THE GEOMETRY OF NETWORKS

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Abstract. We illustrate how geometric ideas can be applied to networks. The mathematics of networks has applications in computer graphics, polymer physics, nanotechnology, web tomography, social networks or loop quantum gravity. In mathematics it allows to look at old questions in a discrete setup and allow to reflect on the question what "geometry" is.

Networks

Mathematicians describe networks by **graphs**, structures of nodes which are hold together by **connections**. The individual nodes are called **vertices**. The connections between the nodes are called **edges**. In computer networks, the nodes are computers or routers and the connections are cables or wireless connections. In social networks, the nodes are people and the connections are friendships or family relations. In nature, the nodes are atoms or neurons, the connections are chemical bonds or synapses. Physicists like Feynman have used graphs to handle complex mathematical expressions, in spin network approaches to quantum gravity, graphs are models to describe space without running into divergence problems. Also artificial intelligence uses graphs for various talks, to solve problems, to analyze games like checkers or puzzles like the Rubic cube or find optimal strategies to find paths through a network or labyrinth.

Historically, the study of networks started with the birth of **topology**. It was Euler who lead the first foundations of graph theory, the problem of the "seven Bridges of Königsberg" was an optimization challenge. Since then, graph theory appears in all parts of mathematics like in combinatorics, probability, algebra, topology or analysis. Networks are also important in geometry as we will see in this talk. When we triangulate space, we get polyhedral structures which can be studied with combinatoric methods. Since its appearance at the birth crib of topology, network theory can serve as skeleton structures to describe geometries. It is also a tool in physics. There are flavors of fundamental physics like **loop quantum gravity** which views space as a spin network of quantized "spin foam". Along string theory, loop quantum gravity is

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a candidate for combining relativity with quantum mechanics. One of the advantages of discrete spaces is that one has not to worry about divergences and the approach is well suited for simulation in a computer, which is a discrete neural network itself.

The analysis of networks is a vibrant research area. Like "fractals", "complexity" or "chaos", "connected", "sync", "burst", "social graphs", "small worlds" are the terms which are hip. Mathematics is at the center of in this commotion. While networks are intrinsically discrete, ideas from calculus or geometry are pivotal in understanding networks. Why is the science of networks suddenly in fashion? It is not only that the world wide web, social networks, research on proteins, or nano-technology fuels the area, the subject also is extremely practical. The applications in computer science, operations research or the analysis of complex systems are piling up.

Here are examples:

- swarms of robots
- social networks
- network of business connections
- chemistry as networks of atoms
- computer graphics: networks to model surfaces
- biology as networks of biological components
- routing optimization problems in a street network
- parallel computing with a network of computers
- games and puzzles as Cayley graphs
- word connections and associations
- the brain as a neural network
- artificial intelligence questions like games or puzzles

A view on the bookshelf about networks show how much interest there is today in the subject:

Geometric ideas

Lets look at some notions in geometry and see what they could mean in networks:

Spheres What is the radius of "facebook", "linkedin", or "googleplus"? If the people I'm connected with have distance 1 and the union of all friends of my friends is the disc of radius 2. How does the number $B_r(x)$ grow. What is the radius r such that the ball $B_r(x)$ contains all people? One only recently realized that networks are usually small. We live in a small world. How many steps do we need to get from one place to an other? The maximal distance between two nodes is called the diameter of the

FIGURE 1. Some books related to the science of networks.

network. What is the diameter of the web or facebook? What is the diameter of the situation graph of a game like the Rubik cube? The later is called the "God number". It is known that it is 20. For smaller puzzles like the floppy one can count it: it is 6 for the "floppy" puzzle which consists of one layer of the Rubik cube.

Cliques Groups of people, where everybody is friend to each other are called **cliques**. A set of 3 people form a triangle. A group of 4 friends is a tetrahedron. A gang of 5 people is geometrically a four dimensional tetrahedron. You see that graphs allow to see higher dimensional spaces. Cliques are important to express geometric information like the Euler characteristic which for surfaces is the number of components minus the number of holes. We can now look at the number v_k of cliques generalizing the order $v_0 = |V|$ (number of people) and the size $v_1 = |E|$ (number of friendships). The "super cardinality" $\chi(G) = \sum_{k=0}^{\infty} (-1)^k v_k$ is called the Euler characteristic of G. It is an important quantity and is a topological invariant.

