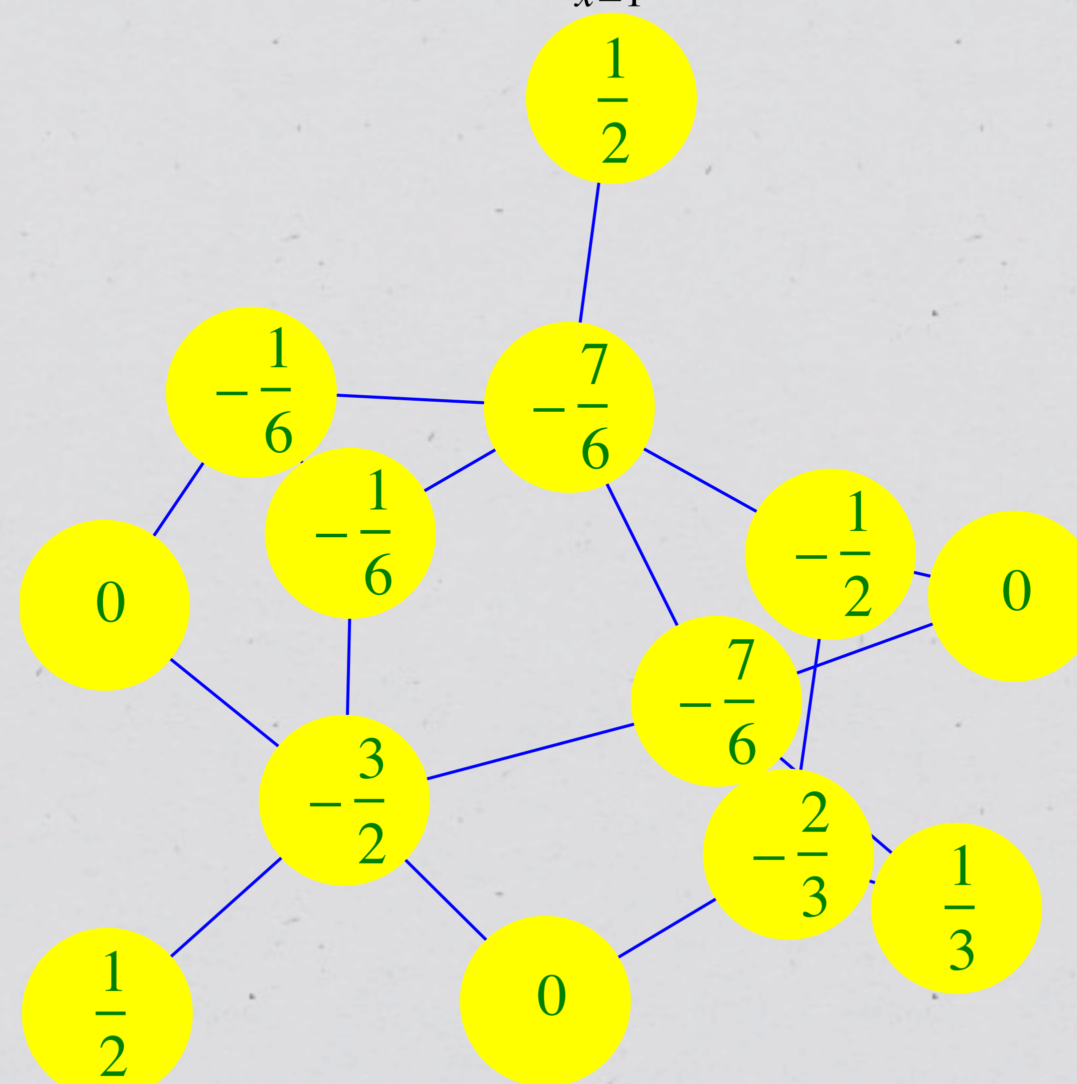


General Graphs

$$1=\chi=\sum_{x=1}^{|V|}K[x]$$



$$-4=\chi=\sum_{x=1}^{|V|}K[x]$$



This is teachable!

Spring 2011: Math E320

Spring 2012: Math E320

Spring 2012: Math Circle

Lesson 1

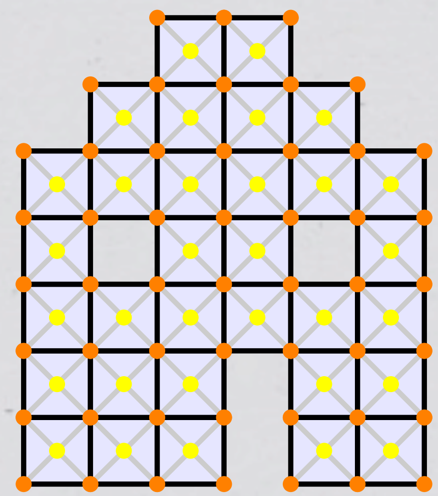
A mathematical theorem

Objective

In this worksheet, we work on a particular result in mathematics. Our task is to understand the theorem, place it into the landscape of mathematics and gain insight why the theorem is true.

Bricks, Windows, Houses and Towns

A unit square is called a **brick**. We draw the diagonals and call their intersection the **center** of the brick. Each brick has **4 corners**. If two bricks have a common corner, they **touch**. Bricks with two common corners **face.** Their intersection is then called a **face**. Two bricks are **connected**, if you can go from one brick to the other by crossing faces between other bricks in the house. A **house** is a finite union of bricks such that any two bricks in the house are connected. A **town** is a finite union of houses. A brick-free region which is enclosed by bricks of the house is called a **window** of the house.

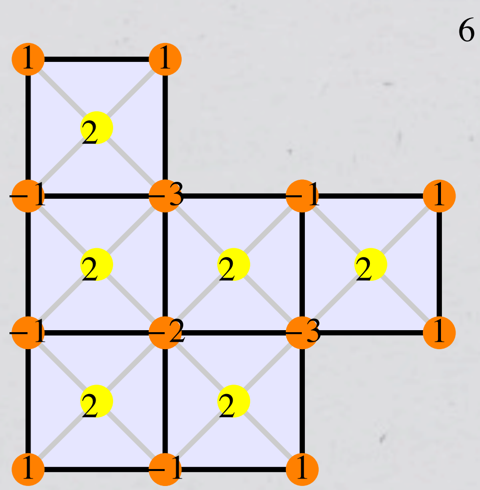


Rules

We assign numbers called **curvatures** to all the nodes of a house. Here are the rules:

- At an intersection of 4 bricks meet, put -2.
- At an edge where 2 bricks meet, put -1.
- At an inner corner, where 3 bricks meet, put -3
- At an outer corner, where only 1 brick is, put 1
- At a center of a brick, put 2.

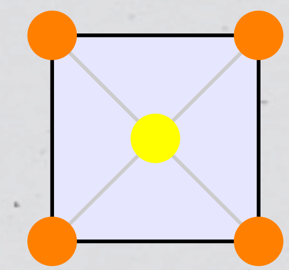
The sum of all these curvatures is called the **total curvature** of the house or town.



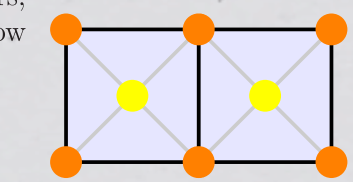
Examples of houses

Lets add up some curvatures:

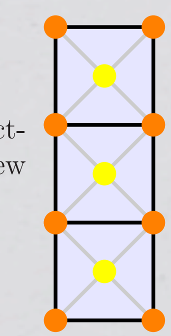
1. For one brick alone, we have 2 in the center and 1 at each corner. This adds up to 6.



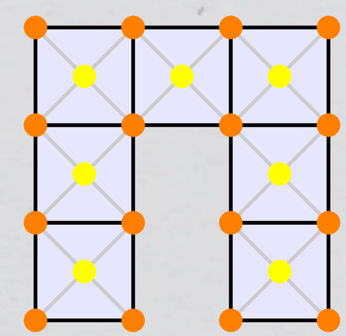
2. If two bricks face each other, we have 2 centers. This is 4. We have 2 flat corners, this adds -2 so that we are down to 2. Now we have 4 corners and this adds up to 6.



3. Assume now, three bricks forming a rectangle of size 3×1. We have added a new center and 2 flat corners. We still have 6.



4. Eight bricks form an arch.



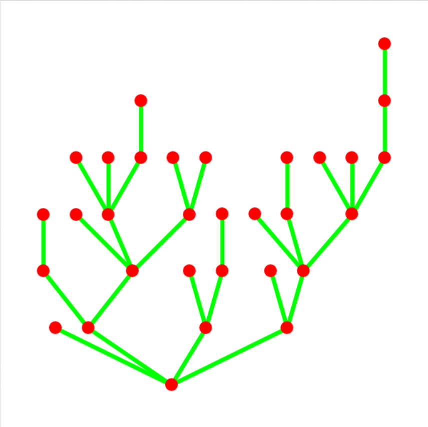
A mathematical theorem

Objective

In this worksheet we want to understand a statement in mathematics, understand the result and see why the result is true.

Trees and Forests

A **graph** is a pair (V, E), where V is a finite set of and where E is a set of vertices, each connecting two different vertices. The set of edges is denoted by E. We assume that edges connect different nodes only once that that no multiple connections appear.
A **closed path** in a graph is a sequence of three or more different vertices, where neighboring vertices are connected by edges and such that at the end we reach the same point again. A graph is called **forest** if it contains no closed path. Part of a graph, where we can go from any vertex to any other is called a **connected component**. A connected component of a forest is called a **tree**.



Trees appear often in applications. Which of the following are trees or can be trees?

- A genealogy map
- A hierarchy in an organization
- A directory structure in a computer
- A protein
- A water molecule
- A DNA string
- Relations between friends
- A computer network
- The internet
- The freeway network

Lesson 2

Rules

We assign now numbers called **curvatures** to every node of the tree. We use the following rule:

- At every vertex with only one neighbor (leafs or trunc) put 1/2.
- At a limb, where two branches come together we put 0.
- At a crotch, where $d = 3$ or more branches meet put 1-d/2.

When summing all these curvatures we call this the **total curvature** of the tree. We can summarize this rule as follows:

The curvature of a vertex v is $K(v) = 1 - d(v)/2$ where $d(v)$ is the number of neighbors of v.

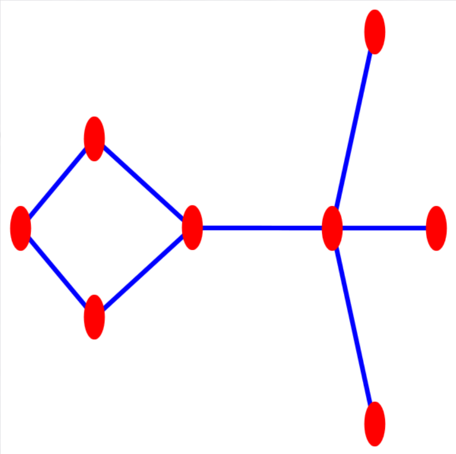
A theorem about trees

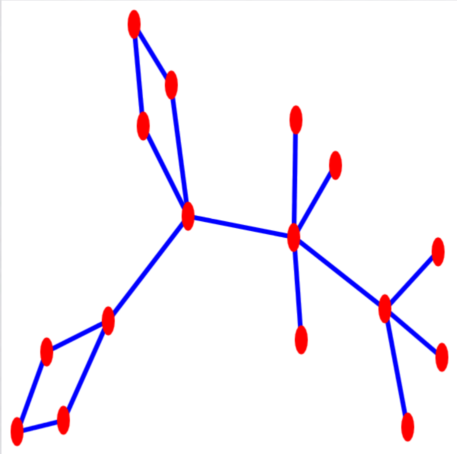
Theorem:
For any forest G, the total curvature is equal to the number of trees.

- Experiment with different trees and forests.
- Start with very simple cases, like graph with 2 or 3 vertices.
- Attempt a proof.

A theorem about gardens

A graph in which no triangles exist is called a **garden**. A connected component of a garden is called a **plant**. Unlike for trees, we can now have closed loops of length larger than 3, which we call **flowers**.





Theorem:
For any garden, the total curvature is the number of plants minus the number of flowers.

- Experiment with different gardens and plants.
- Start with very simple cases, like a plant which has only one flower and no stem.
- Add a stem and see what happens.
- Can you find a proof?

Problems

- Meyers-Bonnet: there are only finitely many graphs of positive sectional curvature density. How many two dimensional graphs are there? Can they all be deformed to become a sphere?

- Hopf problem: does positive sectional curvature of a four dimensional geometric graph imply positive Euler characteristic?

