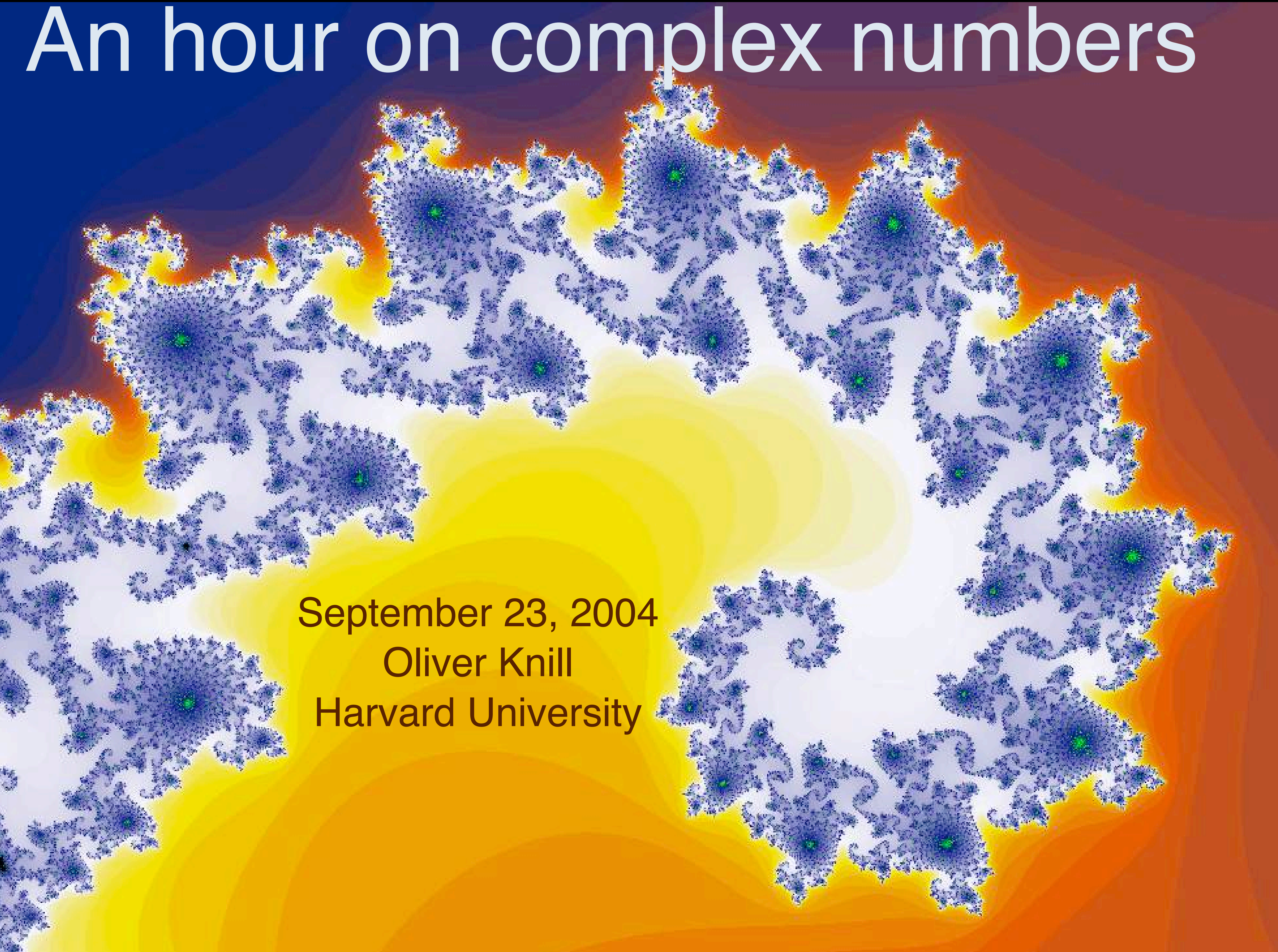
# An hour on complex numbers



September 23, 2004

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Harvard University

# Content

- The algebra of complex numbers
- Exp, Log and roots
- Applications: complex dynamics, gaussian integers

# Who knows what this is?

$$(-1)^i = \boxed{?}$$

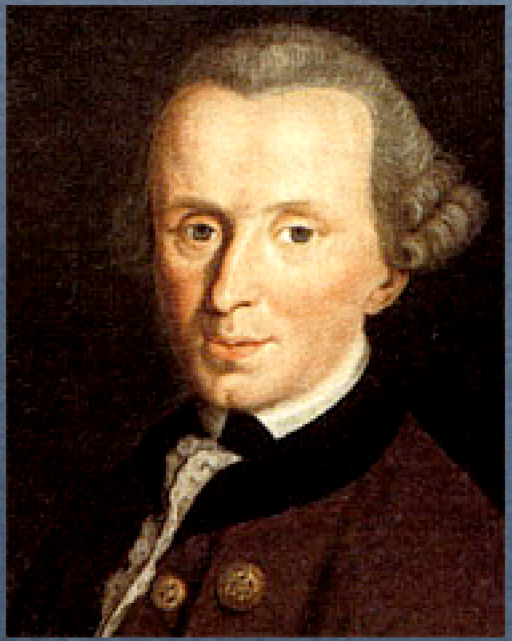
Our goal is to review from the beginning and also to develop the subject so far, that you will see what is the answer.

# Square root of -1

$$i = \sqrt{-1}$$

Gauss in 1825 : “The true metaphysics of the square root of -1 is elusive”.

# Apropos Metaphysics:



If you wonder about the onthological undecidability of non-existing objects in the real of the "possibility" in contrast to "actuality", then you will have to consider the Kantian view of phenomenal objects as the result of interaction between external and internal or to postulate the metaphysics for non-existent objects as a complete equality between subject and object.

# Everything clear?

# That reminds me:

What is the difference between a mathematician and a philosopher?

In the autobiographic essay “Young Toerless”,the austrian writer Robert Musil treats the pain of adolescent self-discovery and also the difficulty to understand

$$i = \sqrt{-1}$$

# Robert Musil (1880-1942)


Studied Mathematics and Psychology

# Euler Formula

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

Is the gateway to most secrets in complex numbers.

# Proof:

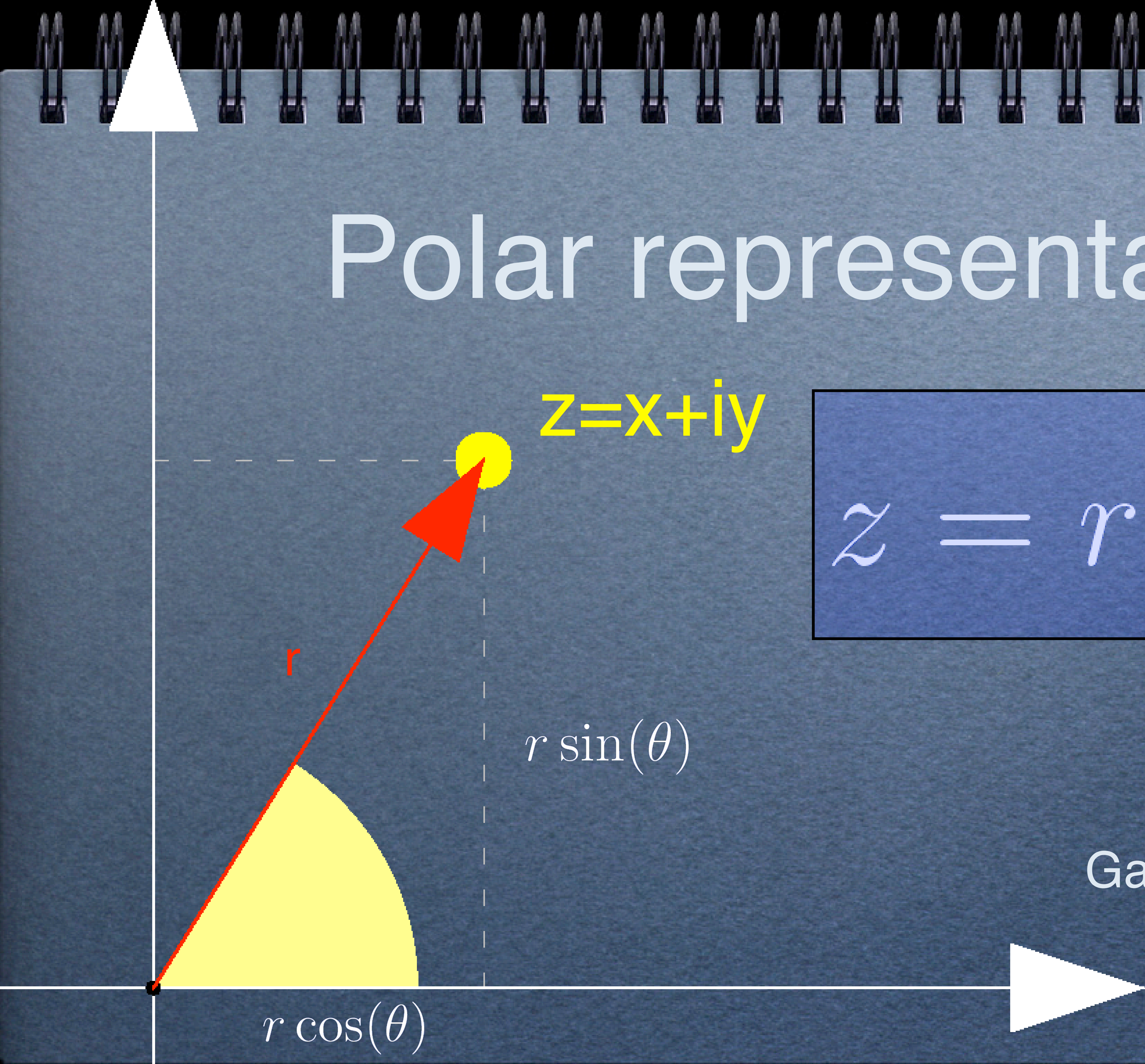
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