Dimension What is the dimension of the internet? The dimension of networks can be defined as in geometry: a point is d dimensional, if its unit sphere is $d-1$ dimensional. We can define the dimension of a graph as the average over all dimensions of a point. Graphs have in general a fractal dimension. It is interesting to measure dimension for random graphs or concrete networks. A fractal is a graph for which the dimension is constant everywhere but such that it is not an integer. The dimension in a graph can be defined inductively as 1 plus the dimension of the unit sphere. For a truncated cube the dimension is already a fraction. Graph theoretical fractals rarely have connections with fractals we usually consider like the Shirpinsky carpet. But the dimension is a metric which can be used in many different ways.

Curves A curve is a sequence of points in the graph such that successive points are connected by an edge. Of special interests are shortest connections. They are also called **geodesics**, shortest paths between two points. In graph theory, Eulerian or Hamiltonian paths are especially important. The cover the graph and minimize the number of edges or vertices which are crossed. Given a network, is there a path which visits every edge exactly once? This is called an Eulerian path. A closed Eulerian path is called an Eulerian circuit. An other problem is the problem to visit every vertex exactly once. A Hamiltonian cycle is a closed Hamiltonian path.

Symmetries Classically, one has classified geometries according to the symmetric which make up the object. In planimetry we distinguish for example transformations which preserve length and angles. This is a group of symmetries which allow translations, rotations or reflections. An larger group of transformations only asks angles and lines to be preserved. This allows for similarities. We can then look for a group

of transformations which preserve angles or the transformations which preserve lines. One can look at various symmetry groups on a network also. One can for example look at all the transformations of the network which preserves friends.
Topology is an area of mathematics which is also called "rubber geometry". A doughnut is topologically the same than a coffee cup because we can deform one into each other. The letter P can be deformed into the letter Q but not into B, nor into E. How can one define deformations of networks so that essential topological properties are preserved?

Spectra If we hit the membrane of a drum, we hear a sound. The possible frequencies are called the **spectrum** of the drum. We can look at the spectrum of geometric objects like regions in the plane, surfaces, more general manifolds. The spectrum encodes information about the geometric object, but many questions remain. One does not know whether the spectrum determines a convex drum for example. Spectra are also defined for networks. It is an active area of research. How much geometric information can we gain by knowing the spectrum? This question is very close to questions in quantum mechanics and related questions appear in very practical reasons. The "eigenvector" of the maximal spectral value of the world wide web network leads to the notion of "page rank". It is a billion dollar object, because that is the concept on which Google started its success.

Inverse problems Geometric spaces can be explored with light or sound because if we know the speed of the signal, we can use it to find distances. Light travels along the shortest possible path. These are the "lines" in the space. A fundamental result in geometry assures that lines between two points exist and are unique. What about in networks? This question is a bit more difficult because there are in general many shortest lines even if points are close. Light behaves more like in quantum mechanics in a network. The world wide web is so large that one can not measure it out completely or put the entire graph into the computer. In this respect it is a bit like the Rubik cube which is too large to be measured out directly. It is also similar to the problem to see into the inside of our body if only scans in different directions are known. An other analogy is the problem to map the geology below the earth surface using sonar methods. Researchers also have to rely on "tomographic methods" to probe the internet. Send some packages and see the backscatter.

Geodesy We can do geometry in a given graph. A good model for planimetry is the triangularization of the plane. We can also do geometry on a sphere like graph like an icosahedron, we have spherical geometry. Or doing geometry on a tree, where we have hyperbolic geometry. A hexagonal lattice graph allows to simulate planimetry. How do circles or ellipses look like in that graph? How would you define a notion of an ellipse in an arbitrary graph? A line is called a straight line, if it locally minimizes distance and is extended as much as possible. Already simple cases like a hexagonal graph show that shortest connections are not necessarily unique. Two lines are called parallel, if they do not intersect. On an icosahedron, there are no parallel lines. On the hexagonal graph, the Euclidean axioms hold. Given a point P and a line L, there is exactly one line through P which is parallel to L.