- Find a way to use the graph theoretical theorem to prove the classical Gauss-Bonnet-Chern Theorem.

An other quote:

The passing of mathematics on to subsequent generations is essential for the future, and this is only possible if every generation of mathematicians understands what they are doing and distills it out in such a form that it is easily understood by the next generation.

Michael Atiyah, Notices AMS February 2005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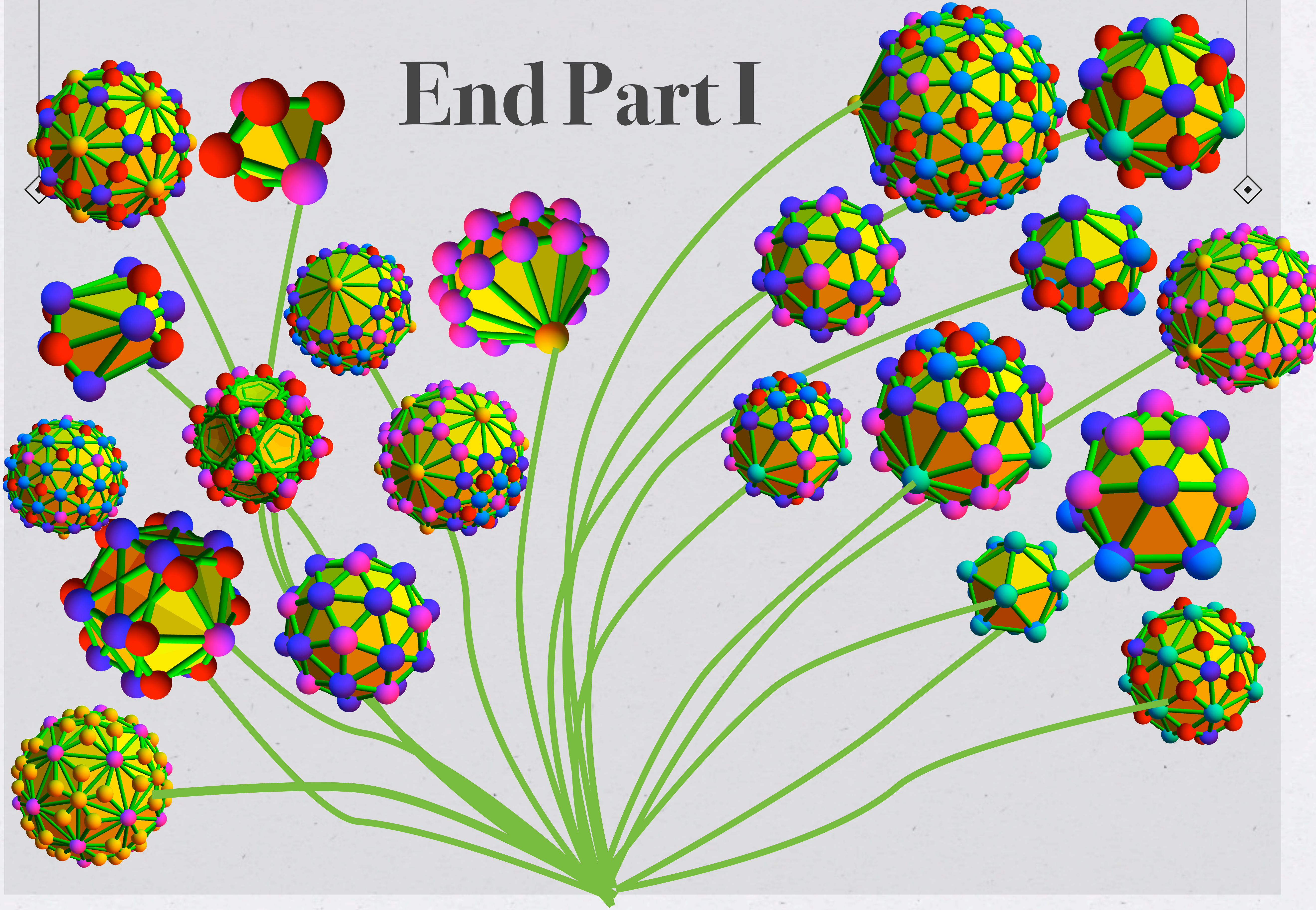
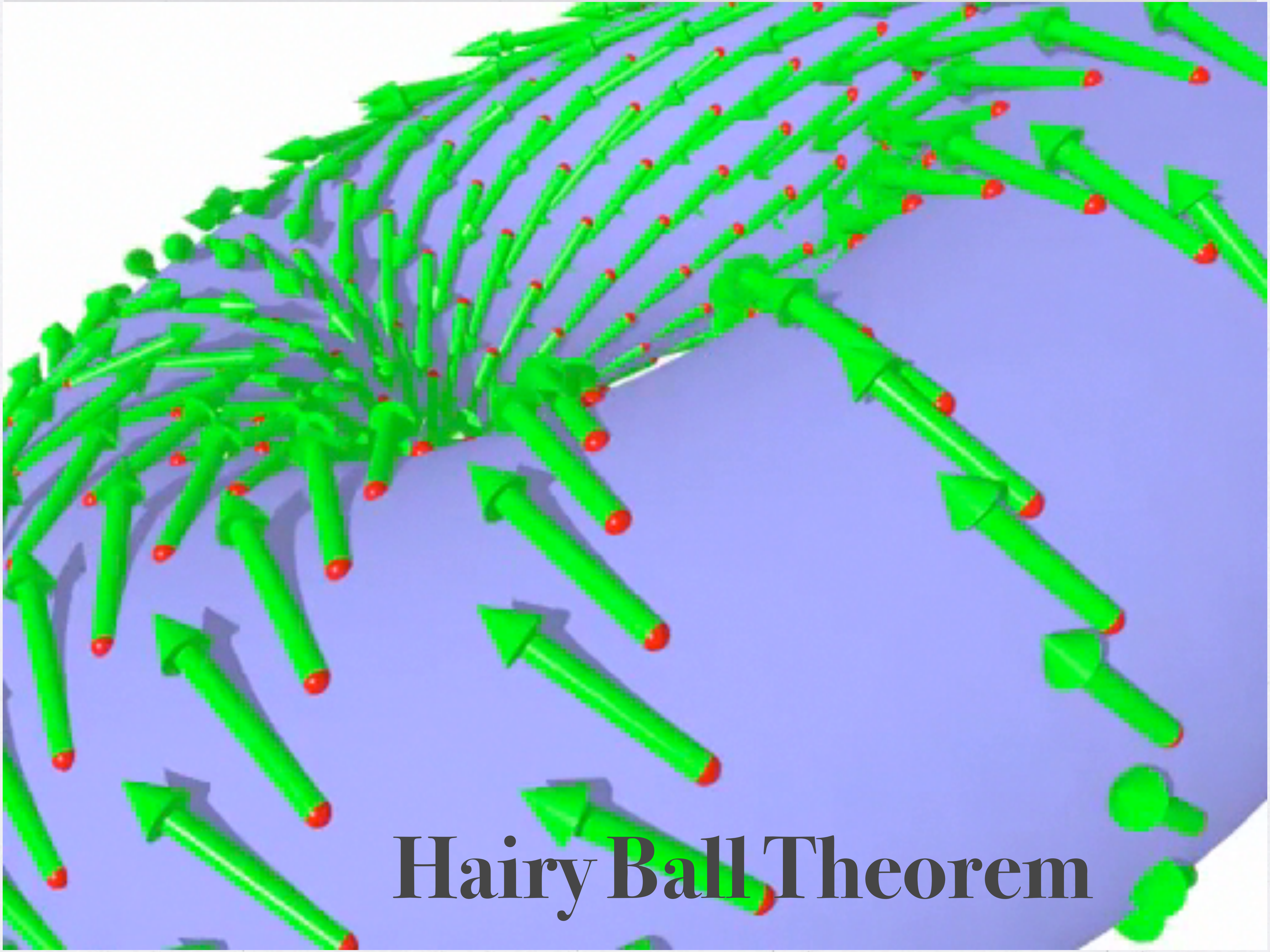
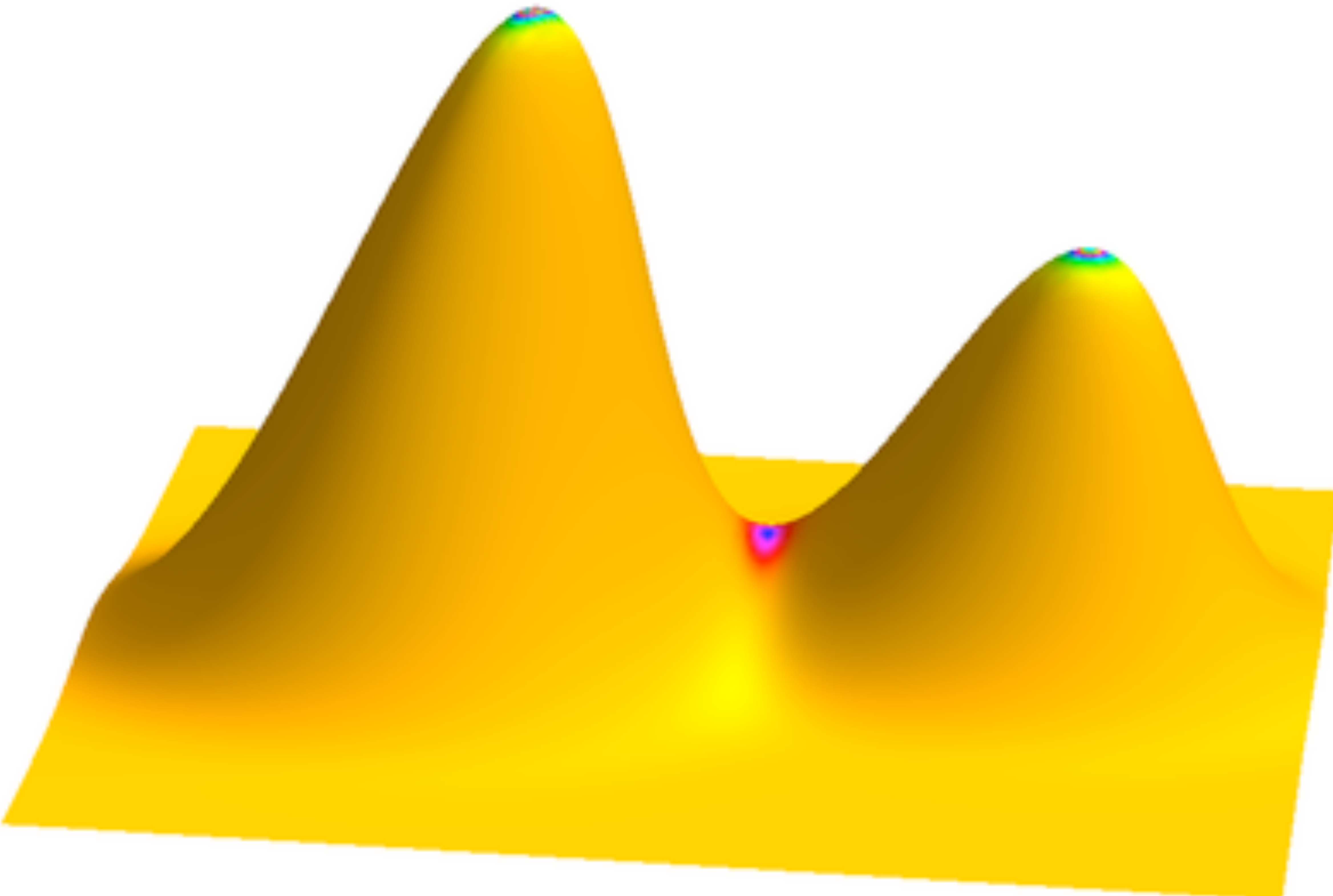


Part II: Chickens and Islands

Hairy Ball Theorem





Classical Poincaré-Hopf

The sum of indices of equilibria of a vector field on a manifold M is the Euler characteristic of M