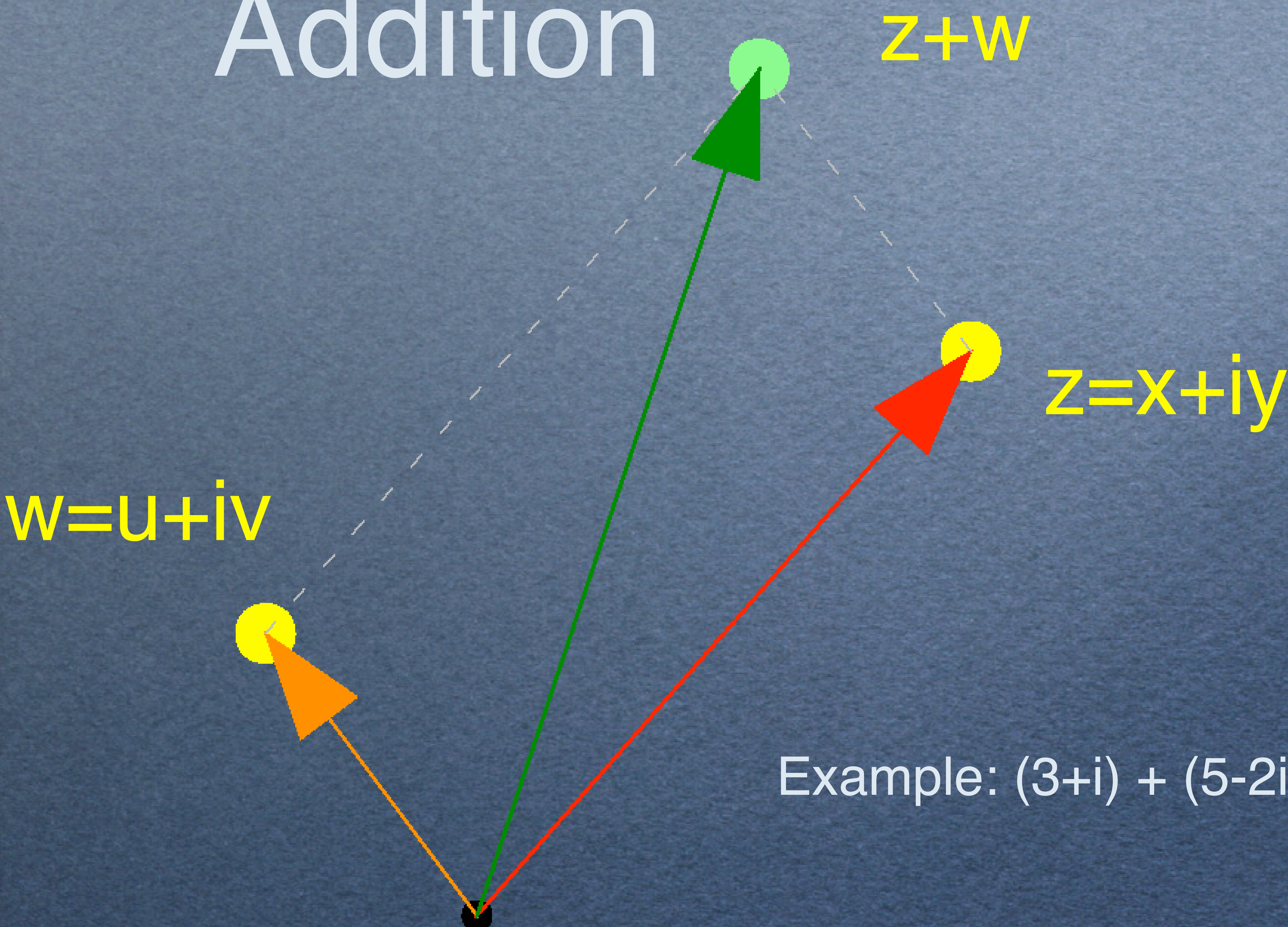
# Polar representation




$$z = re^{i\theta}$$

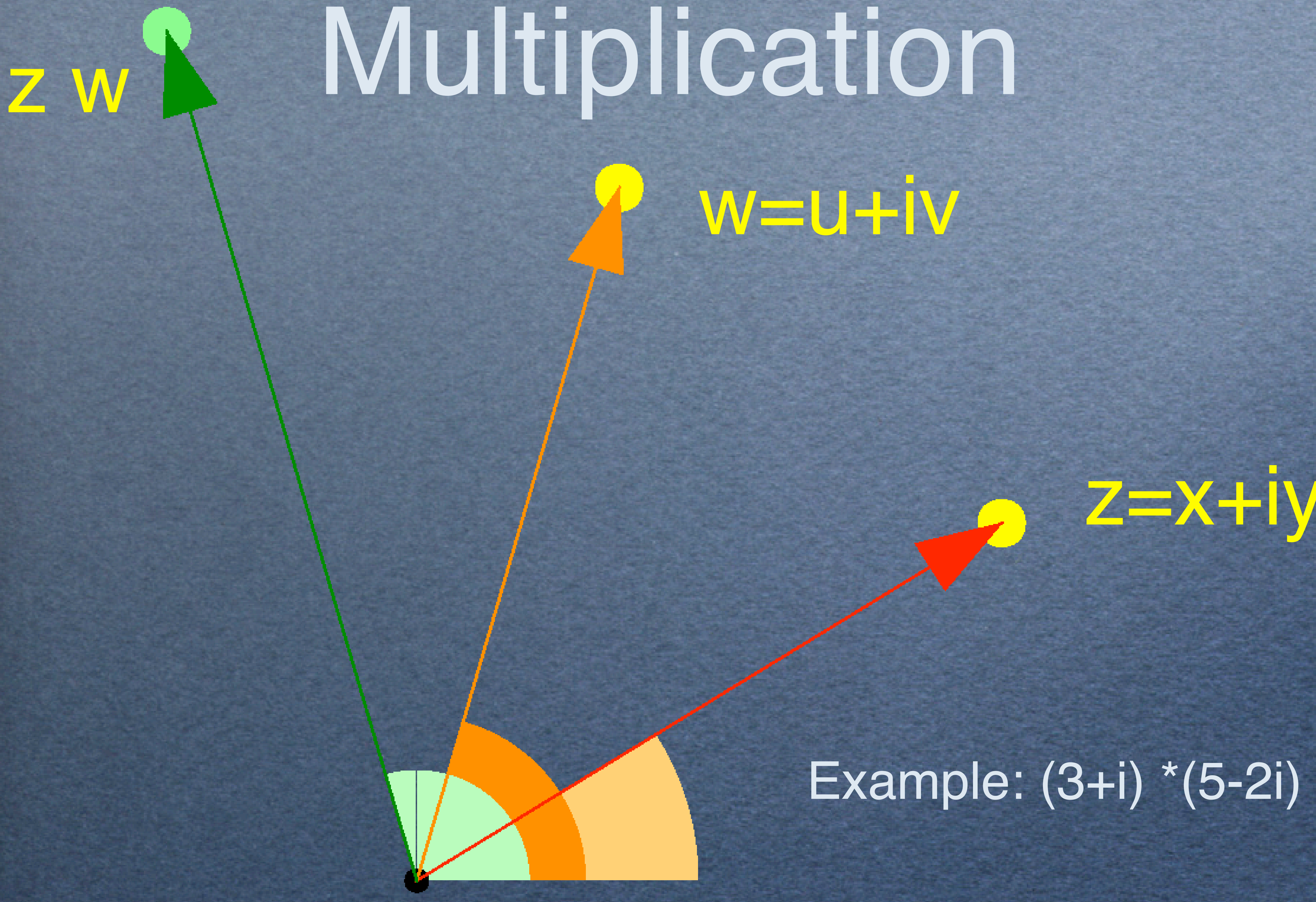
Gauss Plane

# Addition




Example: (3+i) + (5-2i)

# Multiplication



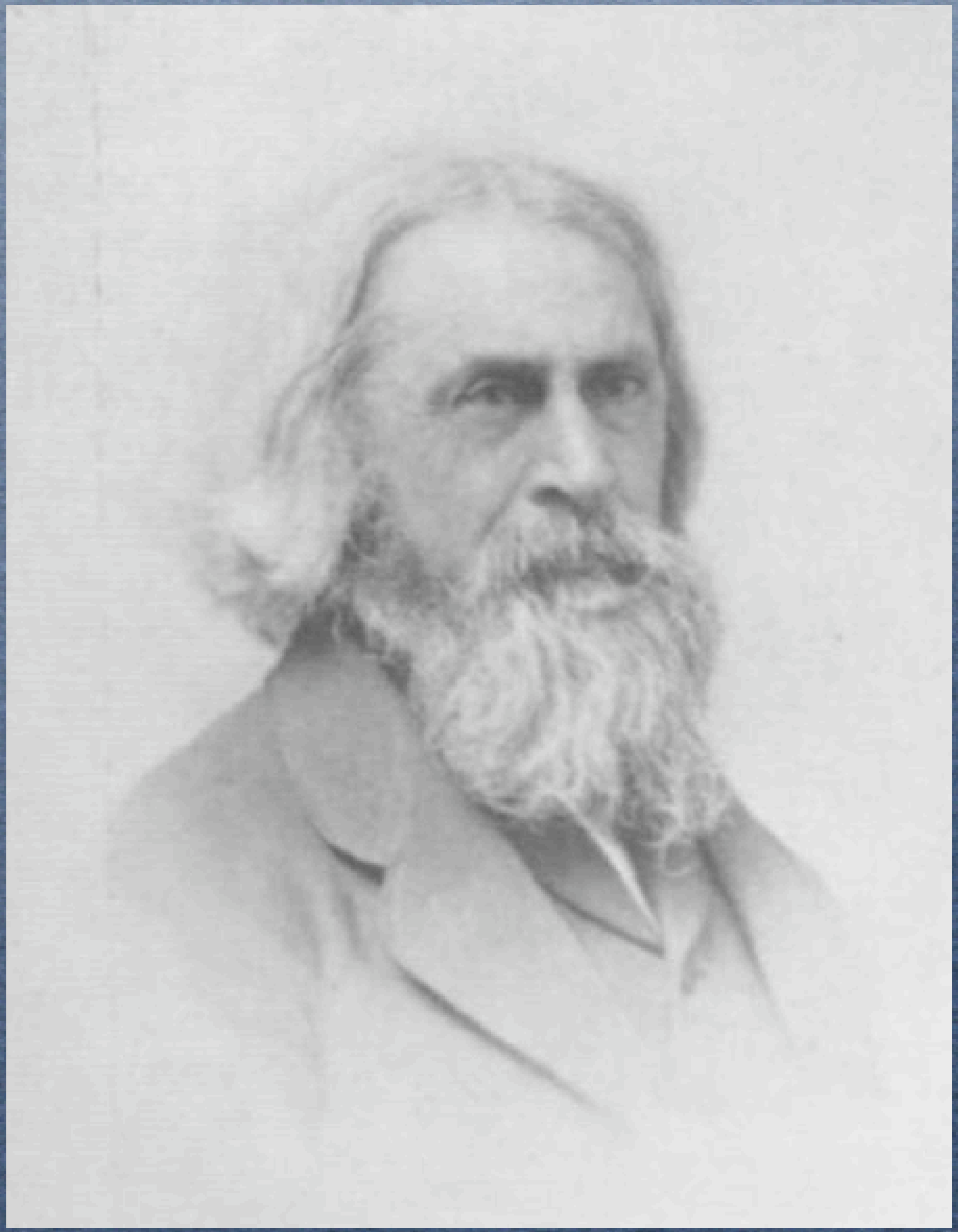

Example: (3+i) *(5-2i)

