Mathtable of 2/24/2004, "Applications of Subharmonic Functions"   Oliver Knill

INTRODUCTION. While working on my undergraduate senior theses at ETH, I stumbled over a paper of the French mathematician M. Herman which introduced a new method for estimating Lyapunov exponents. I was impressed and naively felt that this method could even help to solve one of the major open problems in ergodic theory.

SUBHARMONIC FUNCTIONS. A $R \cup \{-\infty\}$ valued function on the complex plane C is called **subharmonic** in G if it is upper semi-continuous and $C_{f,z}(r) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta}) \, d\theta \geq f(z)$ for all $r > 0$ and all $z \in G$.

EXAMPLES. Harmonic functions are subharmonic, the sub-mean inequality being an equality. Any function $h(|z|)$ with convex h is subharmonic. Indeed, subharmonicity is a notion of "convexity". If $g(z)$ is an analytic function, then $f(z) = \log|g(z)|$ is subharmonic because it is clear if $g(z) = 0$ and because otherwise, f is the real part of an analytic function and so harmonic. More generally, if $A(z)$ is a matrix-valued analytic function, then $f(z) = \log||A(z)||^2$ is subharmonic. Proof: f is continuous. There are vectors v, w of length 1 such that $g(z) = (v \cdot A(z)w) = ||A(z_0)||$ for $z = z_0$. Because $g(z)$ is an analytic map in z, $\log|g(z)|$ is subharmonic and so $2\pi \log||A(z_0)|| \leq \int_0^{2\pi} \log|v \cdot A(z + re^{i\theta})w| \, d\theta \leq \int_0^{2\pi} \log||A(z + re^{i\theta})|| \, d\theta$.

RIESZ. If f is subharmonic, then the Laplacian of f, Δf is a nonnegative measure. One can write it as a sum of a potential and a harmonic function: $f(z) = \int \log|z - w| \, d\mu(w) + h(z)$ where μ is called the **Riesz measure**. Because subharmonic functions solve the **Poisson equation** $\boxed{\Delta f = 2\pi\mu}$, they are important in physics. The measure μ can be thought of as a a charge distribution or vorticity distribution. If $f(z) = \log|p(z)|$, where p is a polynomial, then the Riesz measure is supported on the roots of $p(z)$. Any positive measure μ defines a subharmonic function $f(z) = \int \log|z - w| \, d\mu(w)$. The Poisson-Jensen formula $C_{f,z}(r) = f(z) + \int_0^r \log(r/t) \, d\rho_f(t)$, where $\rho_f([a,b]) = \int_{a \leq |w-z| \leq b} \, d\mu(w)$ quantifies the submean inequality.

AN OPEN PROBLEM. Consider the nonlinear recursion $\boxed{x_{n+1} - 2x_n + x_{n-1} = \lambda \sin(x_n)}$, where x_n are real numbers taken mod2π. This second order difference equation is equivalent to the first order recursion $\boxed{T(x,y) = (2x - y + \lambda\sin(x), x)}$ on $\mathbb{T}^2$ which is called the **Standard map**. If $T^n(x,y) = T(T^{n-1}(x,y))$ are the iterates, one can ask, how the Jacobean $dT^n(x,y)$ grows with n.
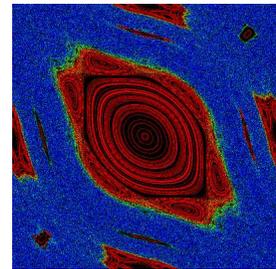
The **Lyapunov exponent** $\lambda(T_\lambda) = \liminf_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}^2} \log||dT^n|| \, dy dx$ measures the exponential growth of dT^n. It is known to be the **entropy** of T, a number of physical interest. All numerical measurements so far indicate that $\gamma(T) \geq \log(\lambda/2)$. But it is not known, whether there is a positive $\gamma(T_\lambda) > 0$ at all. This is a major open problem in ergodic theory, mentioned in countless articles, books, reviews and talks. Actually, the measurements indicate that for all n the **finite time Lyapunov exponents** satisfy $\boxed{\frac{1}{n} \int \int_{\mathbb{T}^2} \log||A^n(x,y)|| \, dxdy \geq \log(\frac{\lambda}{2})}$. This could be verified in principle for every n since the left hand side is an integral of an explicitly given function. However, already for small n, reliable numerical computations are difficult.