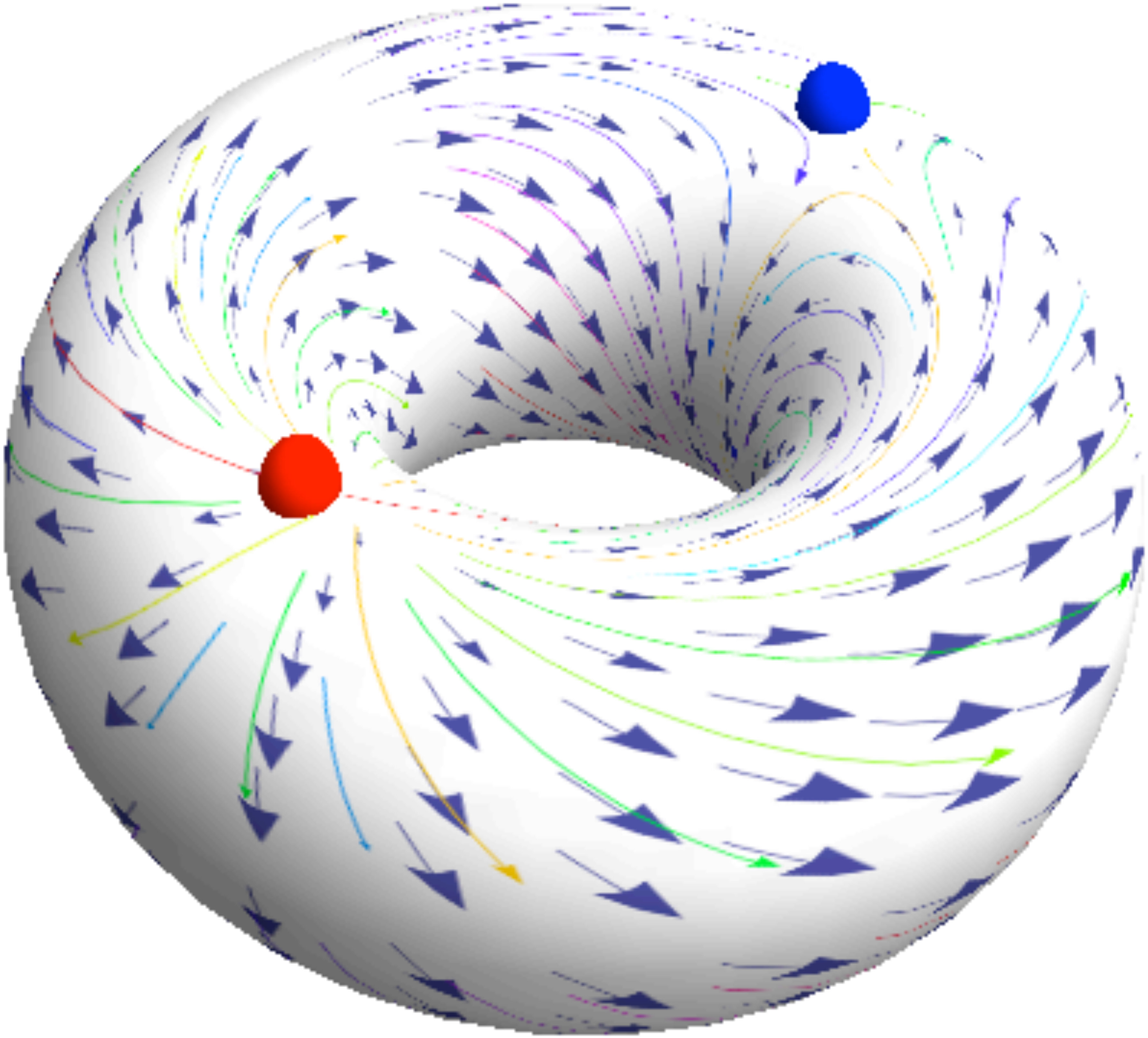
Poincaré-Hopf

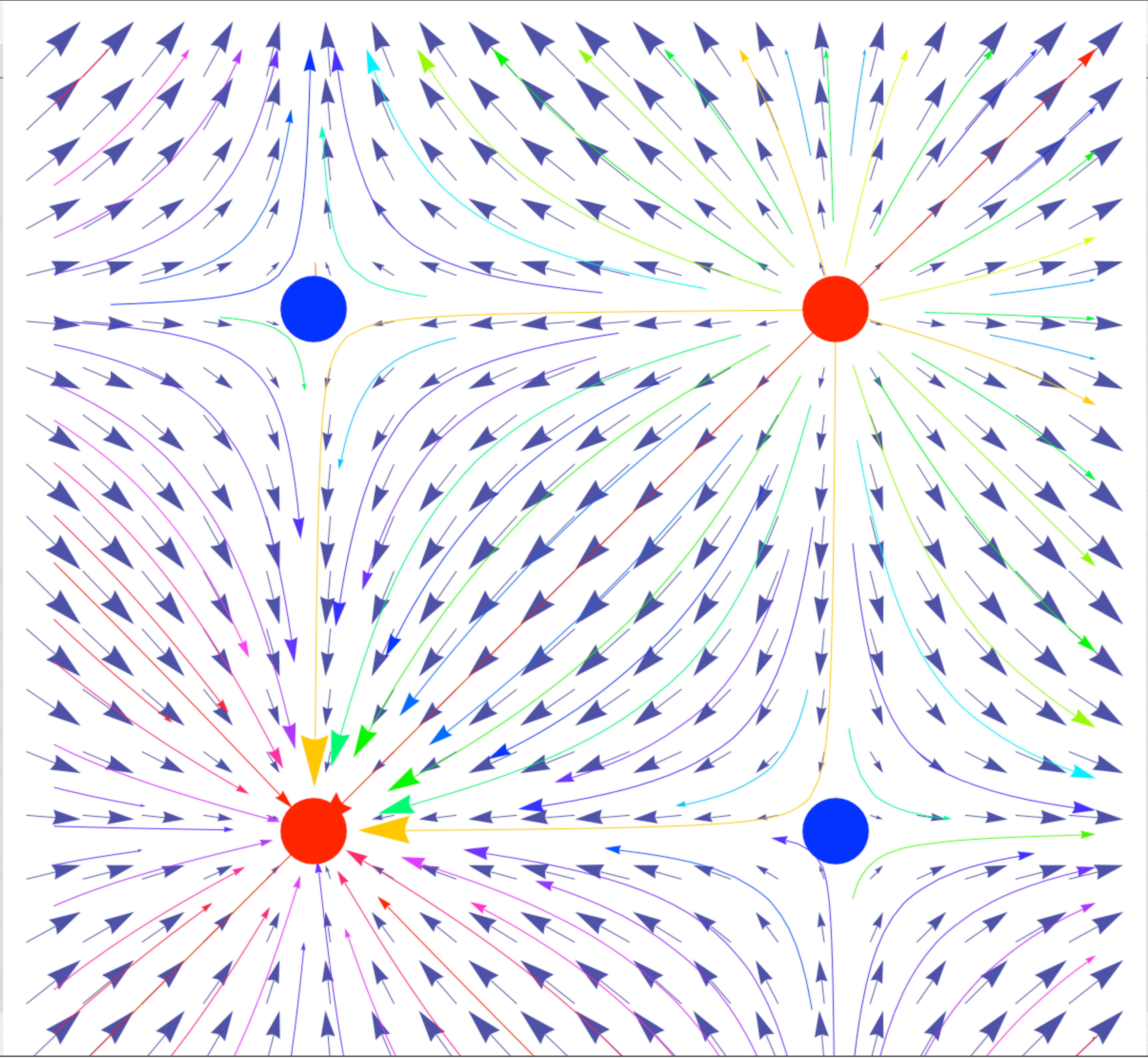
Henry Poincaré

1854-1912

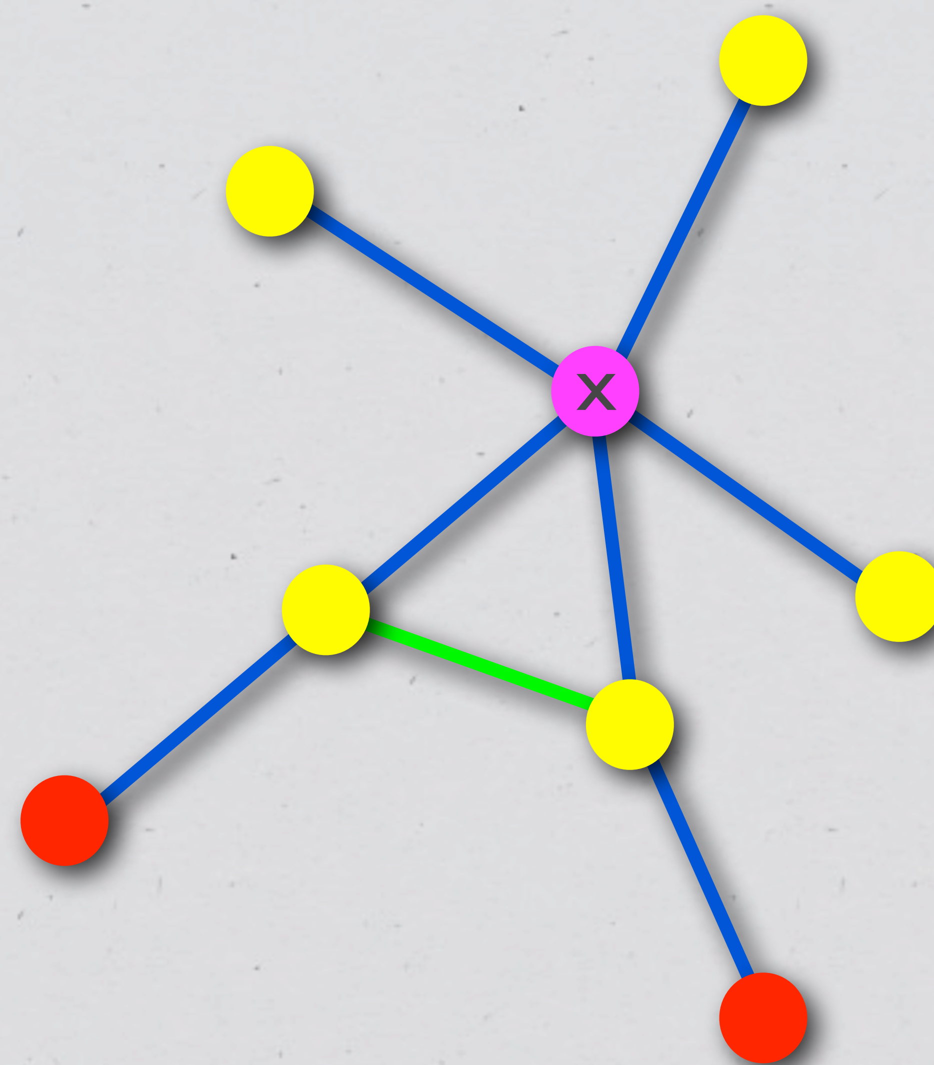
Heinz Hopf

1894-1971

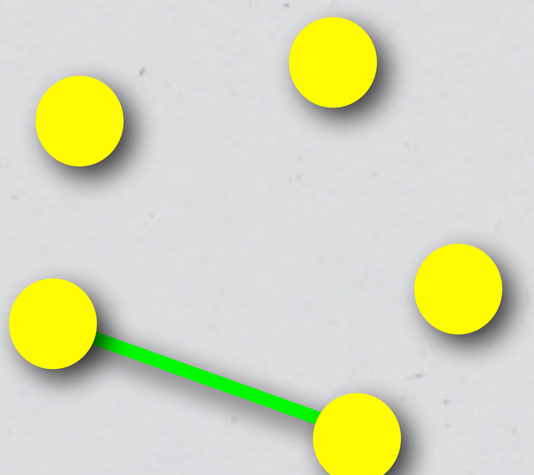




Sphere

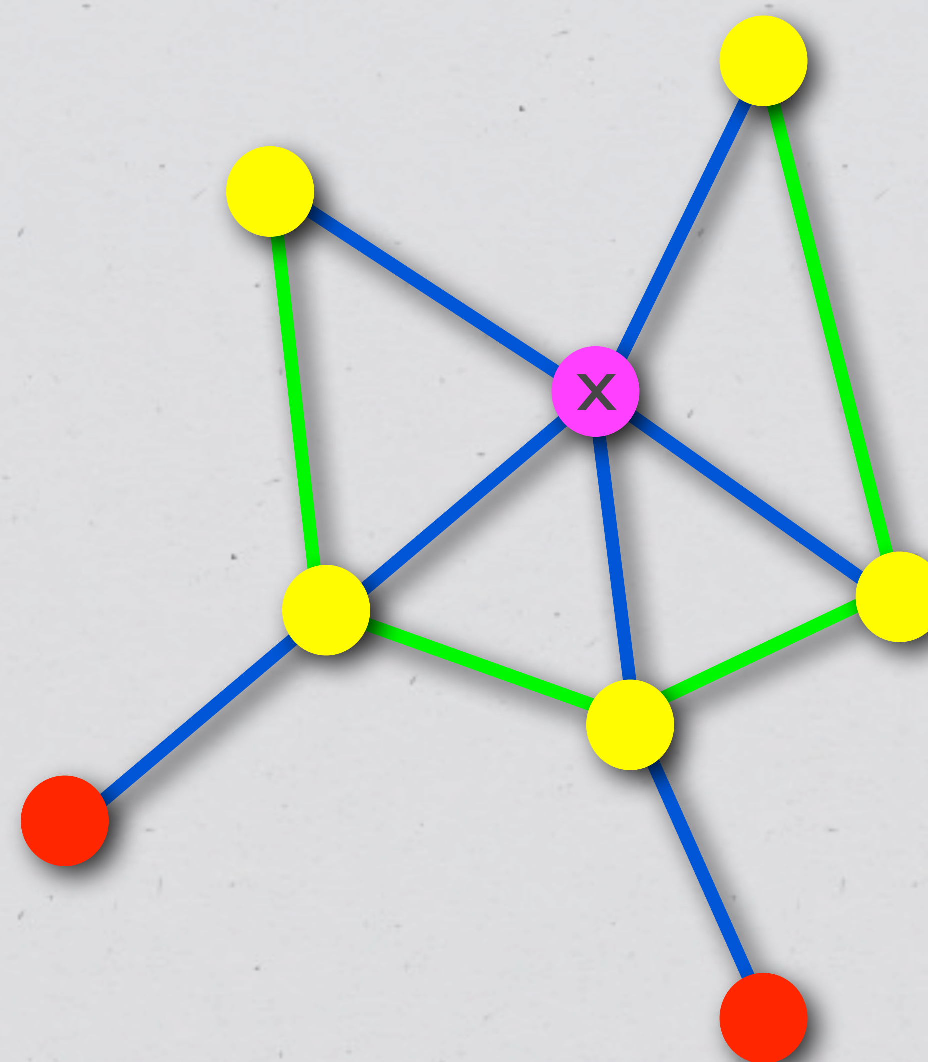


S(x) =



Injective Function

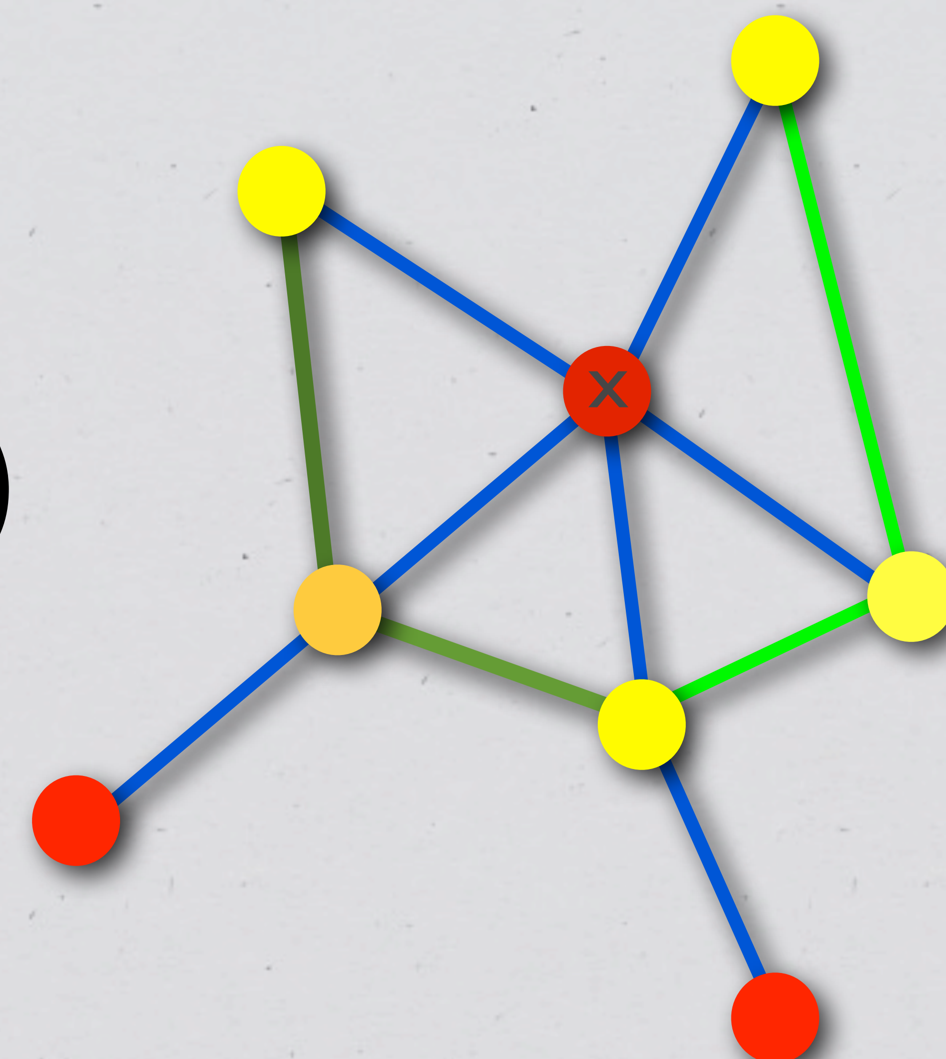
$f{:}V \to R$ injective



$$S^{-}(x) = \{\, y \,\varepsilon\, S(x) \mid f(y) < f(x) \,\}$$

