The oldest open problem of mathematics

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at Math Circle, Northeastern,
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Perfect numbers

Perfect number

the sum of the proper divisors is the number itself.

6 = 1 + 2 + 3

28 = verify yourself

496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248

Why relevant?

Related to largest prime numbers known.

$2^{p-1}(2^p - 1)$ perfect

$2^p - 1$ prime

Mersenne prime

p	*digits*	*discovered*	*mersenne*
32582657	9808358	2006	44?
30402457	9152052	2005	43?
25964951	7816230	2005	42?
24036583	7235733	2004	41?
20996011	6320430	2003	40?
13466917	4053946	2001	39

Open problems

Is there an odd perfect number?

Are there infinitely many perfect numbers?

No other open problem in mathematics is older. The second problem is thousands of years old, the first might too but has been written down by Dequartes.

The sigma function

$$\sigma(n) = \sum_{d|n} d$$

d|n means
d divides n

p,q are prime

$$\sigma(p) = 1+p$$

$$\sigma(p^n) = 1+p+p^2 + \dots p^n$$

$$\sigma(p^n q^m) = (1+p+p^2+\dots+ p^n)(1+p+p^2+\dots +p^m)$$

$$\sigma(n) = \prod_{i=1}^{k} (1+p_i+p_i^2+\dots +p_i^{m_i})$$

this is a multiplicative function:

$$\sigma(ab) = \sigma(a)\,\sigma(b)$$

if a,b have no nontrivial common divisor.