A CALCULATION OF HERMAN. Consider for a fixed irrational number α and for a fixed real number θ, the linear recursion $\boxed{x_{n+1} - 2x_n + x_{n-1} = \lambda\cos(\theta + n\alpha)x_n}$. It can be written as a first order recursion $A(x,y) = (2x + \lambda\cos(\theta + n\alpha) - y, x) = A(\theta + n\alpha)(x,y)$. The exponential growth of x_n is called the Lyapunov exponent of A over the dynamical system $\theta \mapsto \theta + \alpha$. is equal to the asymptotic growth of the product of matrices $\boxed{A(\theta + n\alpha)...A(\theta + \alpha)A(\theta)}$. With $z = e^{i\theta}$ and $w = e^{i\alpha}$, one write this as $A(zw^n)...A(z)$. Now, $B(z) = zA(z) = \begin{bmatrix} 2z - (\lambda/2)(z^2 + z) & -z \\ 0 & z \end{bmatrix}$ is an analytic matrix-valued map and because $f_n(z) = \frac{1}{n}\log||B(w^n z)...B(z)||$ is subharmonic, also $\boxed{f(z) = \frac{1}{n}\log||B(w^n z)...B(z)||}$ is subharmonic and the Lyapunov exponent $f(e^{i\theta}) = \int_{\mathbb{T}} f(e^{i\theta}) \, d\theta/(2\pi)$ is bounded below by $f(0) = \log(\gamma/2)$.

MULTIDIMENSIONAL COMPLEX ANALYSIS. The map $U(z,w,u,v) = (zwe^{z-u}, we^{z-u}, uve^{u-z}, ve^{u-z})$ has a one-parameter family of invariant real tori $S_\lambda = \{(z,w,u,v) \in \mathbb{C}^4 \mid |z| = |u| = \frac{\lambda}{2}, |v| = |w| = 1, z = \bar{u}, w = \bar{v}\}$. If U is restricted to S_λ, one obtains the Standard map T_λ. Note that the parameter λ does not appear in the analytic map $U : \mathbb{C}^4 \mapsto \mathbb{C}^4$. Using the map U was one of many attempts to make use of subharmonicity for estimating the entropy.

A DOOMED APPROACH. The Standard map can be written for $\lambda = 0$ as $(z, w) \mapsto (zw, w)$ on the torus $\{(z, w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}$ which is the distinguished boundary of the polydisc $\{(z, w) \in \mathbb{C}^2 \mid |z|, |w| < r\}$. In these coordinates, the Jacobean is $A_\lambda(z, w) = \begin{bmatrix} 1 + \frac{1}{2}(z + z^{-1}) & 1 \\ \frac{1}{2}(z + z^{-1}) & 1 \end{bmatrix}$. The Lyapunov exponent is the same as for the analytic matrix-valued map $B(z) = zA(z)$. We obtain a lower bound from the fact that $(z, w) \mapsto \frac{1}{n} \log \|B_\lambda^n(z, w)\|$ is subharmonic in each parameter. Because the invariant two-dimensional torus S_λ is $\mathbb{C}^4$ is not the boundary of a polydisc, one can not use a plurisubharmonic estimate for T_λ.

A SPECIFIC SUBHARMONIC FUNCTION. We can parametrized in the Standard map the Jacobean matrix $A(x, y, \theta) = \begin{bmatrix} 2 - \lambda \cos(x + \theta) & -1 \\ 1 & 0 \end{bmatrix}$ with a parameter θ. The Lyapunov exponent is now a function of θ, but only the value $\theta = 0$ has the interpretation of an **entropy**. In the complex, we get an analytically parametrized cocycle $A(z) = \begin{bmatrix} 2zw - \lambda(z^2w^2 + 1)/2 & -1 \\ z & 0 \end{bmatrix}$, where $w = e^{ix}$, $z = e^{i\theta}$ for which $f(1)$ is the entropy of the Standard map and $f(0) = \log |\lambda/2|$. Because of the subharmonic property, $(2\pi)^{-1} \int_0^{2\pi} f(e^{i\theta}) \, d\theta \geq \log(\lambda/2)$. It would be nice to know the fluctuations of $f(z)$ on $|z| = 1$, or know the nature of the Riesz measure μ or the Riesz measures μ_n of the finite time Lyapunov exponents.