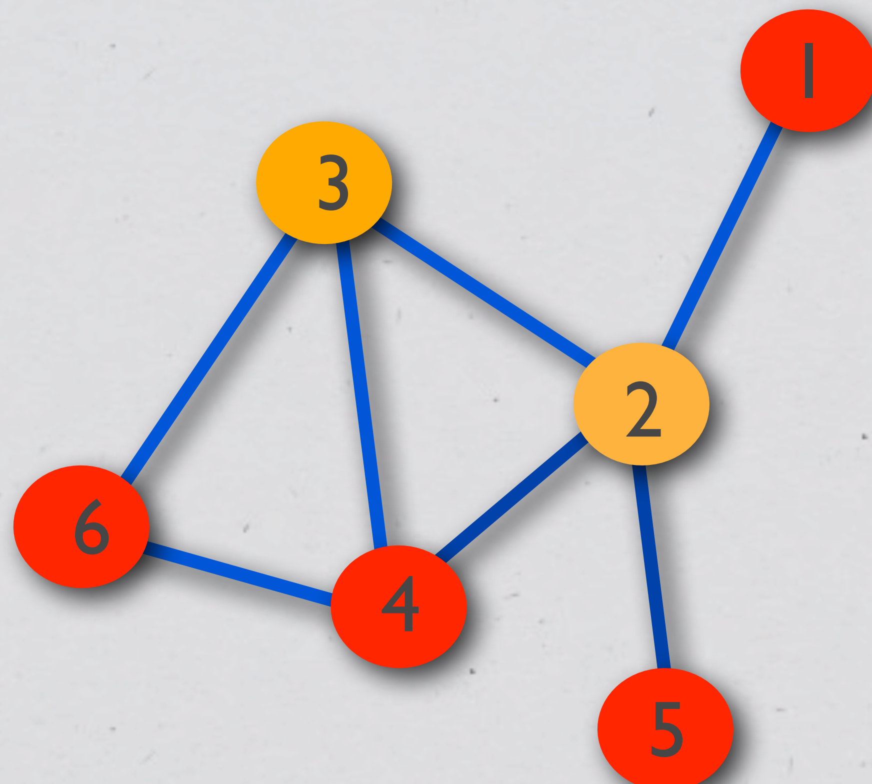
Index

$$i_f(x) = 1 - \chi(S^-(x))$$



$$S^-(x) = \{ y \in S(x) \mid f(y) < f(x) \}$$

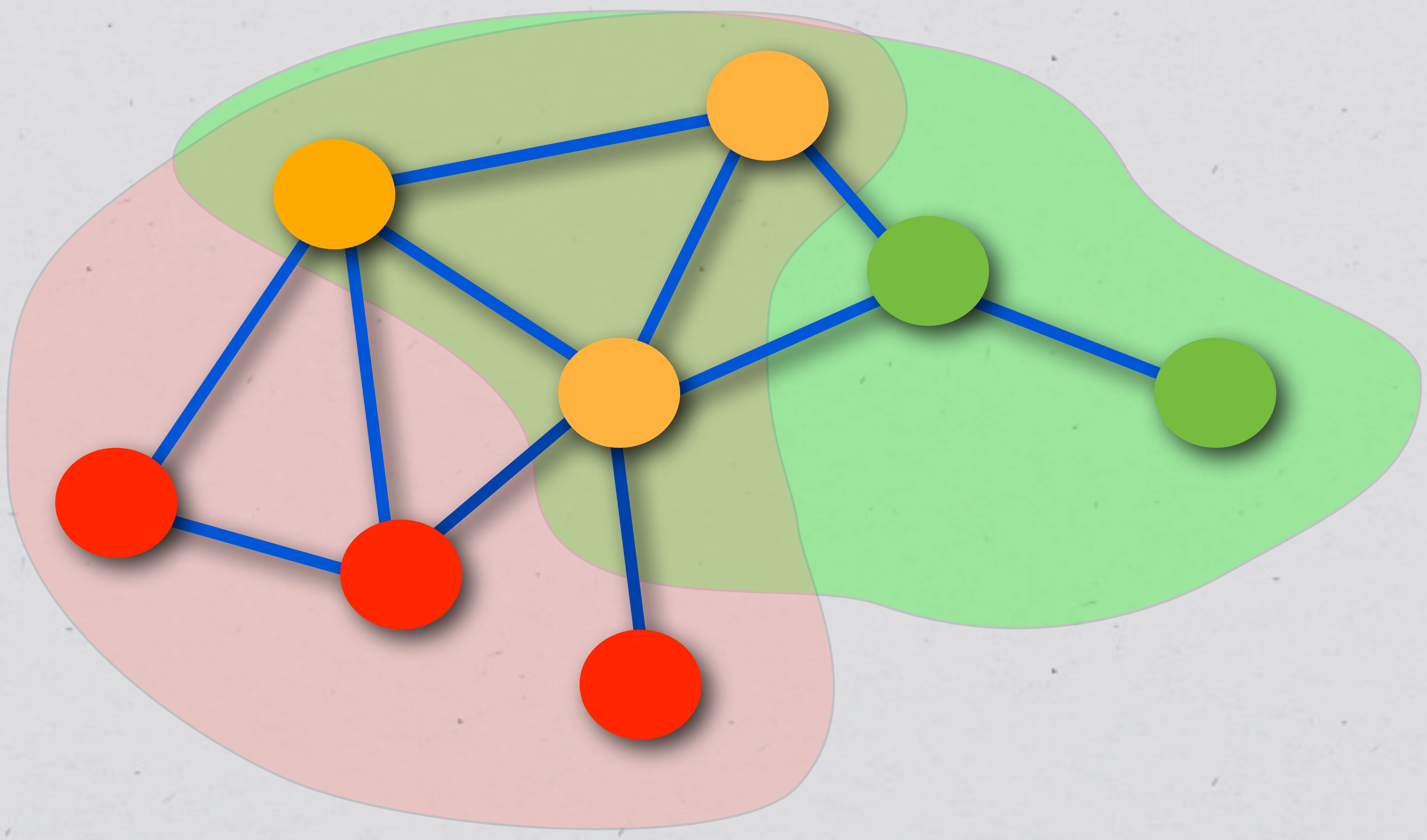
Poincaré-Hopf



$$\sum_{x \in V} i_f(x) = \chi(G)$$

Proof

Lemma: $\chi(G \cup H) = \chi(G) + \chi(H) - \chi(G \cap H)$

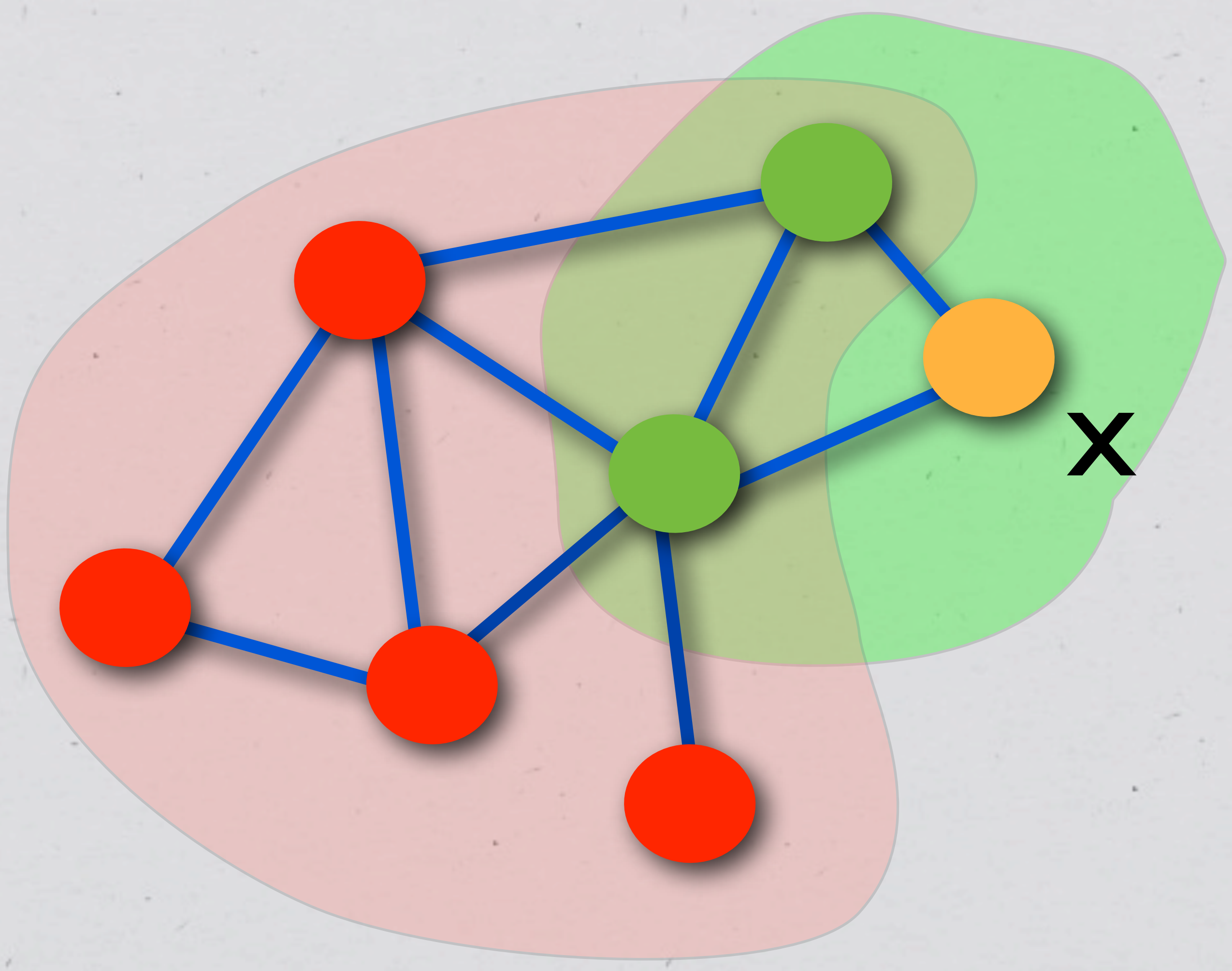


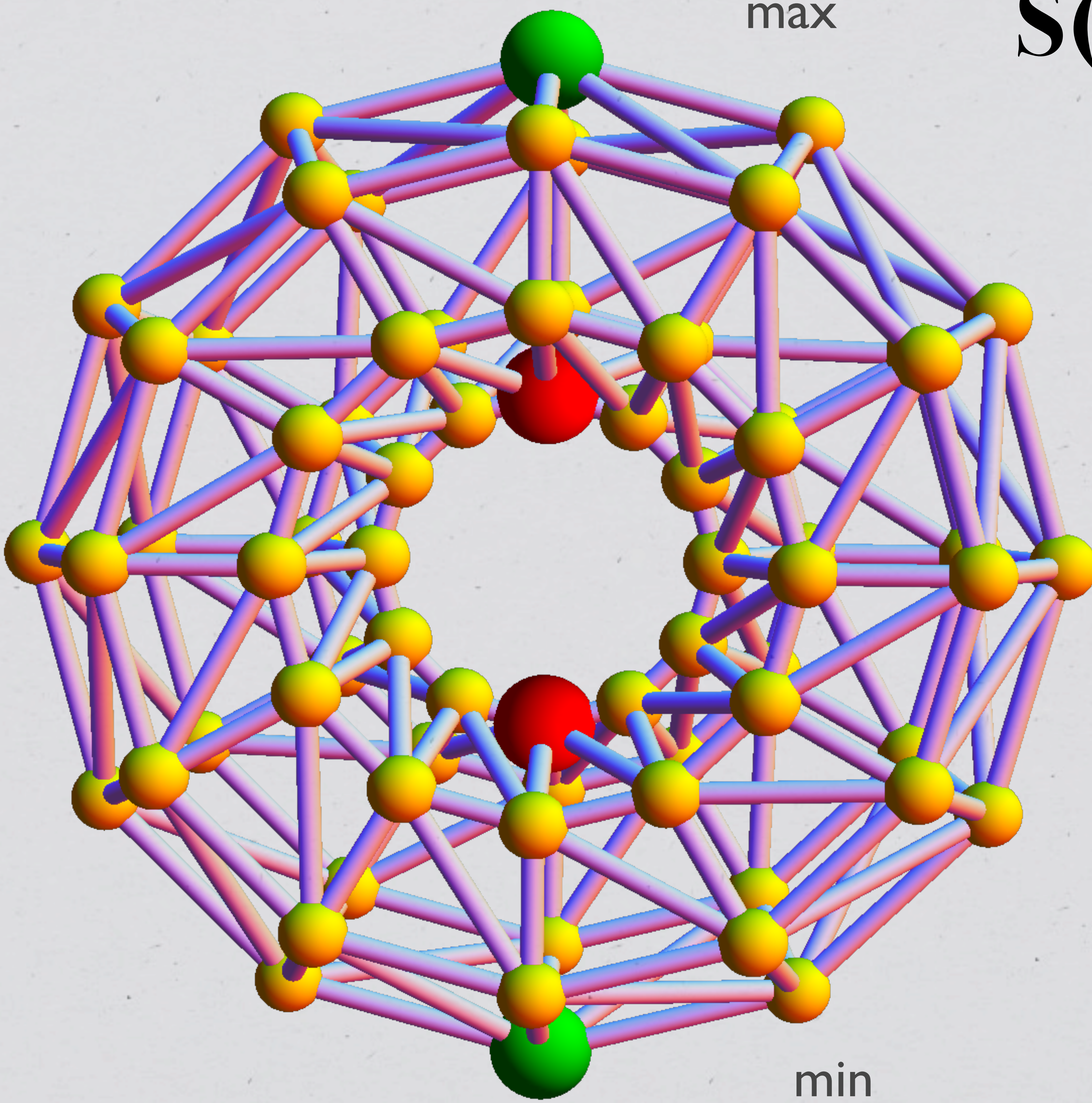
Proof

Build up the graph:

$$1 - \chi(S(x))$$

$$\chi(G \cup B(x)) = \chi(G) + \chi(B(x)) - \chi(S(x))$$





$$S^-(x)=C_8$$

$$1-\chi(S^-(x)) = 1$$

$$S(x)=I_3+I_3$$

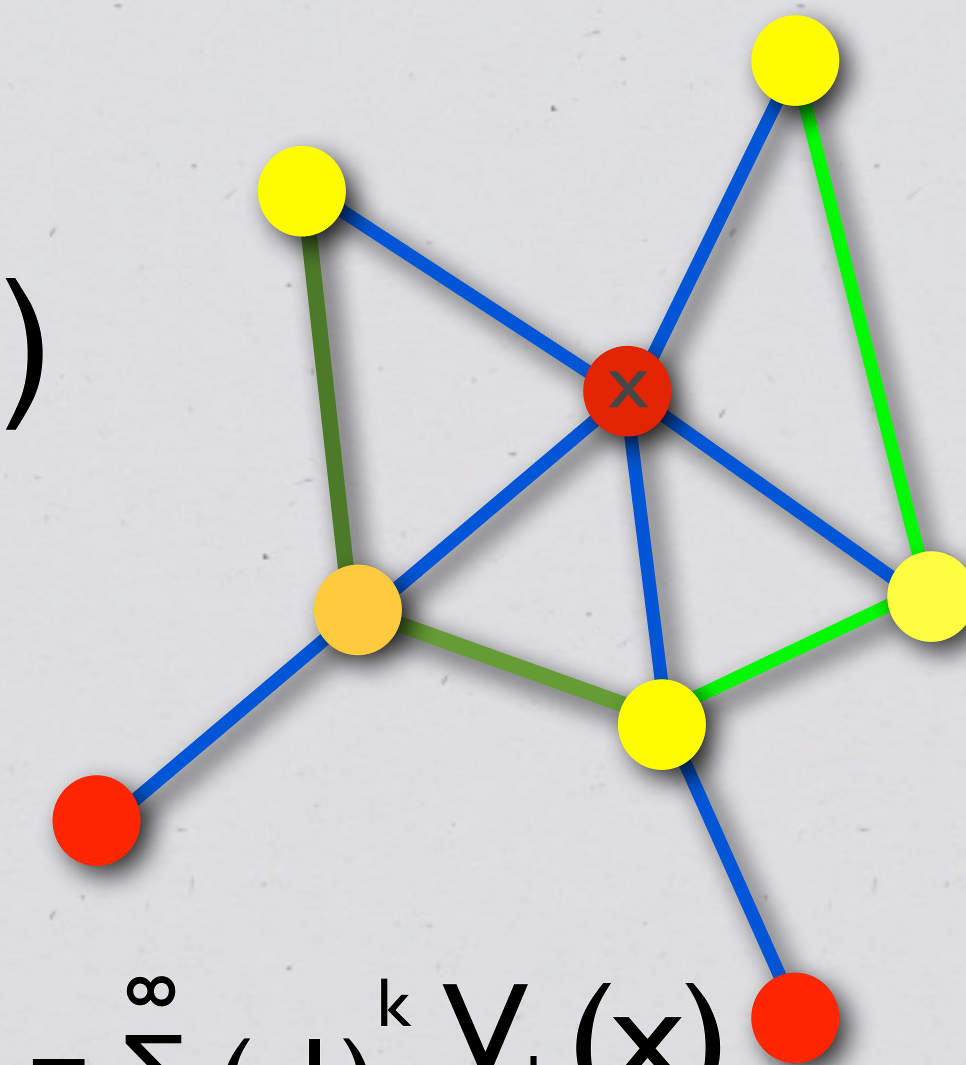
$$1-\chi(S^-(x)) = -1$$

$$1-\chi(S^-(x)) = 1$$

$$S^-(x)=\{\}$$

Index and Curvature

$$i_f(x) = 1 - \chi(S^-(x))$$



$$K(x) = \frac{V_{-1}}{1} - \frac{V_0}{2} + \frac{V_1}{3} \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{V_k(x)}{k+1}$$