Randomness We can look at random subgraphs of a given host graph. What dimensions do we have in average? What is the average order of the unit ball? The average degree. What are the properties of a random function on a network? What happens if we walk randomly around in a graph? How frequently will we return? How many paths are there of length n? These are combinatorial questions which are important also in statistical mechanics.

Curvature The curvature of a vertex is $K(x) = 1 - V_0/2 + V_1/3 - V_2/4 + ...$ where V_k is the number of cliques with $k+1$ members in the circle of friends of x. For a tree, the curvature is $K(x) = 1 - d(x)/2$, where $d(x)$ is the number of friends. In a triangularization of a surface, $K(x) = 1 - d(x)/6$ where $d(x)$ is the number of friends. You see that in a triangular mesh, the curvature is zero if there are 6 neighbors, that the curvature is positive if there are less than 6 neighbors and that the curvature is negative if there are more than 6 neighbors. Positive sectional curvature everywhere means that whenever you see a wheel graph inside the graph, then this wheel graph has maximally 5 spikes. One can say more for geometric graphs: in a 5 dimensional graph for example every unit sphere is 4 dimensional etc. In that case, $K(x) = 1 - V(x)/2 + E(x)/3 - F(x)/4 + S(x)/5 - H(x)/6$ where V the number of vertices, E the number of edges, F the number of triangles, S the number of tetrahedra and H the number of 4D hypertetraedra in the unit sphere $S(x)$.

Calculus A function on V plays the role of a function. The derivative of a function f at a point x in the direction y is $f_y(x) = f(y) - f(x)$. In the discrete derivatives become "differences". Many rules stay the same and one can indeed do calculus as we do it in Euclidean space. The analogies are sometimes surprising. Here is a result which was obtained only recently: [8] one can assign an integer index to each nodes x so that the sum over the graph is the Euler characteristic $\chi(G)$. This index is defined as $1 - \chi(M(x))/2$ where $M(x)$ is the subgraph of the sphere $S(x)$ where $f(y)$ is smaller than $f(x)$. If we add up the indices over a network, we get again the Euler characteristic. This is called the Poincaré-Hopf theorem in geometry. One can define a curvature in a general graph so that adding up all curvatures gives the Euler characteristic. Curvature tells how the graph is "bent" but does not depend on the way the graph is drawn. Curvature depends on the number of neighboring vertices, edges, triangles etc. For a graph which does not contain tetrahedra for example, the curvature is $K(x) = 1 - V(x)/2 + E(x)/3$, where $V(x)$ is the number of vertices in the unit sphere and $E(x)$ is the number of edges in the unit sphere. The Euler characteristic is $v - e + f$ where v is the number of vertices and e the number of edges and f the number of triangles in the entire graph. See [7].

Geometric graphs

1. Literature

[1] [2]

Here are some books from my library: popular books: [14, 15, 1, 12], for networks: [10, 9, 13, 6] and for graphs: [2, 3] and for for polyhedra: [11, 5, 4].

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[1] Answer to Problem 2: v=32,e=35,f=2, chi = -1

[2] The curvatures are $-5/3, -7/6, -1, -1, -1, -1, -1, -2/3, -2/3, -2/3, -1/2, -1/6, 0, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2,$ and add up to -1.