$$h(n) = \frac{\sigma(n)}{n} = 2$$

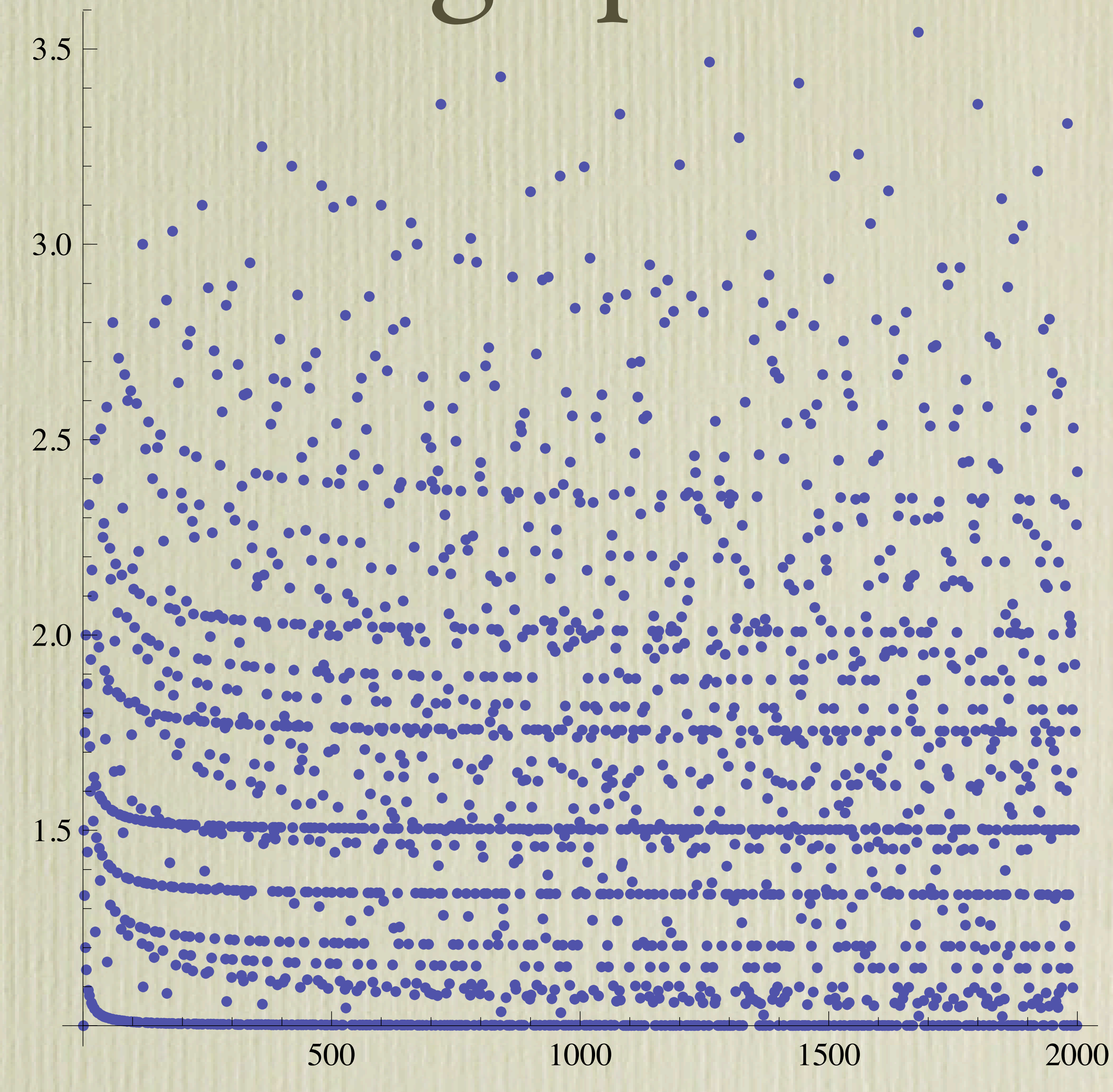
index function

is also multiplicative

$$h(n) = \prod_{i=1}^{k} \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \dots + \frac{1}{p_i^{m_i}}\right)$$

$$h(p) = 1 + \frac{1}{p}$$

graph of h



can be close to 1
can also become arbitrarily large because

$$\prod (1+1/p)$$

p prime

diverges.

Nocco:

An odd perfect number has more than 2 prime factors.

Proof:

use the geometric series formula. A prime factor can contribute only up to 1/(1-1/p) = p/(p-1)

$$h(n) < \frac{p}{p-1} \frac{q}{q-1}$$

$$< \frac{3}{3-1} \frac{5}{5-1} = 15/8 < 2$$

How close can we get to 2 with k prime factors?

Nice research project in experimental mathematics!

Descartes example

$$n = 3^2 7^2 11^2 13^2 \, 22021$$

$$(1 + 3 + 3^2)(1 + 7 + 7^2)(1 + 11 + 11^2)(1 + 13 + 13^2)(1 + 22021) = 2n$$

We have found an odd perfect number, didn't we?