# The most remarkable formula in math

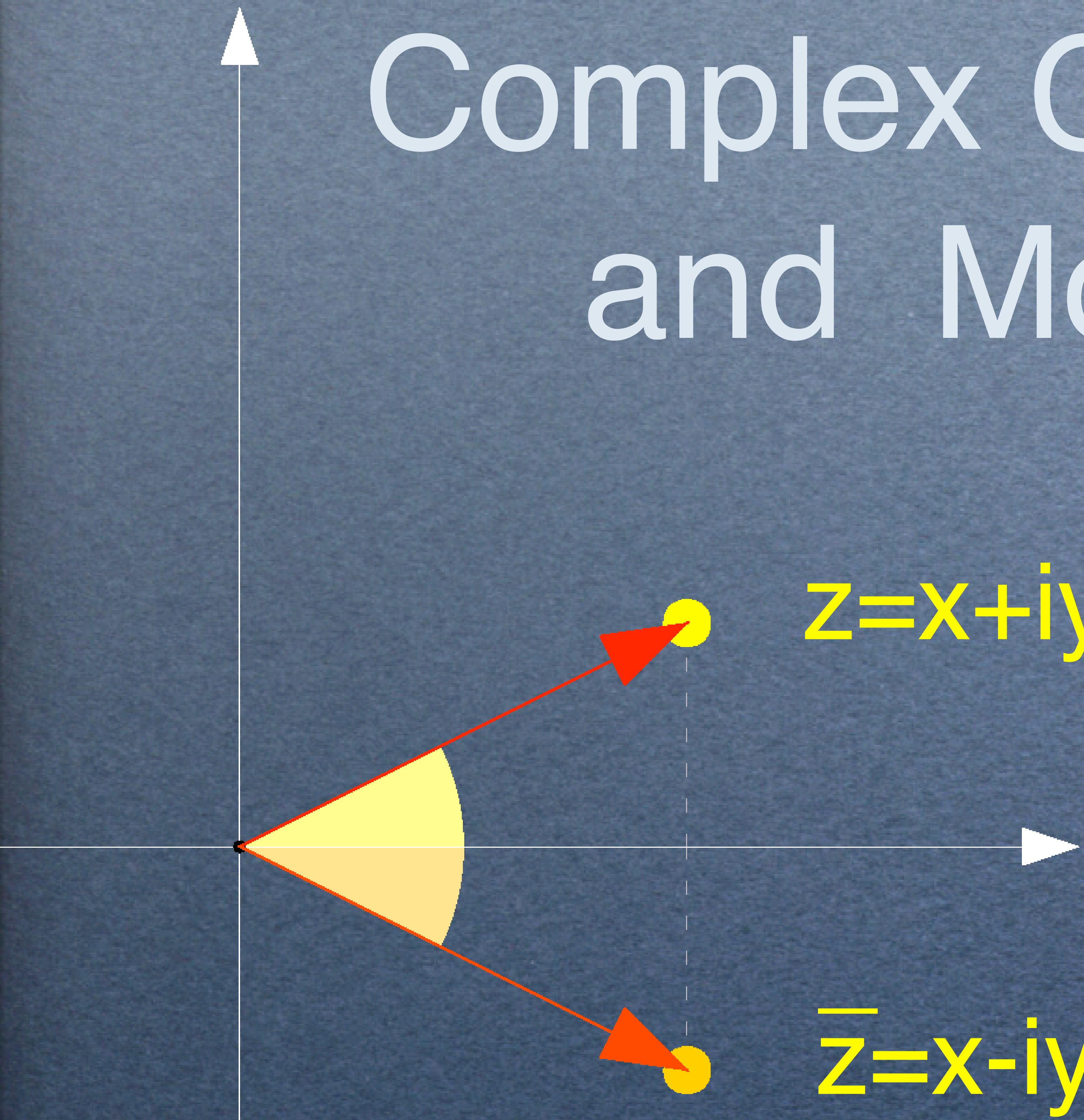
$$1 + e^{i\pi} = 0$$

# Benjamin Peirce: At Harvard

Gentlemen, that is surely true, it is absolutely paradoxical, we cannot understand it, and we don't know what it means. but we have proved it, and therefore, we know it is the truth.



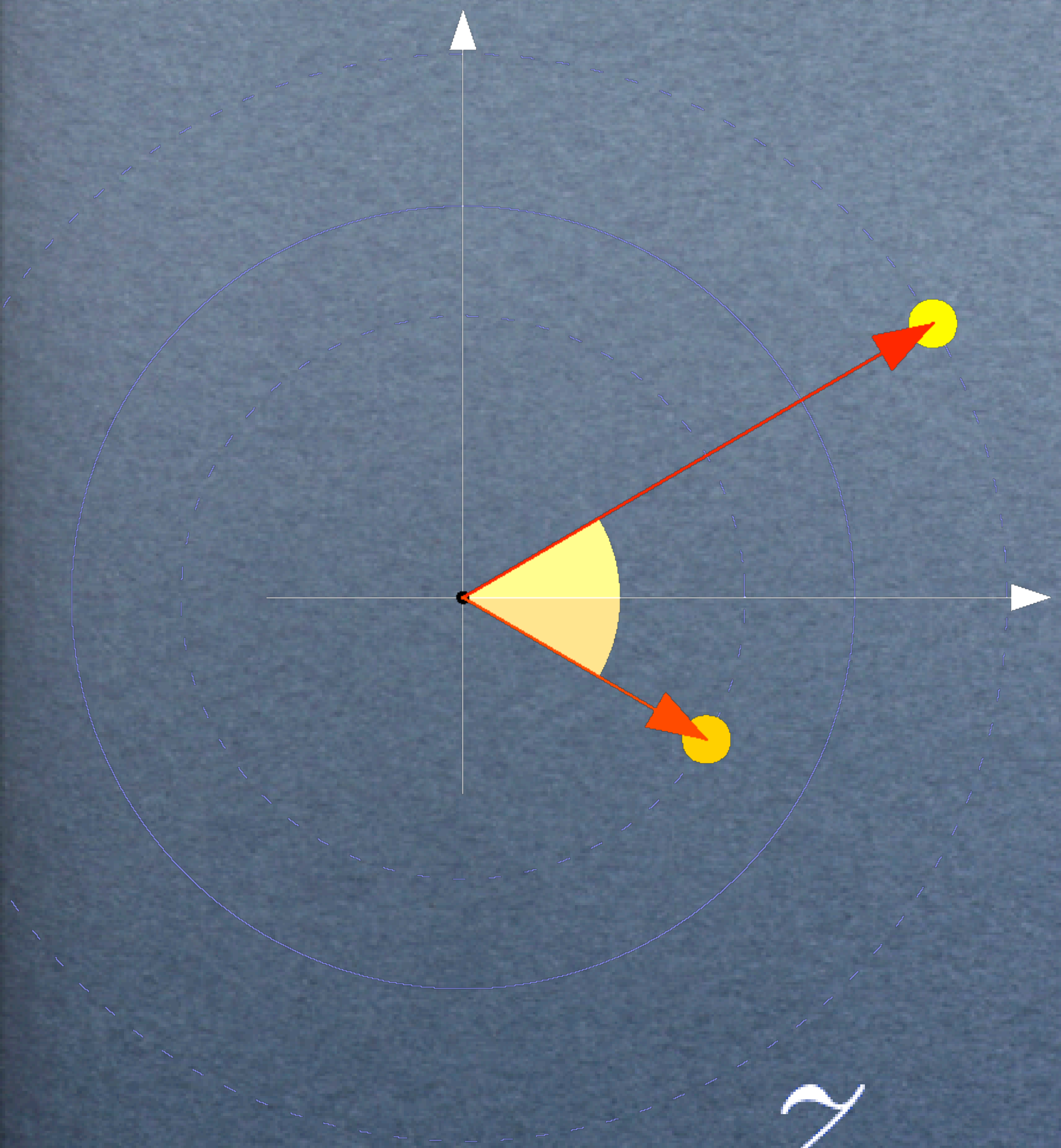
# Complex Conjugate and Modulus



$z=x+iy$

$|z|^2 = z\bar{z}$

$\bar{z}=x-iy$



# Division

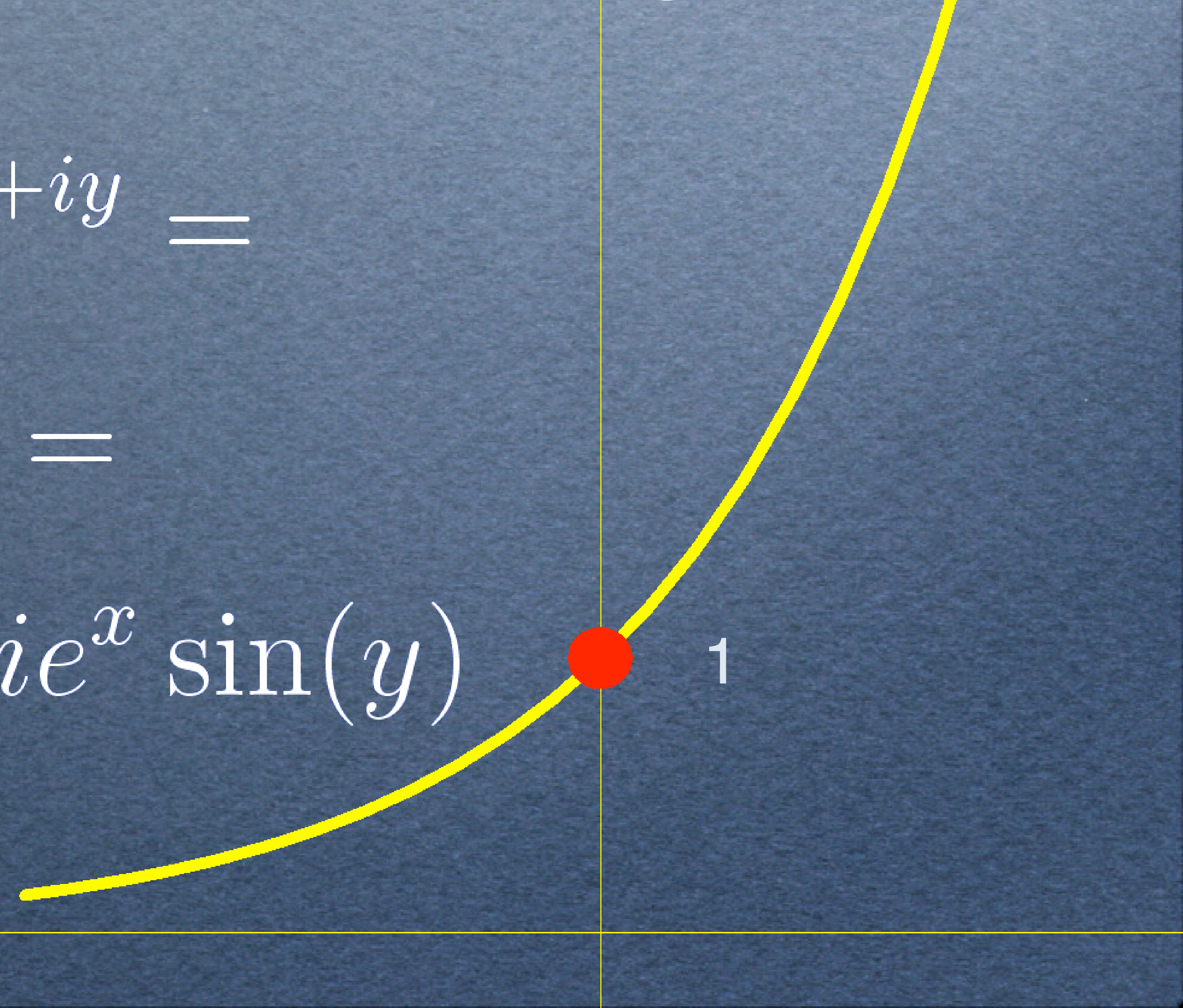
$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