Link

Links Gauss-Bonnet with Poincare Hopf

$$\sum_{x \in V} i_f(x) = \chi(G)$$

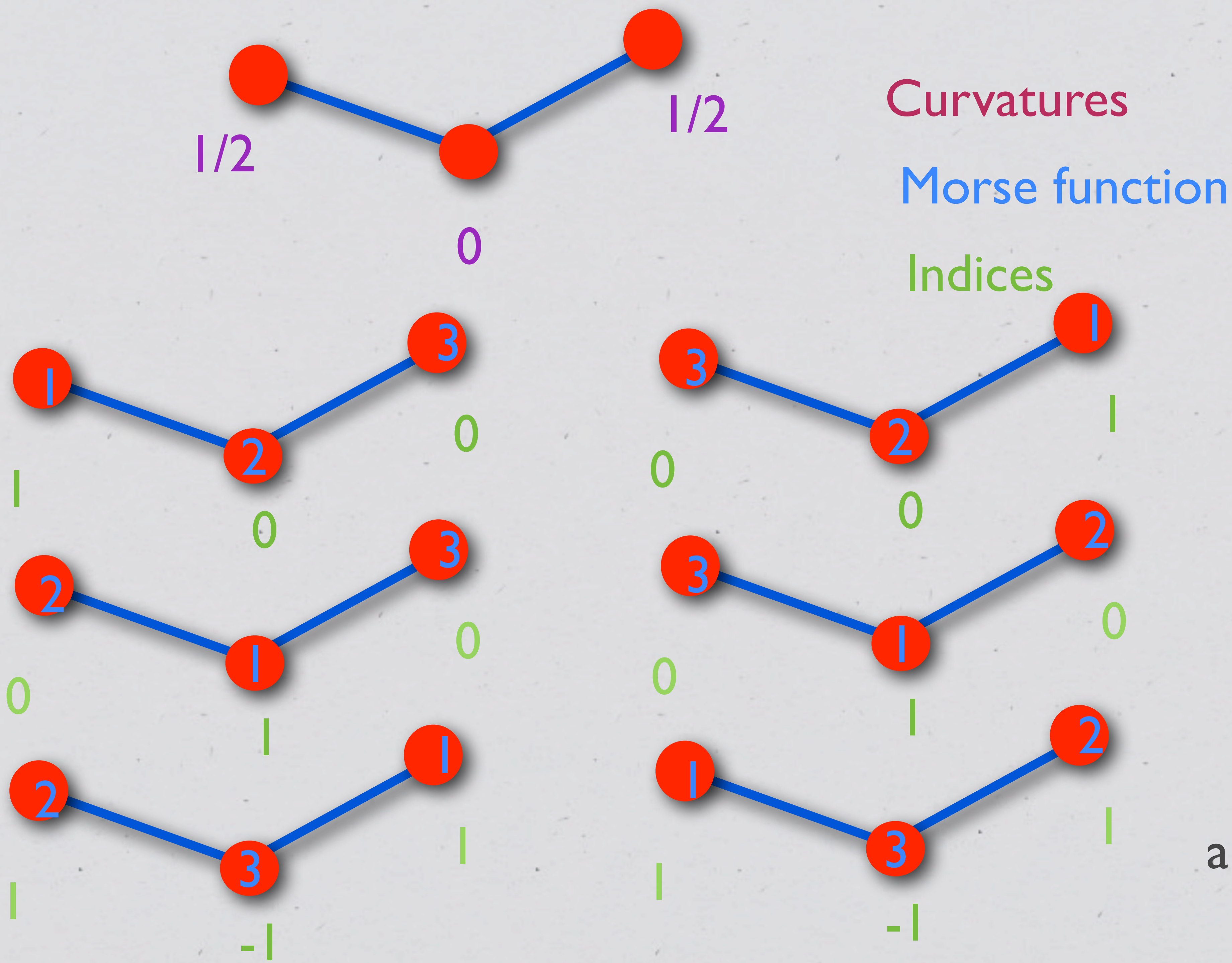
Poincaré-Hopf

Expectation

$$\sum_{x \varepsilon V} K(x) = \chi(G)$$

Gauss-Bonnet

Example



We could

integrate over

the cube of all

Morse functions.

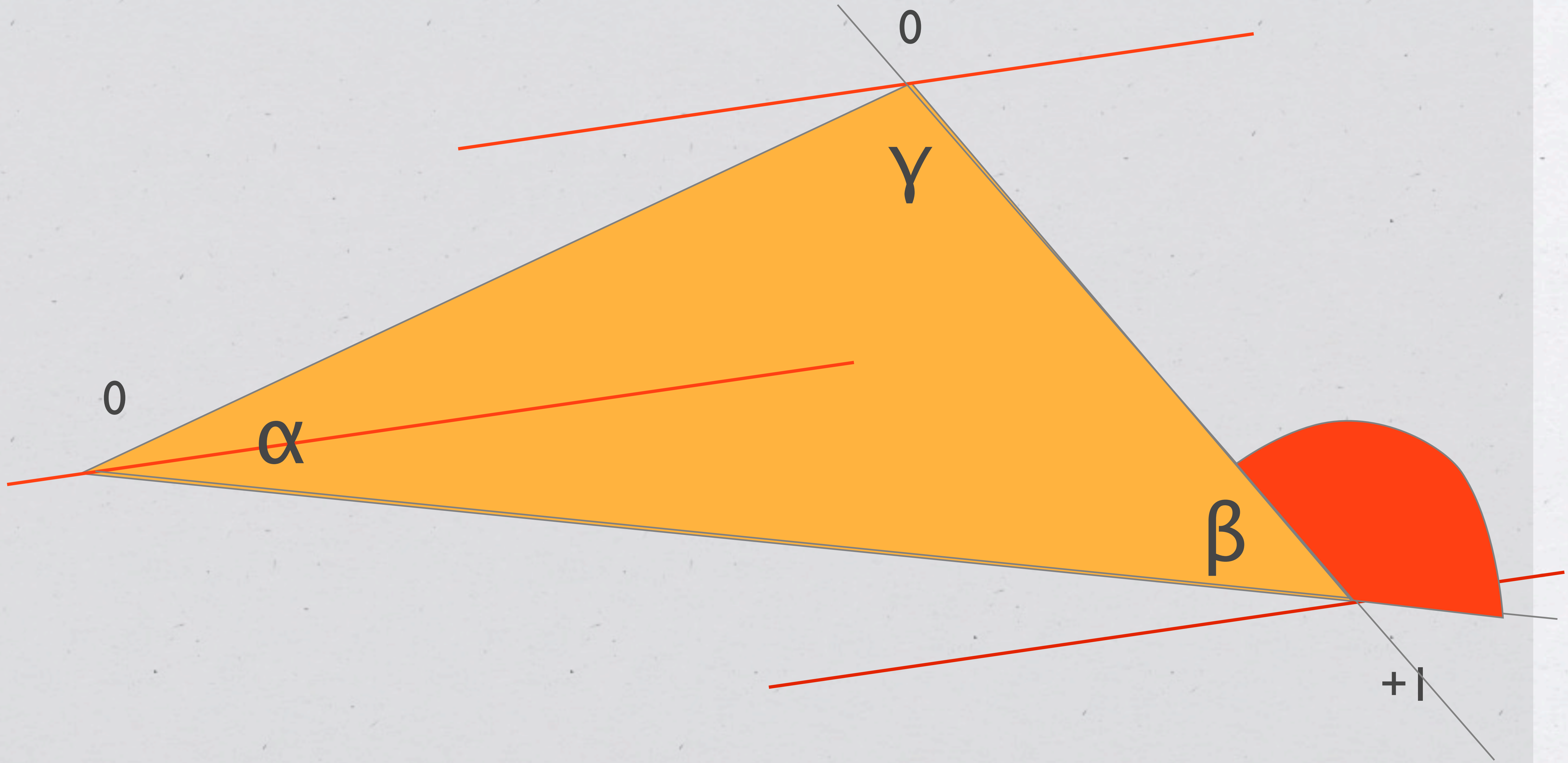
Easier:

average over

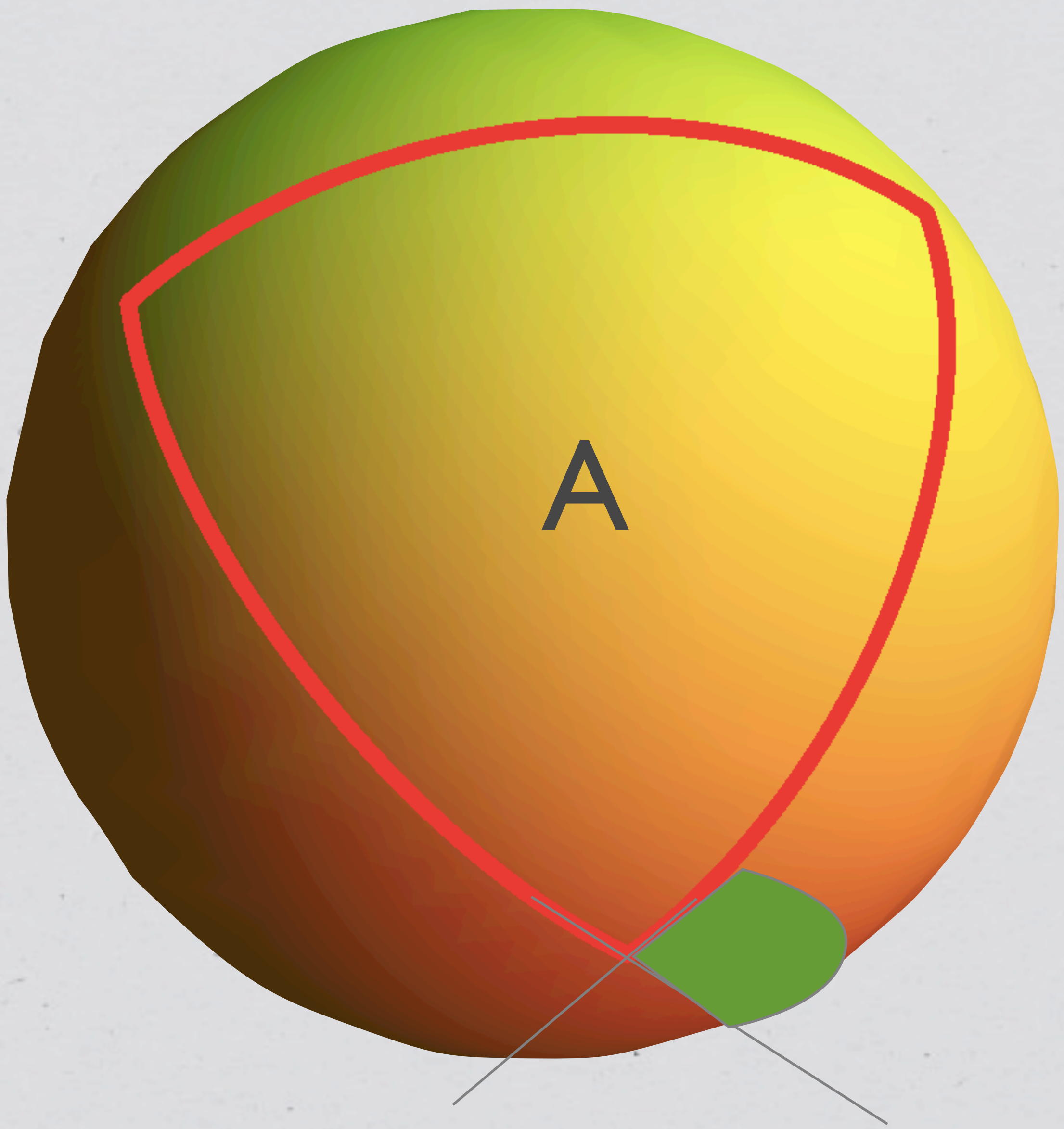
all Morse functions
having 3 values.

Sum of Angles in Triangle

curvature K=(π -α)/(2π)   is the index expectation
summing up curvature is Euler characteristic