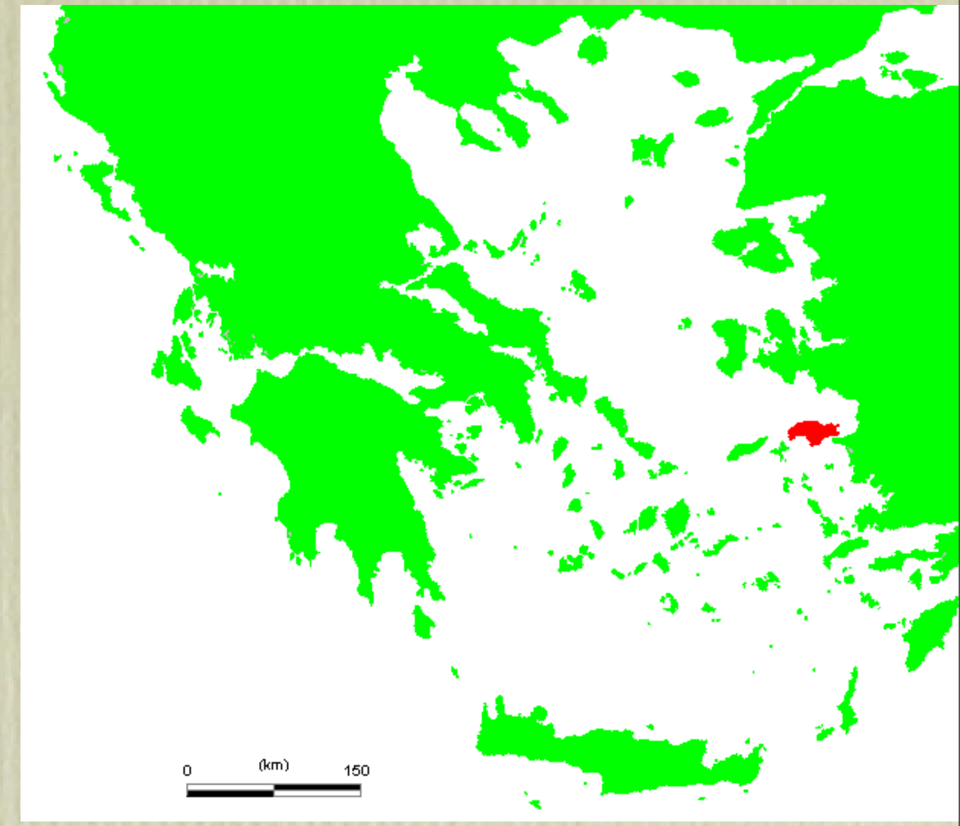
... I think I am able to prove that there are no even numbers which are perfect apart from those of Euclid; and that there are no odd perfect numbers, unless they are composed of a single
prime number, multiplied by a square whose root is composed of several other prime number.

But I can see nothing which would prevent one from finding numbers of this sort. For example, if 22021 were prime, in multiplying it by 9018009 which is a square whose root is composed of the prime numbers 3, 7, 11, 13, one would have 198585576189, which would be a perfect number.

But, whatever method one might use, it would require a great deal of time to look for these numbers...

Mathematicians

Pythagoras



- 580-500 BC, born in Samos
- Mentions perfect numbers

Euclid of Alexandria

- 300-275 BC
- Found structure of even perfect numbers

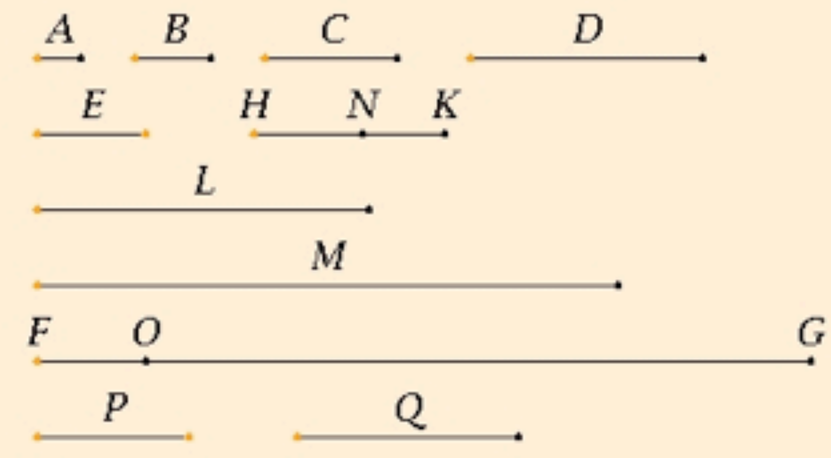
Euclid's Elements
Book IX
Proposition 36

If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes pri sum multiplied into the last makes some number, then the product is perfect.

Let as many numbers as we please, *A*, *B*, *C*, and *D*, beginning from a unit be set out in double proportion, until the sum of all becomes prime, let *E* ec the sum, and let *E* multiplied by *D* make *FG*.

I say that *FG* is perfect.

For, however many *A*, *B*, *C*, and *D* are in multitude, take so many *E*, *HK*, *L*, and *M* in double proportion beginning from *E*.