# Exp(z) for arbitrary z

$$e^z = e^{x+iy} =$$

$$e^x e^{iy} =$$

$$e^x \cos(y) + i e^x \sin(y)$$



# Log(z) for nonzero z


Example:

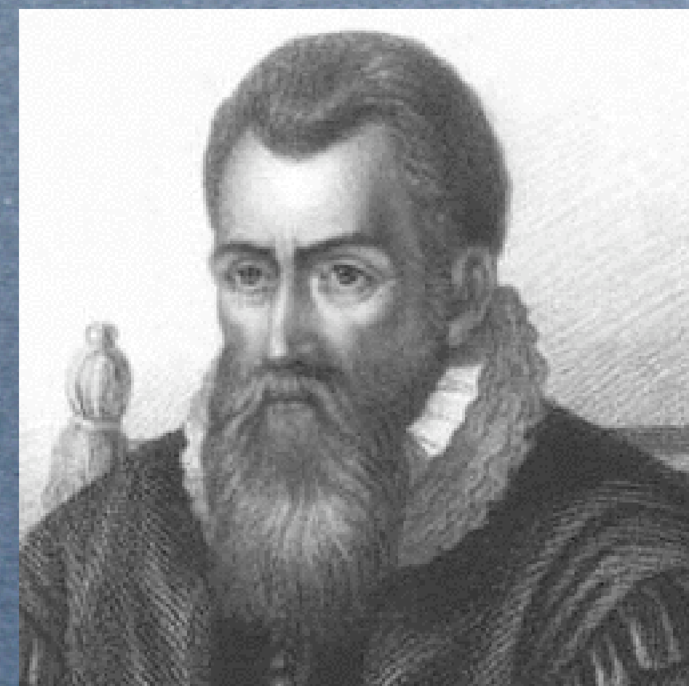
$$\log(i) = \log|i| + i\arg(i)$$

$$= \frac{\pi}{2}\,\mathrm{i}$$

$$\log(z) = \log|z| + i\arg(z)$$

# The Inventors of Log

Switzer-
land

Scott-
land

Joost Buergi

1552-1632

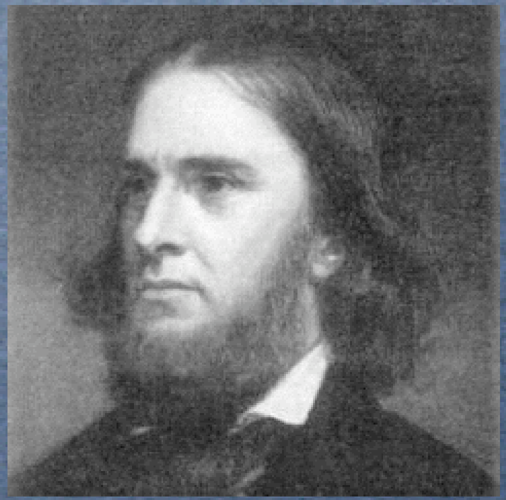
John Napier

1550-1617

Kepler: "... the accents in calculation led Justus Byrgius (Jost Burgi) on the way to these very logarithms many years before Napier's system appeared; but being an indolent man, and very uncommunicative, instead of rearing up his child for the public benefit he deserted it in the birth."

# A “mysterious formula”

$$(-1)^i = e^{-\pi}$$



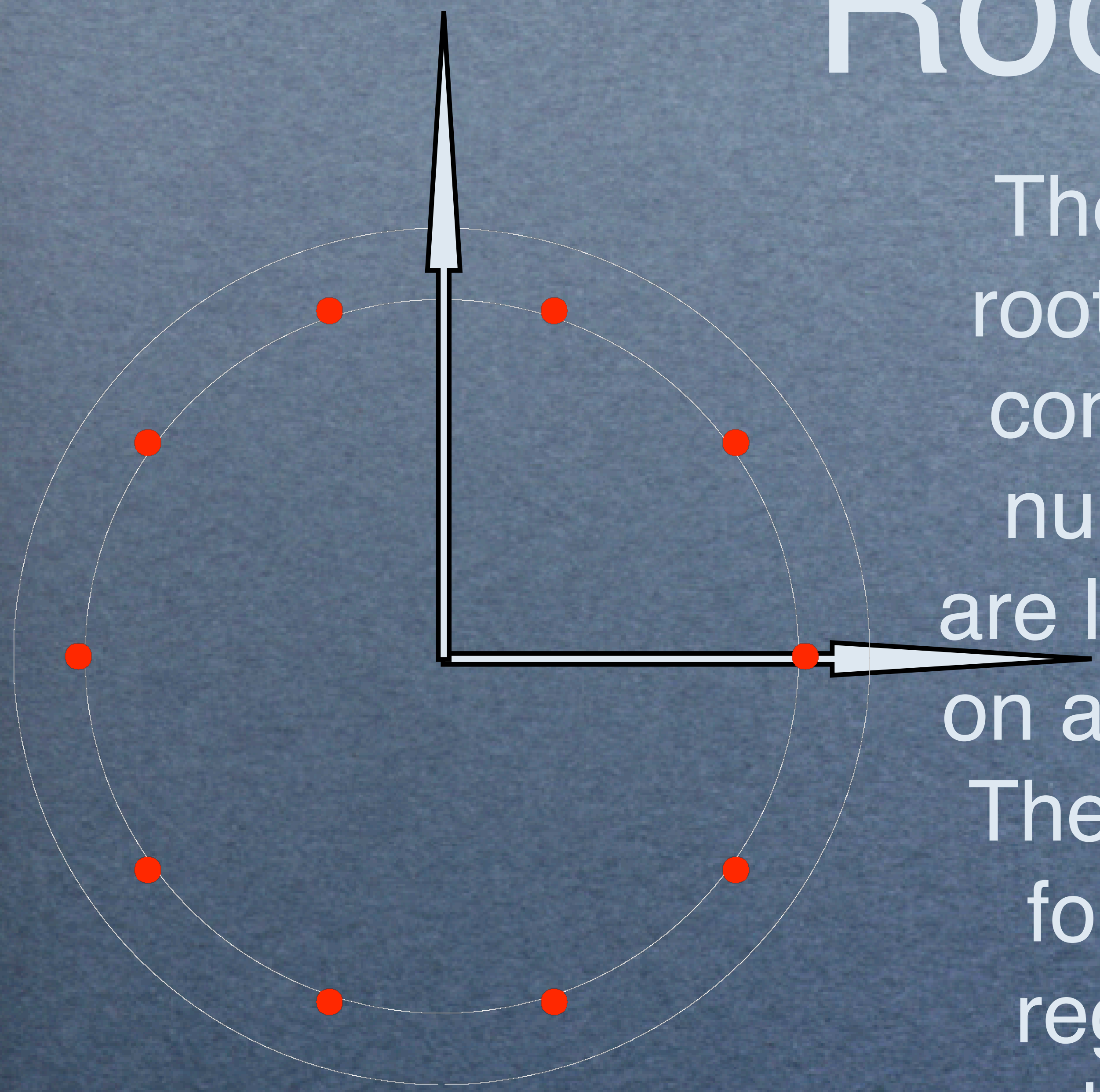
What is $i^i$ ?

# The square root

"Take square root of modulus and divide angle by 2".

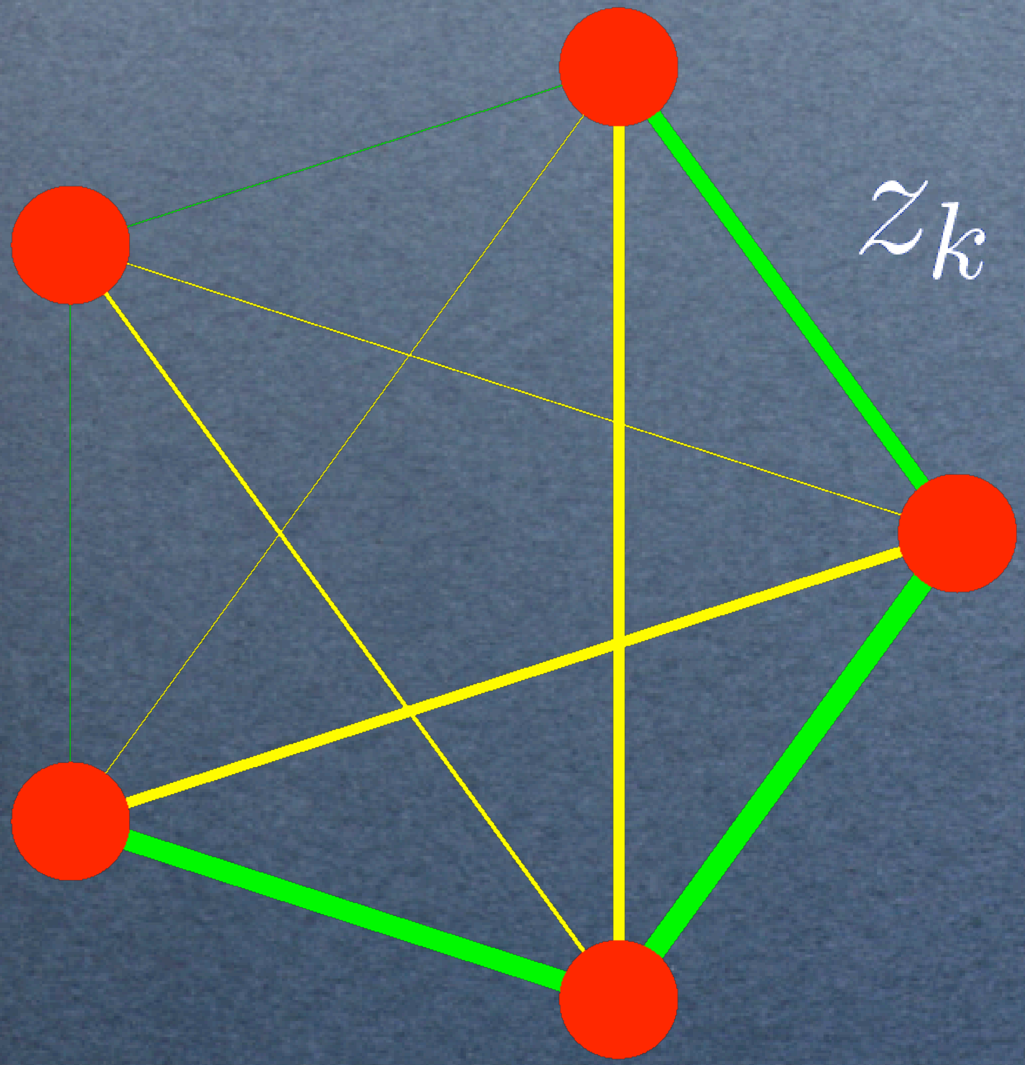
Every complex number different from 0 has exactly 2 square roots.