Harriot

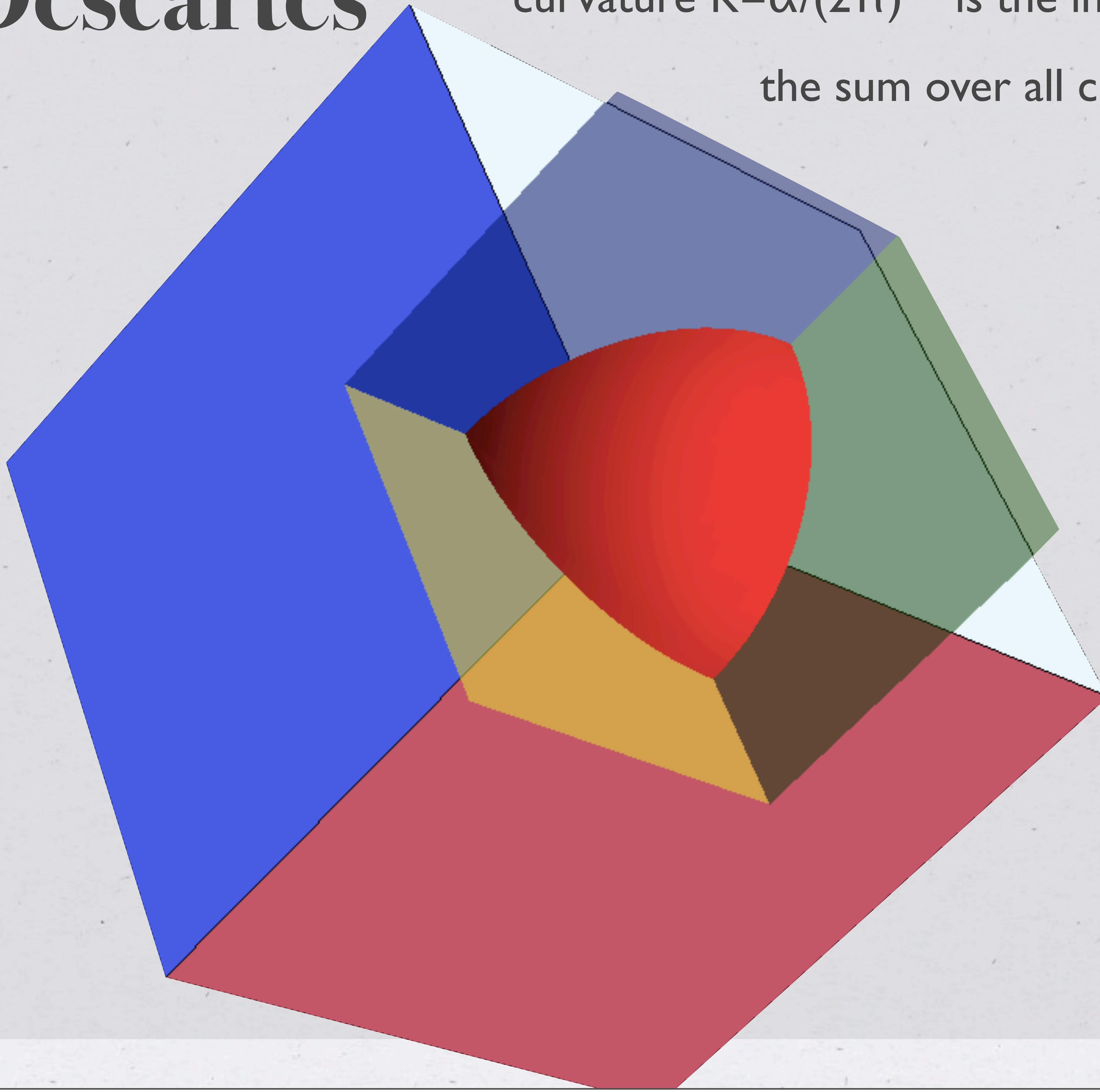


$$\frac{A+E}{2\pi} = \chi(G)$$

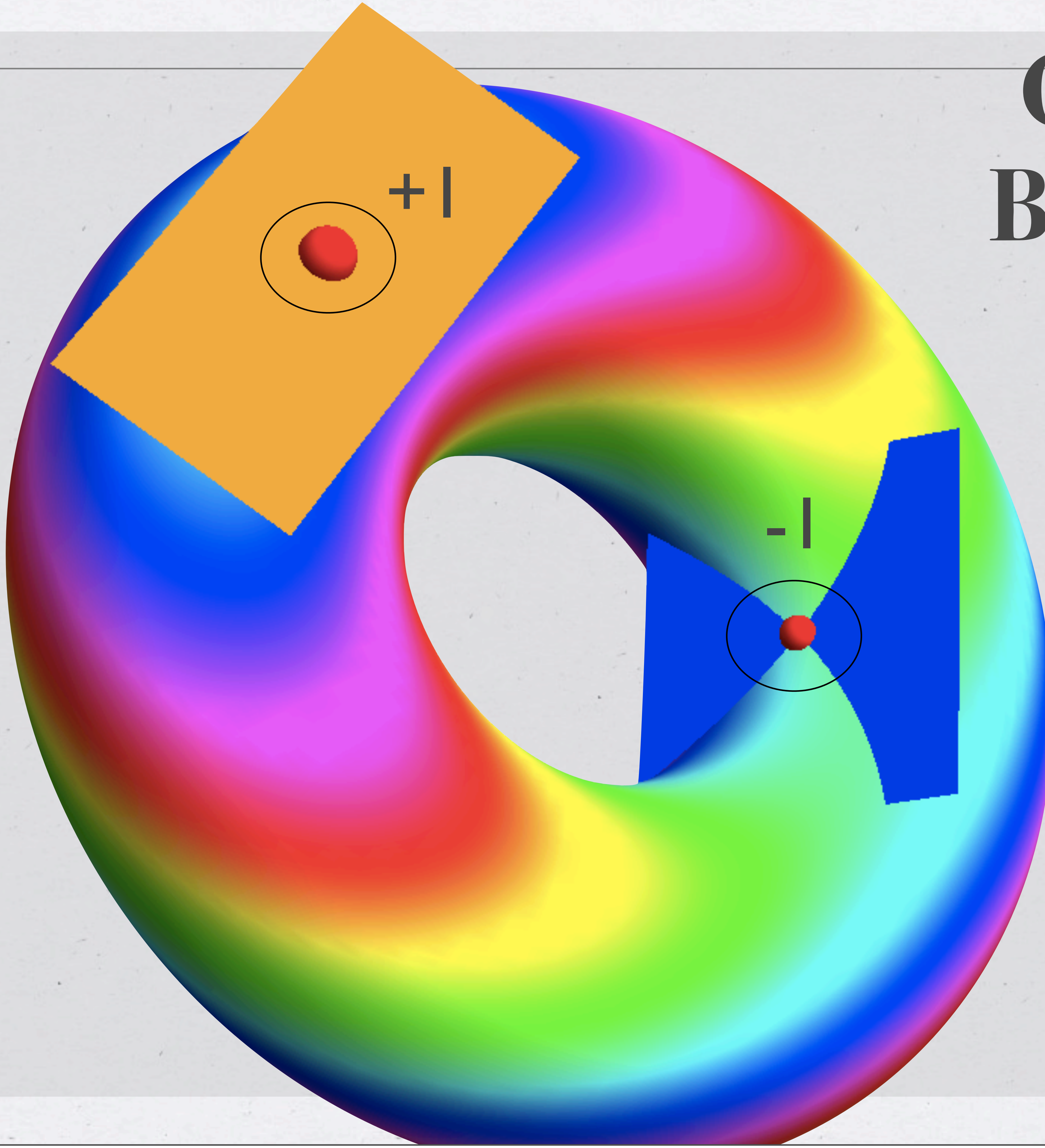
Descartes

curvature $K=\alpha/(2\pi)$ is the index expectation

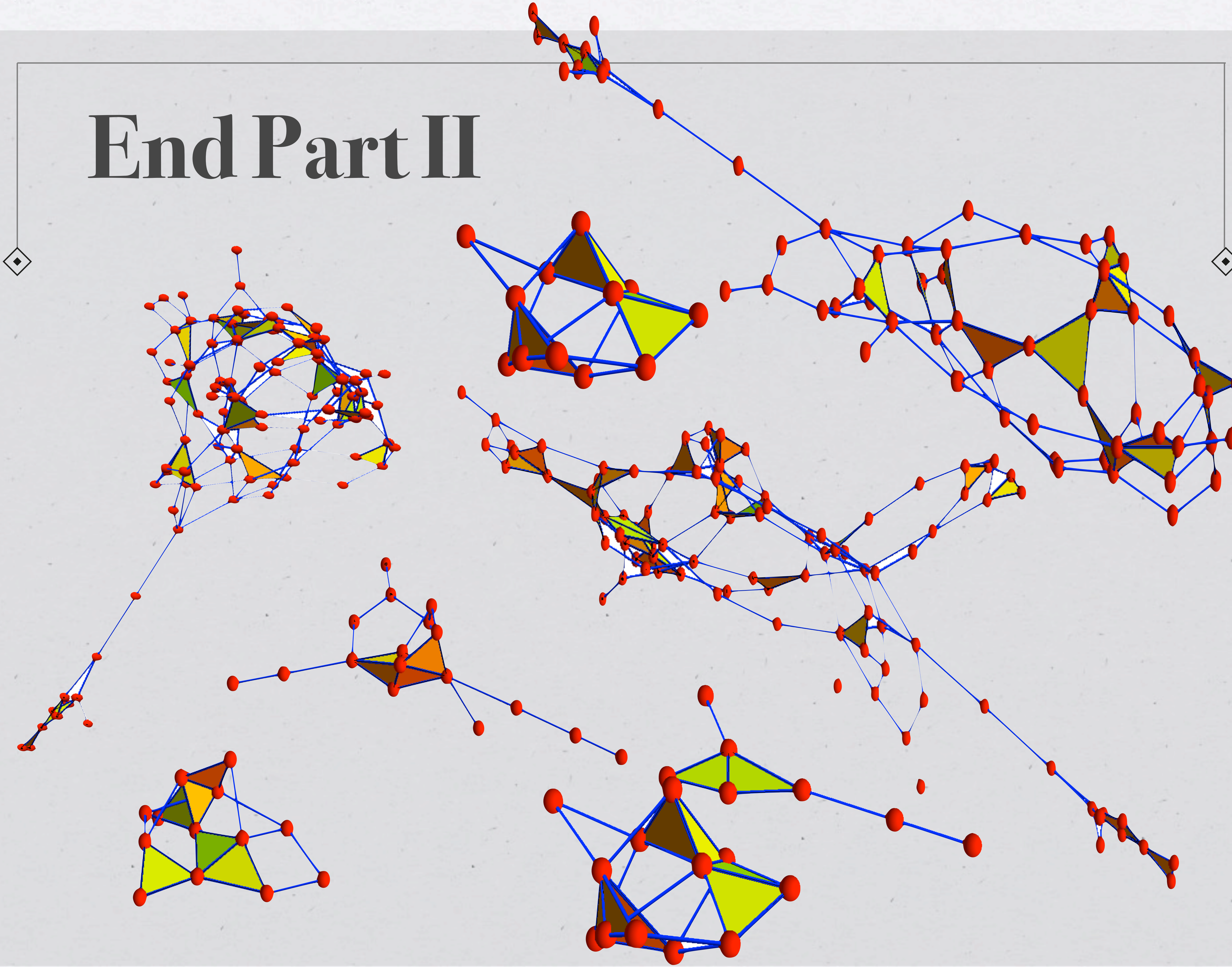
the sum over all curvatures=$\chi(G)$



Gauss Bonnet



End Part II



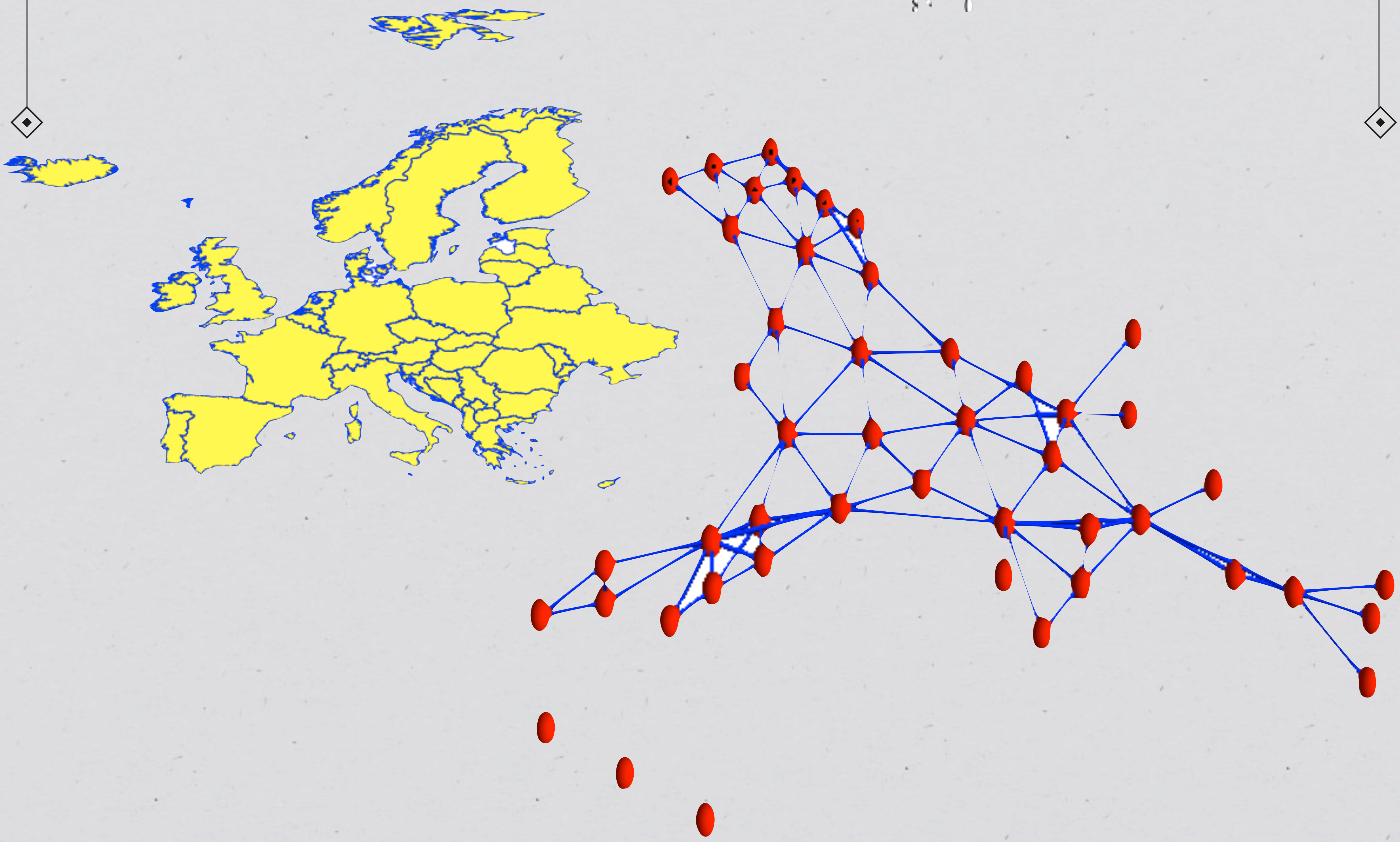
Part III: Problems, Problems ...