Therefore, *ex aequali* *A* is to *D* as *E* is to *M*. Therefore the product of *E* and *D* equals the product of *A* and *M*. And the product of *E* and *D* is *FG*, there the product of *A* and *M* is also *FG*.

Therefore *A* multiplied by *M* makes *FG*. Therefore *M* measures *FG* according to the units in *A*. And *A* is a dyad, therefore *FG* is double of *M*.

But *M*, *L*, *HK*, and *E* are continuously double of each other, therefore *E*, *HK*, *L*, *M*, and *FG* are continuously proportional in double proportion.

Subtract from the second *HK* and the last *FG* the numbers *HN* and *FO*, each equal to the first *E*. Therefore the excess of the second is to the first as excess of the last is to the sum of those before it. Therefore *NK* is to *E* as *OG* is to the sum of *M*, *L*, *KH*, and *E*.

And *NK* equals *E*, therefore *OG* also equals *M*, *L*, *HK*, *E*. But *FO* also equals *E*, and *E* equals the sum of *A*, *B*, *C*, *D* and the unit. Therefore the whole F equals the sum of *E*, *HK*, *L*, *M*, *A*, *B*, *C*, *D*, and the unit, and it is measured by them.

I say also that *FG* is not measured by any other number except *A*, *B*, *C*, *D*, *E*, *HK*, *L*, *M*, and the unit.

If possible, let some number *P* measure *FG*, and let *P* not be the same with any of the numbers *A*, *B*, *C*, *D*, *E*, *HK*, *L*, or *M*.

And, as many times as *P* measures *FG*, so many units let there be in *Q*, therefore *Q* multiplied by *P* makes *FG*.

But, further, *E* multiplied by *D* makes *FG*, therefore *E* is to *Q* as *P* is to *D*.

And, since *A*, *B*, *C*, and *D* are continuously proportional beginning from a unit, therefore *D* is not measured by any other number except *A*, *B*, or *C*.

Nicomachus of Gerasa

- 60-120 AD
- Introduction to Arithmetic
- Abundant and deficient numbers

ΝΙΚΟΜΑΧΟΥ ΓΕΡΑΣΗΝΟΥ

ΠΥΘΑΓΟΡΙΚΟΥ

ΑΡΙΘΜΗΤΙΚΗ ΕΙΣΑΓΩΓΗ.

NICOMACHI GERASENI PYTHAGOREI

INTRODVCTIONIS ARITHMETICAE LIBRI II.

RECENSVIT

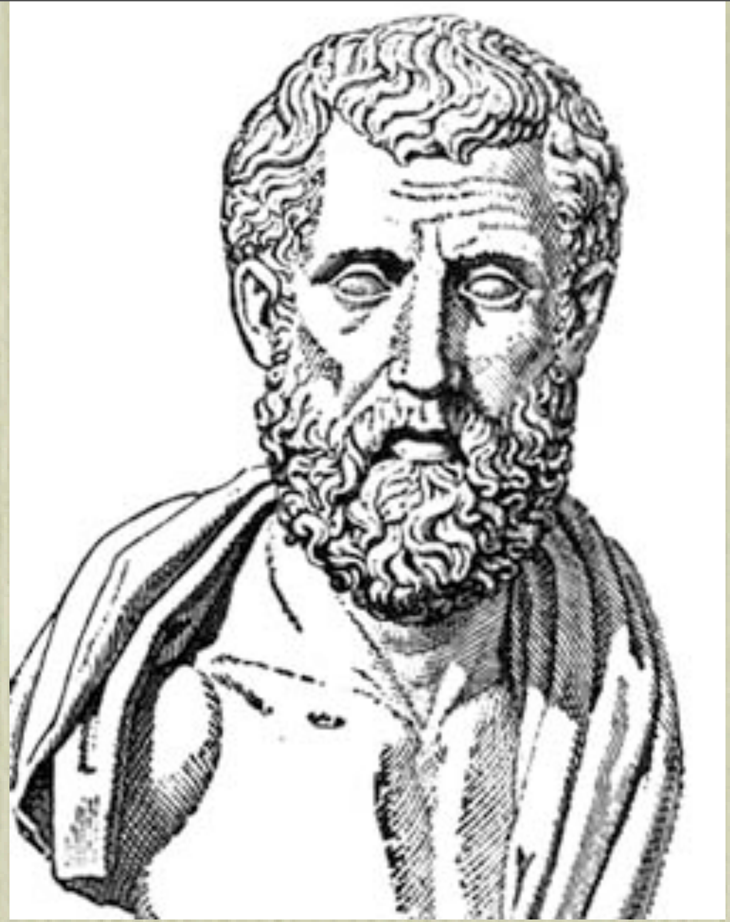
RICARDVS HOCHE.