# Roots



The n'th roots of a complex number are located on a circle. The roots form a regular polygon.

# Application: the pentagon



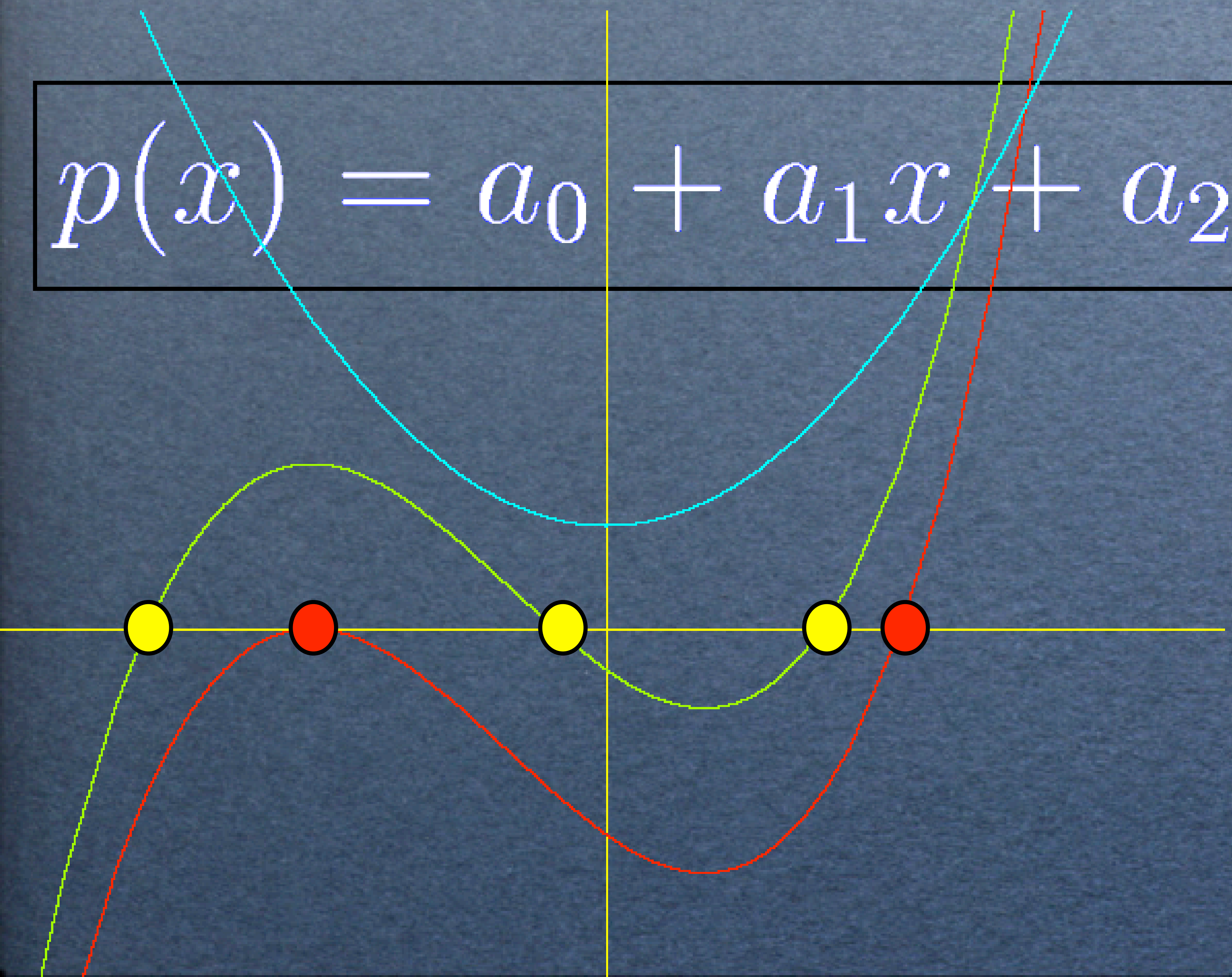
$$z_k = e^{k\pi/5}, k = 1, 2, 3, 4, 5$$

$$\frac{|z_3 - z_1|}{|z_2 - z_1|} = \frac{\sqrt{5}+1}{2}$$

Golden ratio

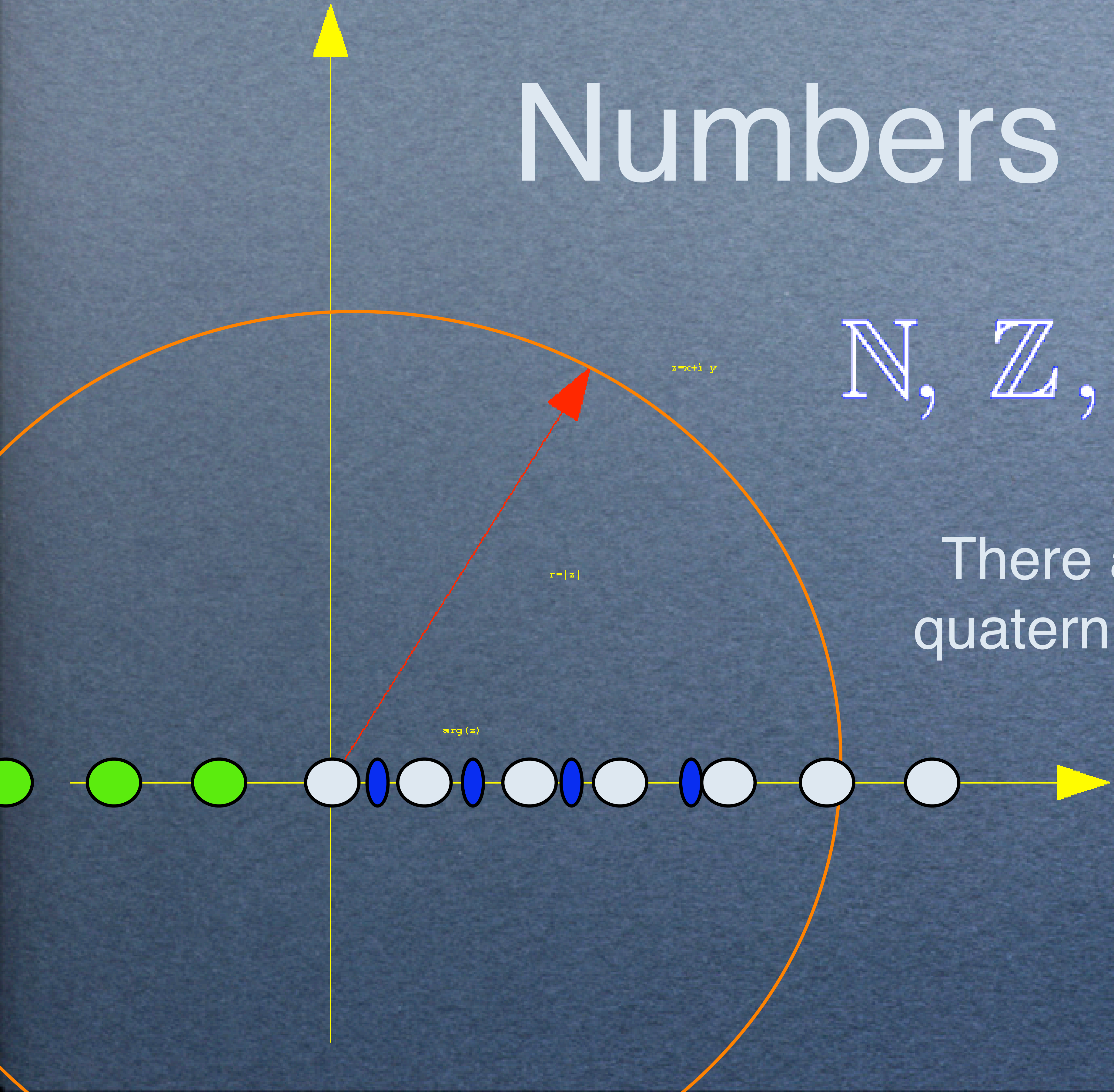
# Fundamental theorem of algebra

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$



A polynomial of degree n has exactly n roots p(x)=0

# Numbers




$\mathbb{N}, \ \mathbb{Z}, \mathbb{Q}, \ \mathbb{R}, \ \mathbb{C}$

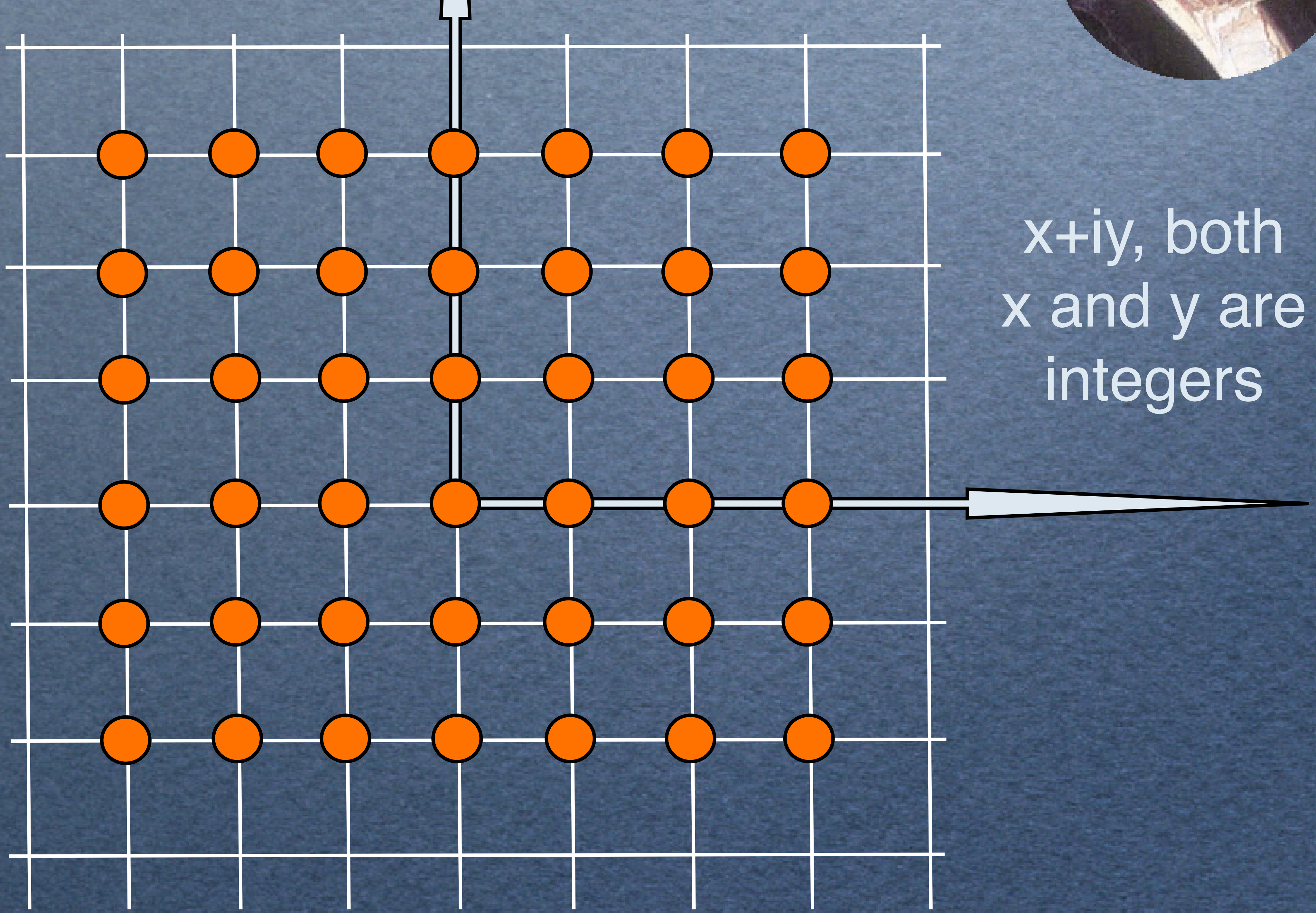