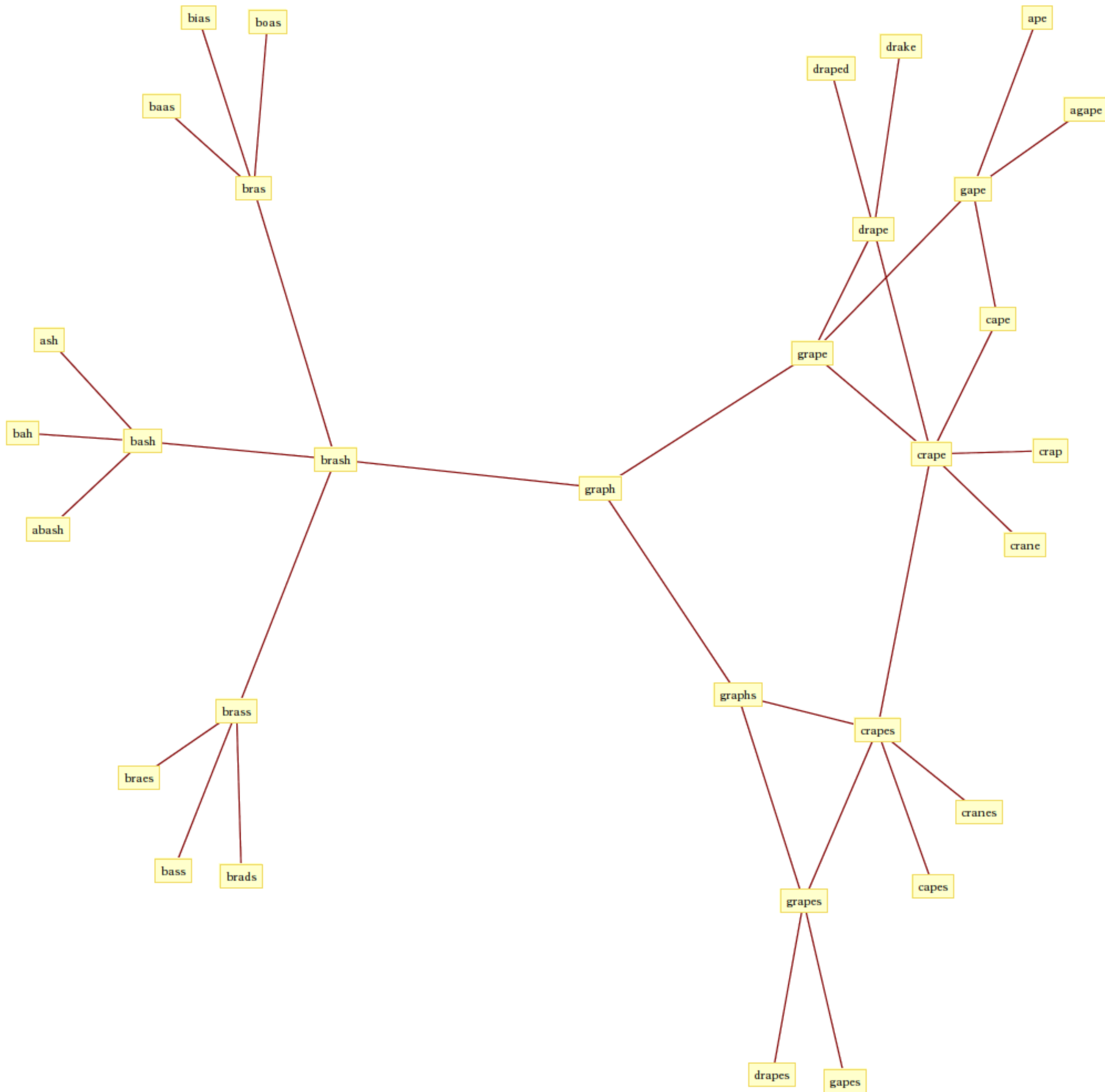

FIGURE 2. Lets look at the words in the neighborhood of the word "graph". **Problem 1:** Can you find the circles of radius 2 of "grape" and "drapes". What is the intersection of these two circles? **Problem 2:** Find the Euler characteristic $\chi(G) = v - e + f$, where v is the number of vertices, e the number of edges and f the number of triangles. **Problem 3:** Compute the curvature $K(x) = 1 - |V(x)|/2 + |E(x)|/3$ at each point where $|V|$ is the number of edges and $E(x)$ is the number of vertices in $S(x)$. Add them up and see whether they agree with $\chi(G)$. This is Gauss-Bonnet.