ACCEDVNT CODICIS CIZENSIS PROBLEMATA ARITHMETICA.

LIPSIAE

IN AEDIBVS B. G. TEVBNERI.

MDCCCLXVI.



Theon of Smyrna

- 70-135 AD
- Music of Spheres
- Abundant and deficient numbers

ליהואחז עם כי אם־חמשים פרשים ועשרה רכב ועשרת אלפים
8 רגלי כי אבדם מלך ארם וישמם כעפר לדש׃ 8 ויתר
דברי יהואחז וכל־אשר עשה וגבורתו הלוא־הם כתובים על־ספר
9 דברי הימים למלכי ישראל׃ 9 וישכב יהואחז עם־אבתיו ויקברהו
בשמרון וימלך יואש בנו תחתיו׃ פ
10 10 בשנת שלשים ושבע שנה ליואש מלך יהודה מלך יהואש
11 בן־יהואחז על־ישראל בשמרון שש עשרה שנה׃ 11 ויעשה הרע
בעיני יהוה לא סר מכל־חטאות ירבעם בן־נבט אשר־החטיא
12 את־ישראל בה הלך׃ 12 ויתר דברי יואש וכל־אשר
עשה וגבורתו אשר נלחם עם אמציה מלך־יהודה הלוא־הם
13 כתובים על־ספר דברי הימים למלכי ישראל׃ 13 וישכב יואש
עם־אבתיו וירבעם ישב על־כסאו ויקבר יואש בשמרון עם מלכי
14 ישראל׃ פ 14 ואלישע חלה את־חליו אשר ימות בו וירד
אליו יואש מלך־ישראל ויבך על־פניו ויאמר אבי | אבי רכב
15 ישראל ופרשיו׃ 15 ויאמר לו אלישע קח קשת וחצים ויקח אליו
16 קשת וחצים׃ 16 ויאמר | למלך ישראל הרכב ידך על־הקשת
17 וירכב ידו וישם אלישע ידיו על־ידי המלך׃ 17 ויאמר פתח החלון
קדמה ויפתח ויאמר אלישע ירה ויור ויאמר חץ־תשועה ליהוה
18 וחץ תשועה בארם והכית את־ארם באפק עד־כלה׃ 18 ויאמר
קח החצים ויקח ויאמר למלך־ישראל הך־ארצה ויך שלש־
19 פעמים ויעמד׃ 19 ויקצף עליו איש האלהים ויאמר להכות חמש
או־שש פעמים אז הכית את־ארם עד־כלה ועתה שלש פעמים
20 תכה את־ארם׃ ס 20 וימת אלישע ויקברהו וגדודי מואב
21 יבאו בארץ בא שנה׃ 21 ויהי הם | קברים איש והנה ראו את־

[9] Mm 2120. [10] Mm 3025. [11] Mm 2178. [12] Mm 1748. [13] Mm 3071. [14] Mm 87. [15] Mm 2121. [16] Mm 2122. [17] Mm 1955.