There are more numbers:
quaternions and octonions...

# A word of wisdom:

The complex numbers are the most natural numbers,
the natural numbers are the most complex numbers.

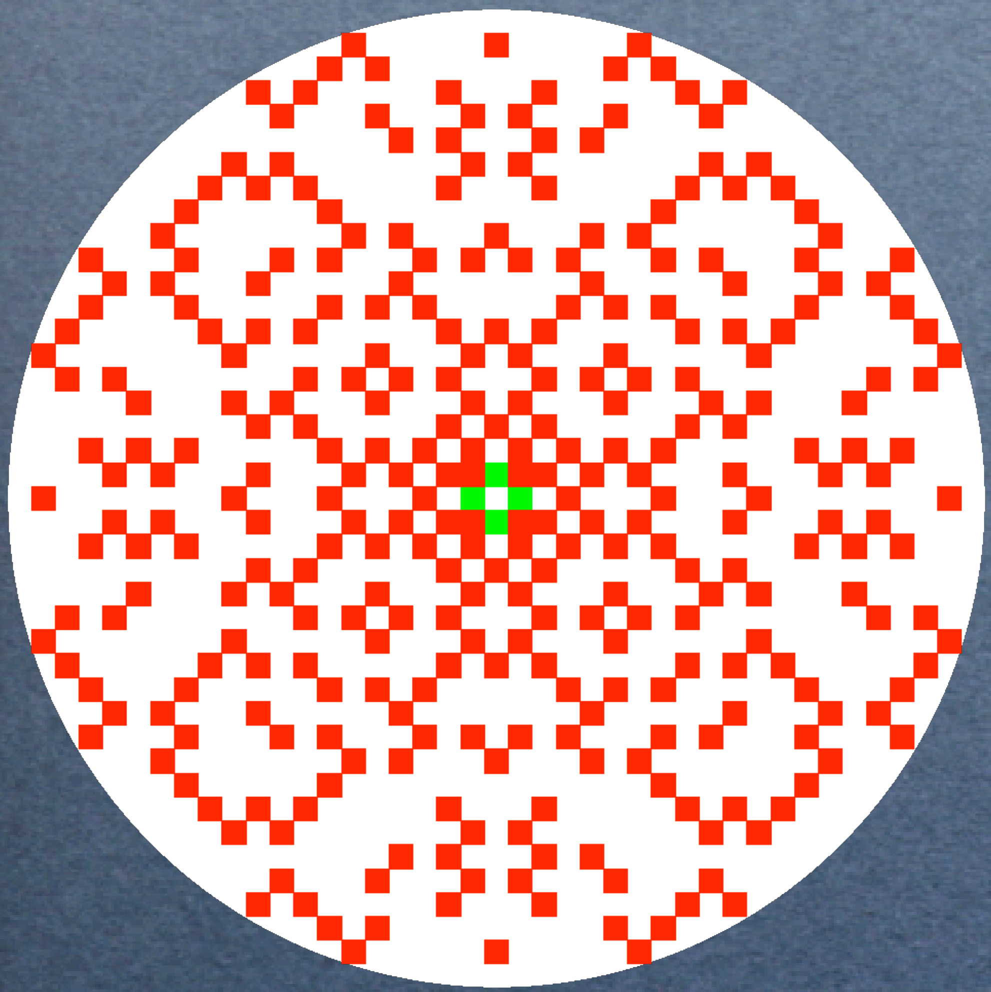
Indeed, working with natural numbers is one of the most difficult things with many open problems.

# Gaussian Integers



x+iy, both
x and y are
integers

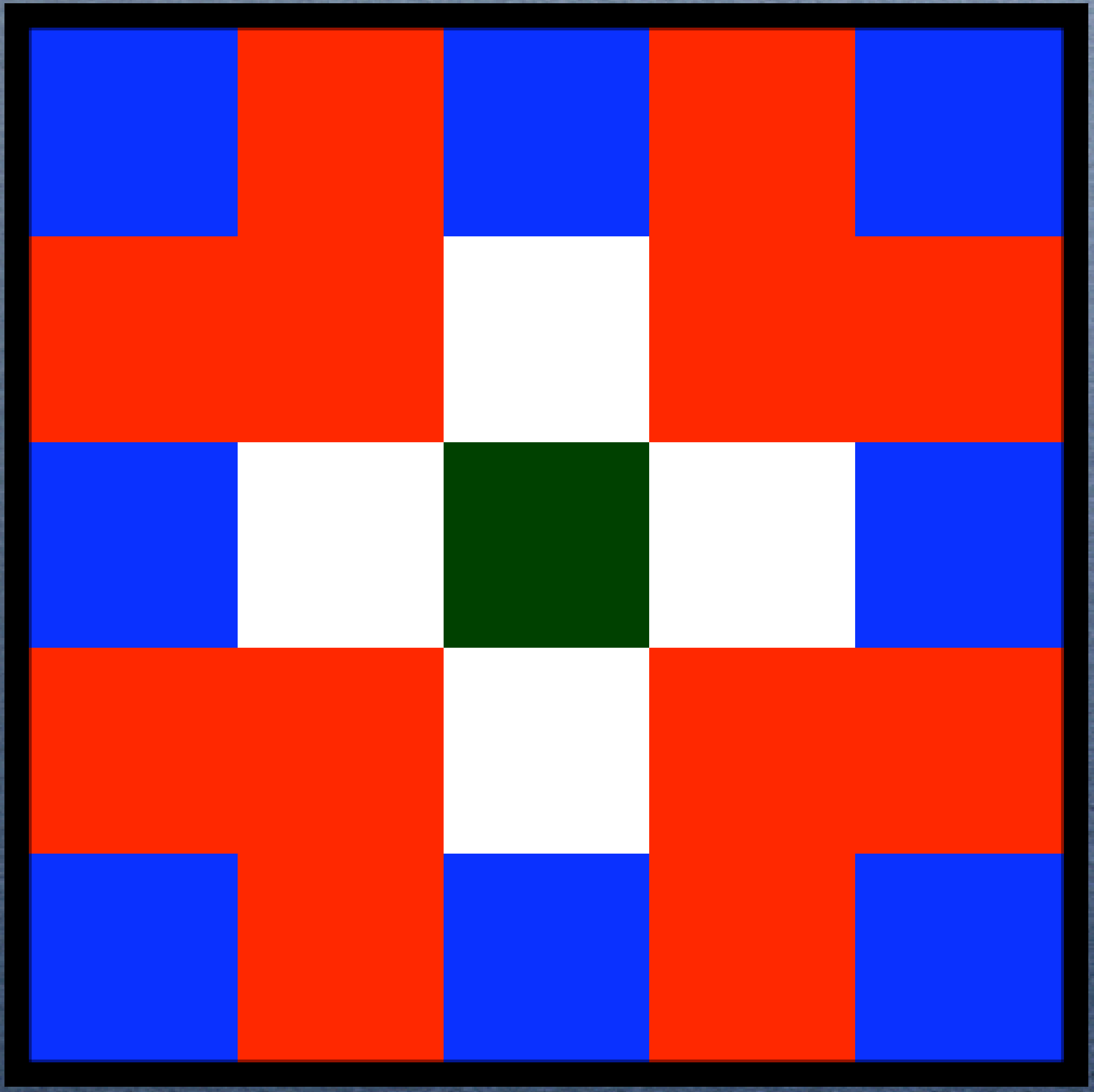
# Which are prime?



5=(1+2i) (1-2i) is not prime

You can check that 3 is prime, or (1+i) is prime by dividing through all integers with smaller norm.