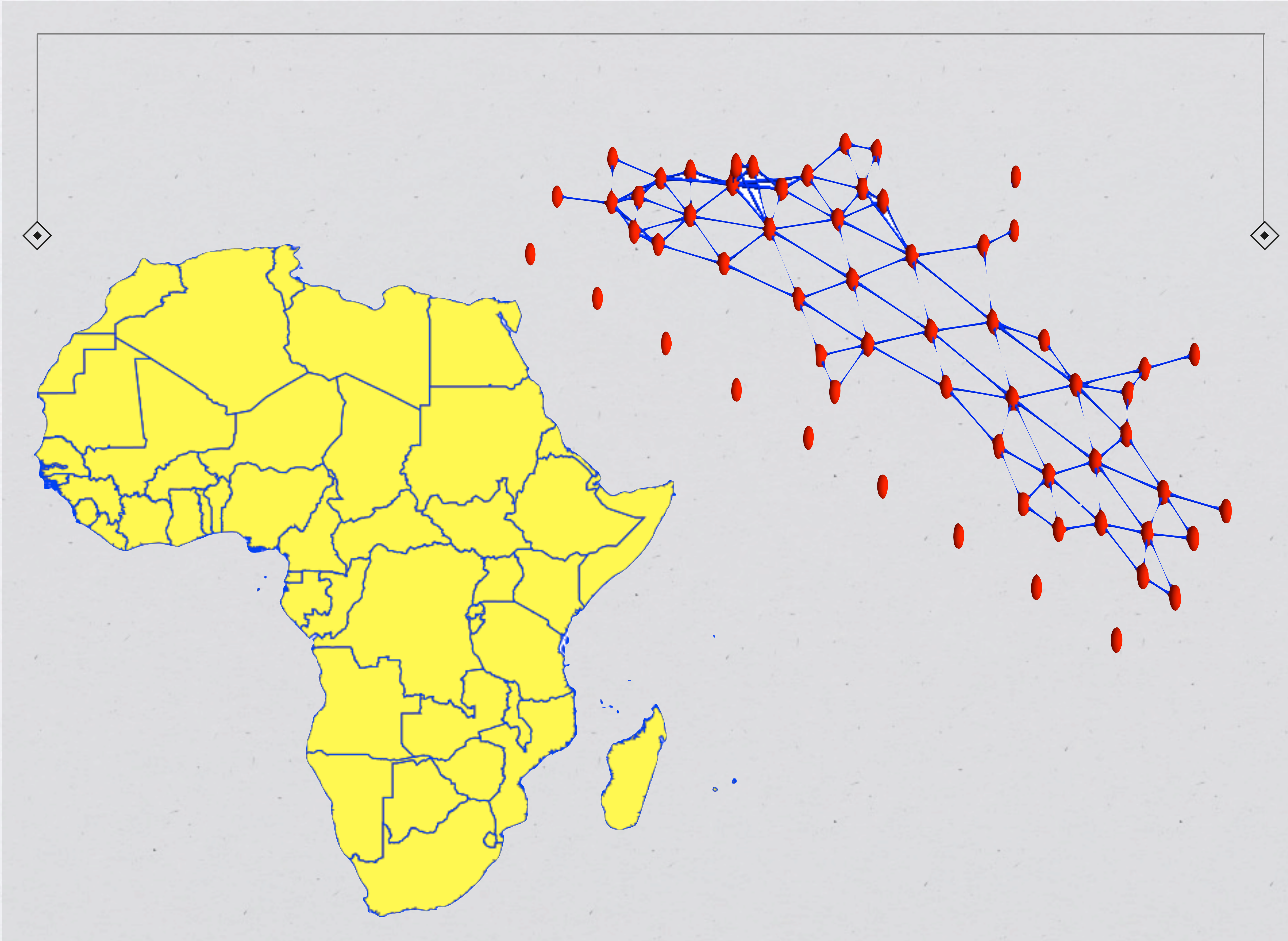
1) Curvature

A 4 dimensional positive curvature graph has positive Euler characteristic.

2) Coloring

Maximal chromatic number of a d-dimensional graph is d+2.





3) Complexity

Find the Complexity of computing the Euler characteristic of a graph of order n.

4) Variational

Find the minimal Euler characteristic of a graph of order n.

5) Simply connected

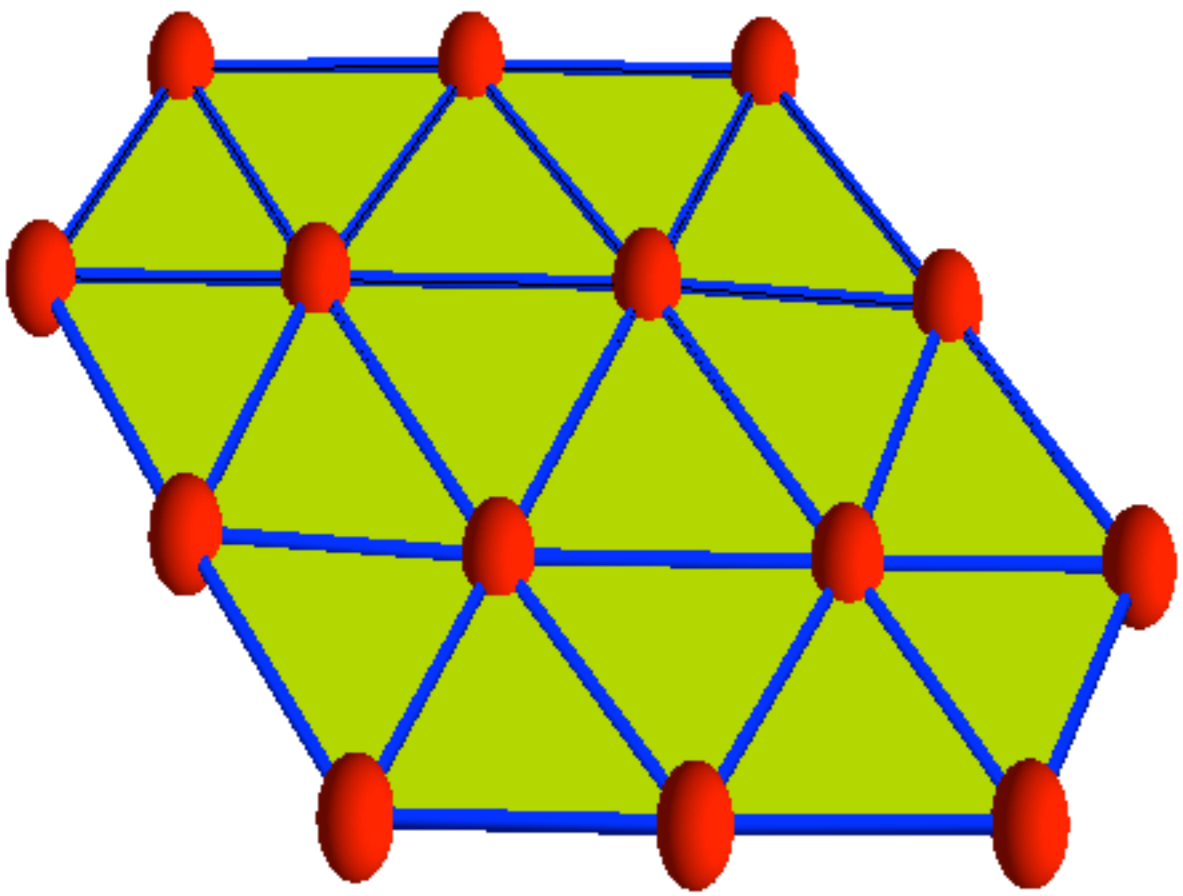
A simply connected graph of dimension d>1 is a sphere.

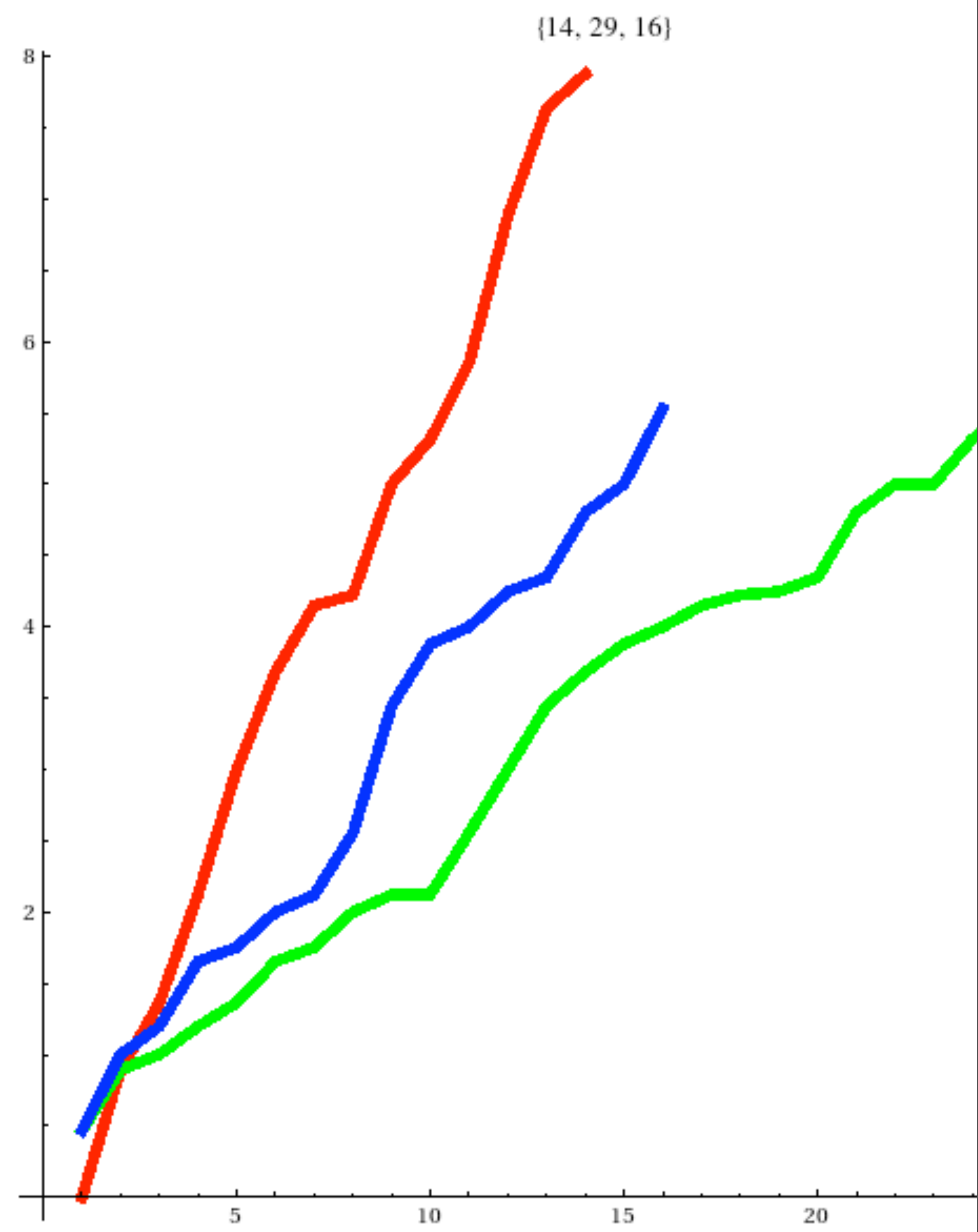
6) Fixed points

Find the minimal number of fixed points for an automorphism preserving some nondegenerate 2 form.

7) Spectra

Are there isospectral convex domains?




{14, 29, 16}
8
6
4
2
5
10
15
20

End