THE HOFSTADTER BUTTERFLY. In solid state physics, one is interested in the subharmonic function $f_\alpha(E) = \log |\det(L - EI_n)|$, where

$$L = \begin{bmatrix} \lambda \cos(\alpha) & 1 & 0 & \cdot & 0 & 1 \\ 1 & \lambda \cos(2\alpha) & 1 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 0 \\ 0 & \cdot & \cdot & 1 & \lambda \cos((n-1)\alpha) & 1 \\ 1 & 0 & \cdot & 0 & 1 & \lambda \cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move as a Bloch wave whenever the determinant is negative. These intervals form the **spectrum** of the quantum mechanical system. A physicist is interested in the rate of change of $f_\alpha(E)$ or its dependence on λ, when E is fixed.
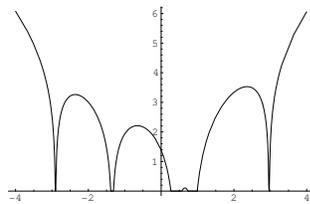
The graph to the left shows the function $E \mapsto \log(|\det(L - EI_n)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture to the right shows the spectrum of the crystal depending on α. It is called the "Hofstadter butterfly" made popular in the book "Gödel, Escher Bach" by Douglas Hofstadter.
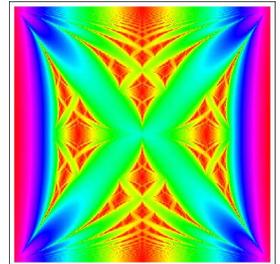
THE LYAPUNOV EXPONENT. The butterfly illustrates the spectrum in the case $\lambda = 2$. The function $f_\alpha(E)$ can be defined for all α, it is the Lypunov exponent of the matrix-valued map $A(\theta) = \begin{bmatrix} E - \lambda \cos(\theta) & 1 \\ -1 & 0 \end{bmatrix}$ over the dynamical system $x \mapsto x + \alpha$ on the circle $\mathbb{T}^1$. The complex parametrization with $z = e^{2\pi i\theta}$ gave a lower bound $\log(\lambda/2)$ for the Lyapunov exponent. A result of in the theory of Schrödinger operators derives from this that the operator has no absolutely continuous spectrum. The Riesz measure of the subharmonic map $f_\alpha(E)$ is called the **density of states**. It is a measure on the real line which has no absolutely continuous component.

OTHER ANALYTIC PARAMETRIZATIONS. Besides z or E, one can also complexity λ and compute the Lyapunov exponent of the matrix-valued map $A_\lambda(x) = \begin{bmatrix} E - \lambda g(x) & -1 \\ 1 & 0 \end{bmatrix}$ for a smooth function $g(x)$. The Lyapunov exponent $f(\epsilon)$ of $B_\epsilon(x) := \begin{bmatrix} \epsilon E + g(x) & -\epsilon \\ \epsilon & 0 \end{bmatrix}$ is $f(\epsilon) - \log |\epsilon| = f(\epsilon) + \log |\lambda|$. If $f(0) = \int_\mathbb{T} \log |g(x)| \, dx > -\infty$ like for $g(x) = \cos(x)$, and because subharmonic functions are continuous "almost everywhere", one can conclude that the Lyapunov exponent of A_λ is often positive and grows like $\log(C\lambda)$ for most λ. Actually, this is true for an arbitrary but **fixed** dynamical system and not only for the irrational rotation $x \mapsto x + \alpha$. Especially, for the Standard map T_κ. We know that for each fixed κ, that there is a set Y_κ of full Lebesgue density at ∞, such that for $\lambda \in Y_\kappa$, the matrix valued map A_λ over the system T_κ has positive Lyapunov exponent. But we would need to find a λ such that A_λ has positive Lyapunov exponent for T_λ. The argument showed that for large λ, a small mismatch of the coupling constant in the map and in the cocycle can lead to positive Lyapunov exponents - it could also indicate that numerics fool us.