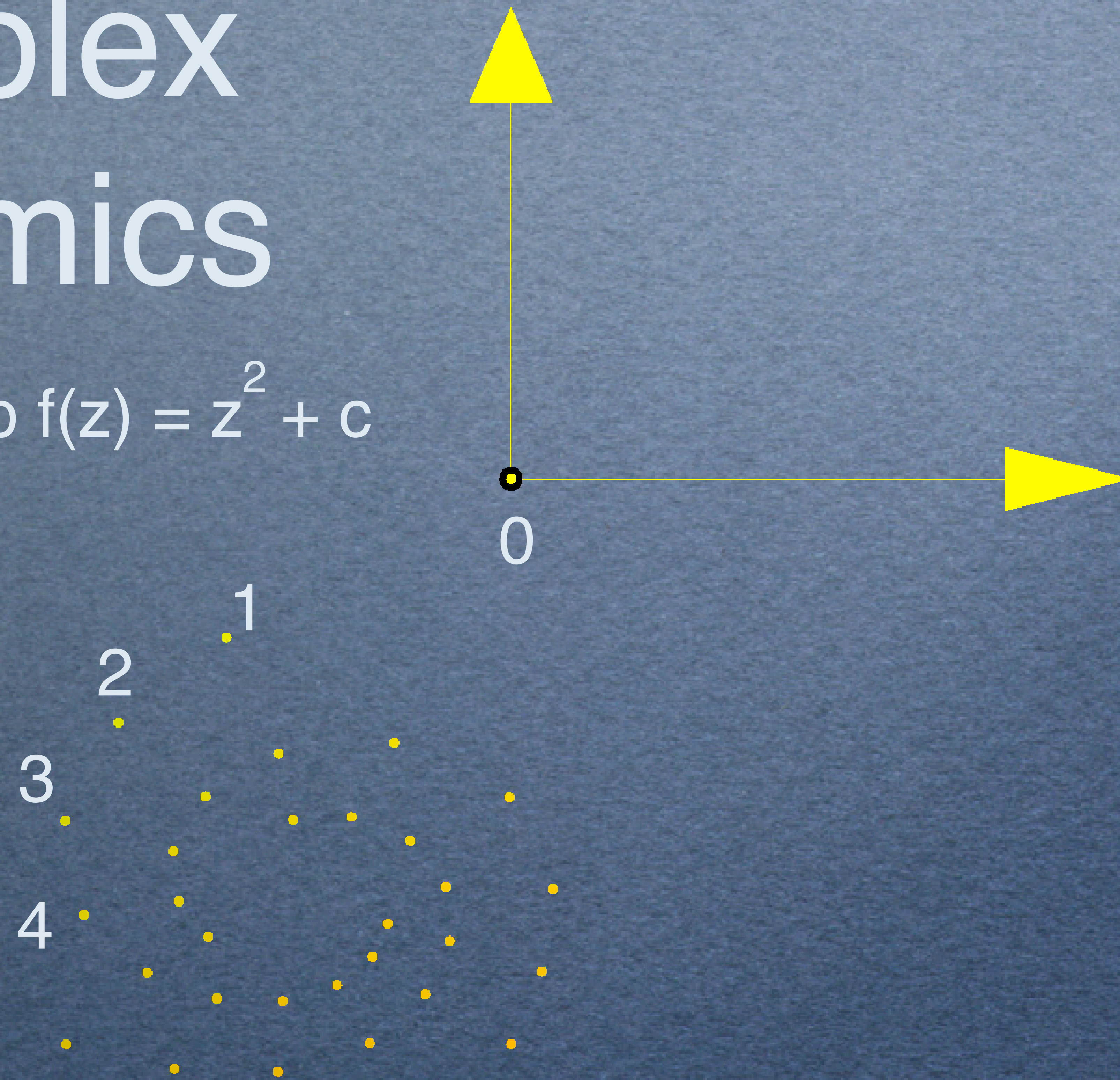
# Gaussian Primes



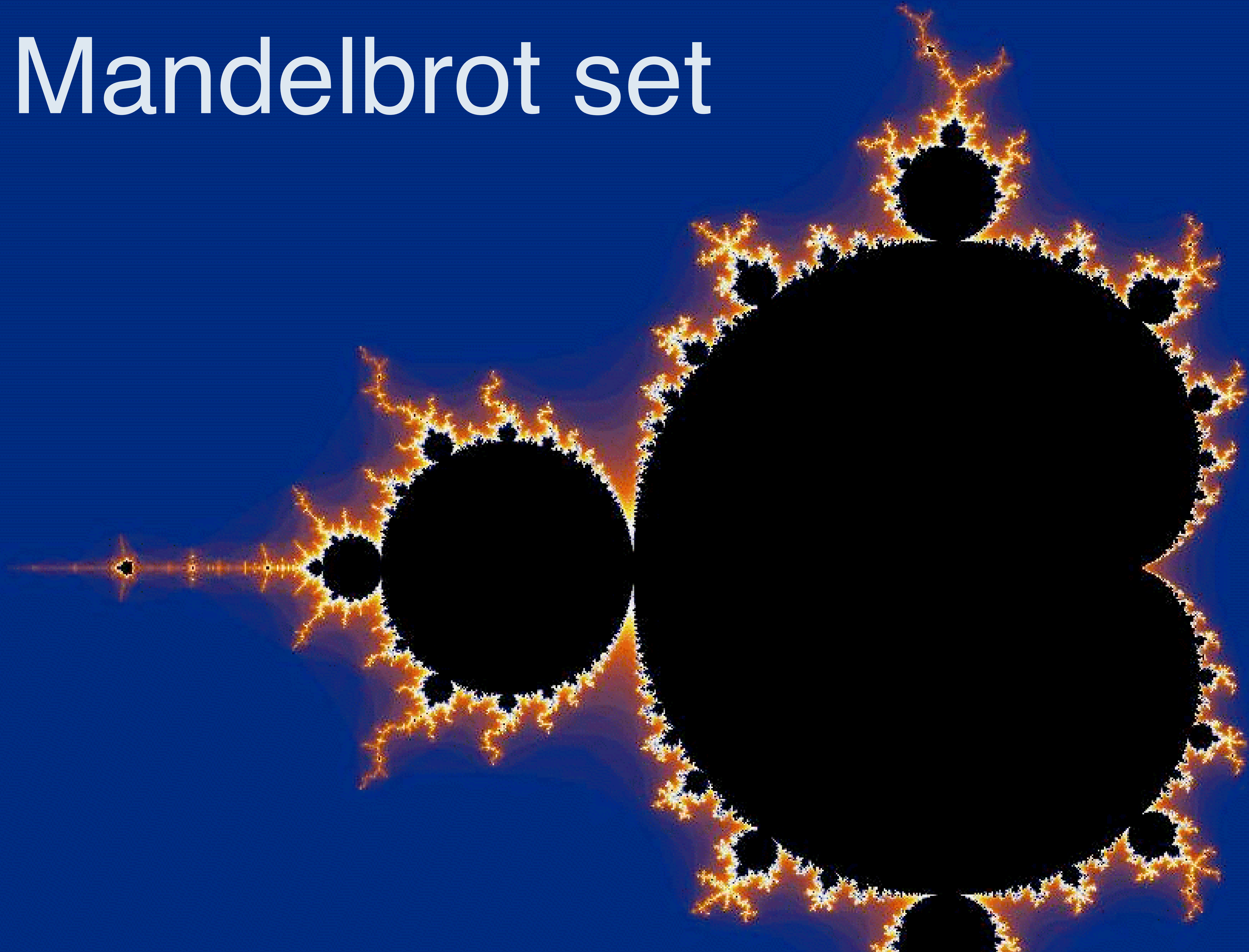
# Complex Dynamics

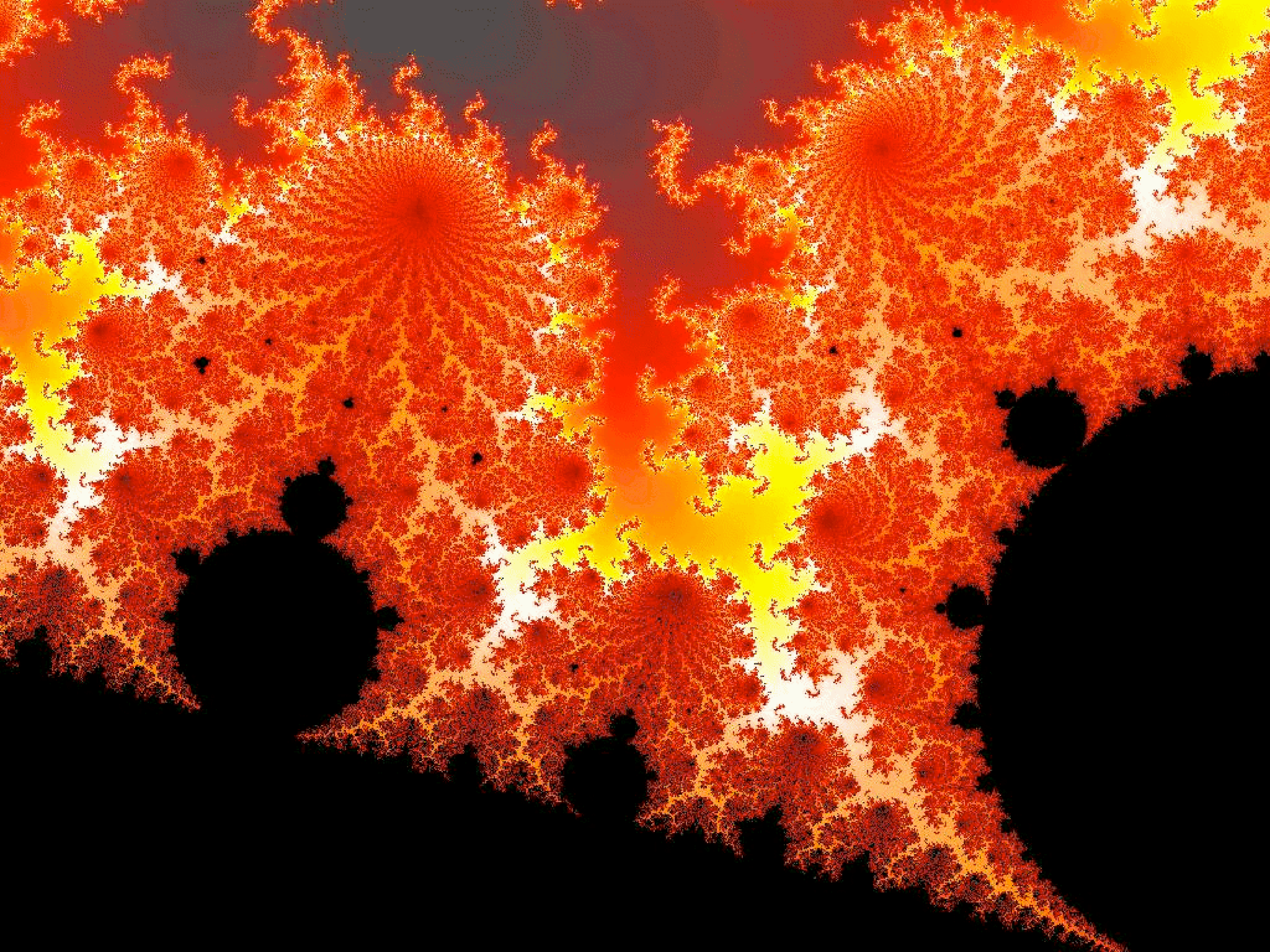
iterate the map $f(z) = z^2 + c$

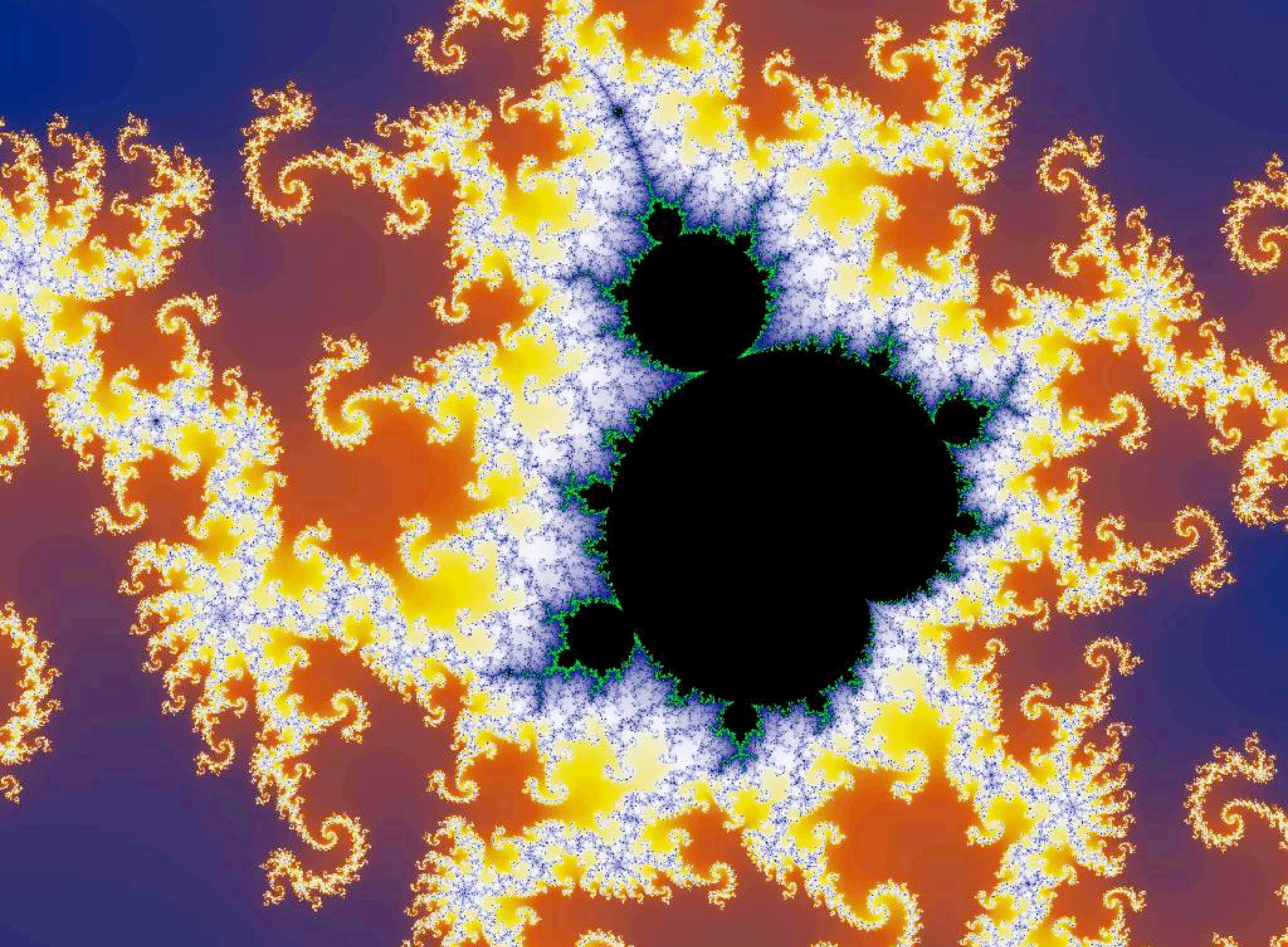
here c=0.6 i

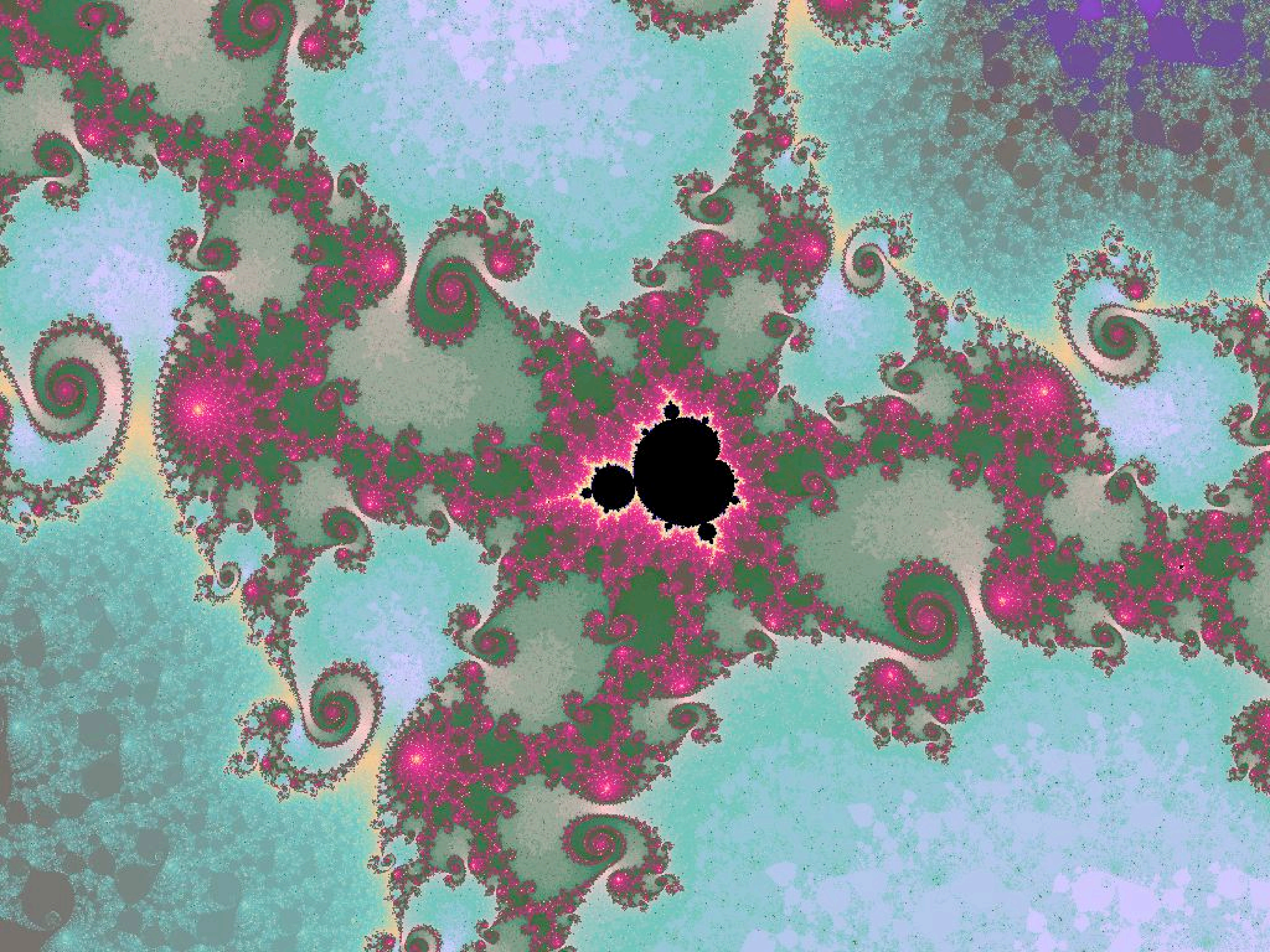


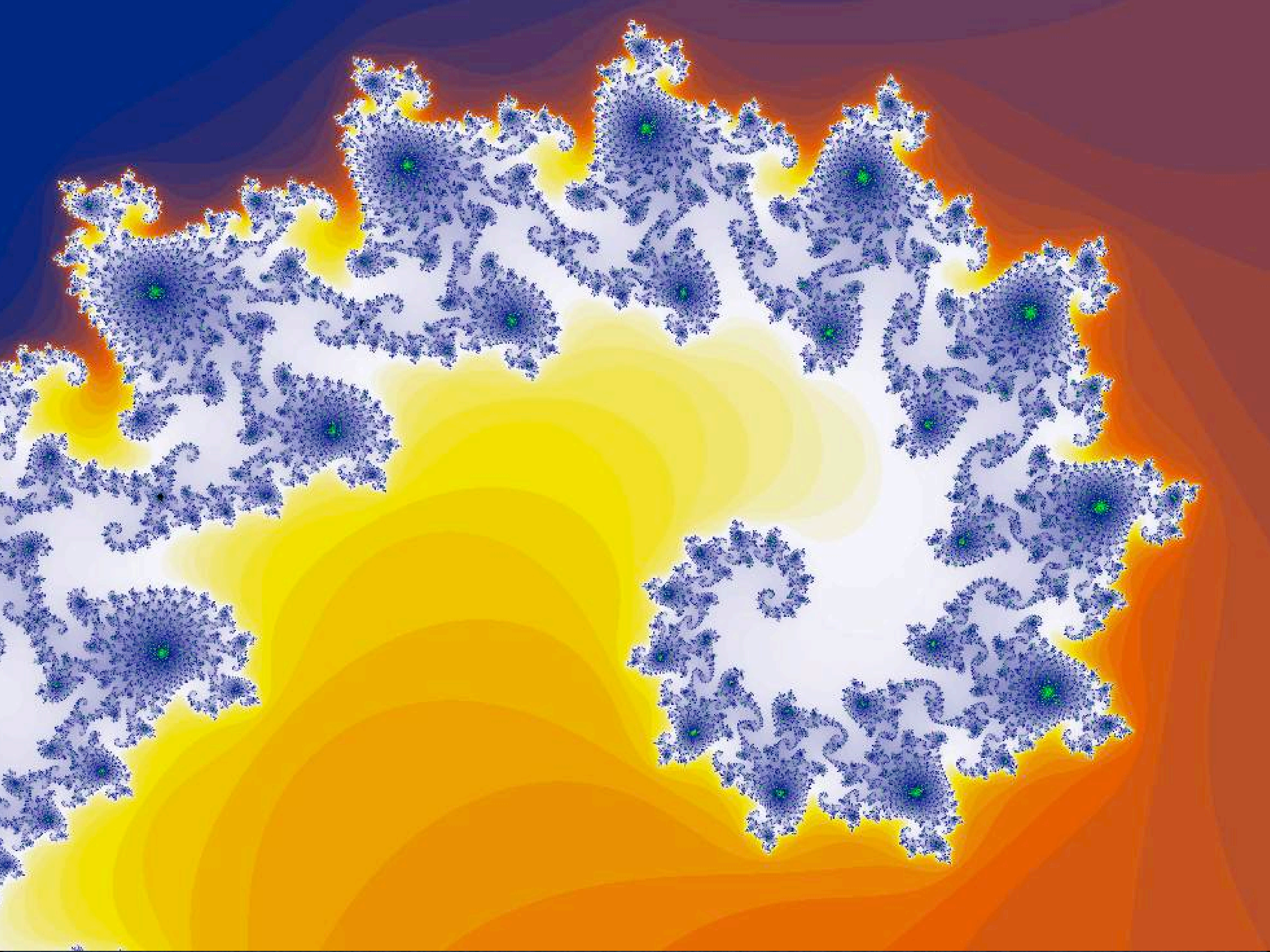
# Mandelbrot set











“The shortest path between two truths in the real domain passes through the complex domain”

Jacques Hadamard (1865-1963)