7 b 𝔊L huc tr 23 ‖ **10** a–a 𝔊127 40, 𝔊245 36; 𝔊N min 39, recte? ‖ b–b 𝔖 13 ‖ **11** a 2 Mss (vid) et Vrs suff pl ‖ **14** a ut 2,12b ‖ **18** a–a > 𝔊107𝔖, dl? ‖ **20** a Ms 𝔗 sg ‖ b 2 Mss יבאו, 𝔗 sg ‖ c–c inc; 𝔊 ἐλθόντος τοῦ ἐνιαυτοῦ cf 𝔗, 𝔖 bh bšnt' hj eodem anno cf 𝔙 ‖ **21** a Seb ויהיו.

Dickson mentions Kings II, 13,19

[9] Elementa, liber IX, prop. 36. Opera, 2, Leipzig, 1884, 408.
[10] S. Rubin, "Sod Hasfiroth" (secrets of numbers), Wien, 1873, Kings, II, 13, 19.

but even the Hebew text does not seem to mention perfect numbers.
Some religious context is because 6 is the number of days in which god created the world.

Some astronomical context comes from the fact that the moon month is 28 days.

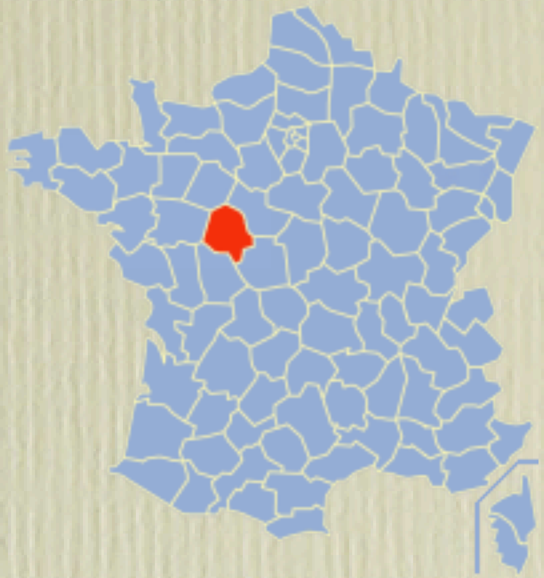
Thabit ibn Qurra

- 836-901 AD
- Amicable numbers

Marin Mersenne

- 1588-1648
- tried to get formula for all primes
- studied primes of the form $2^p - 1$

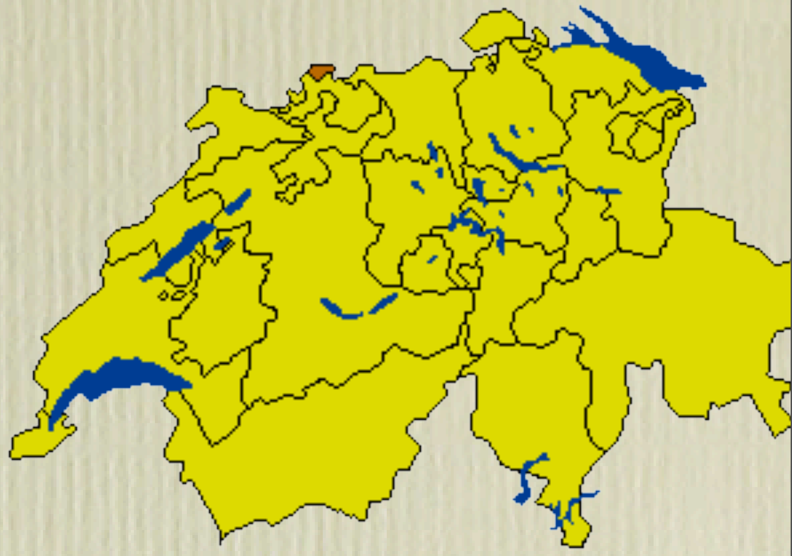
Rene Descartes



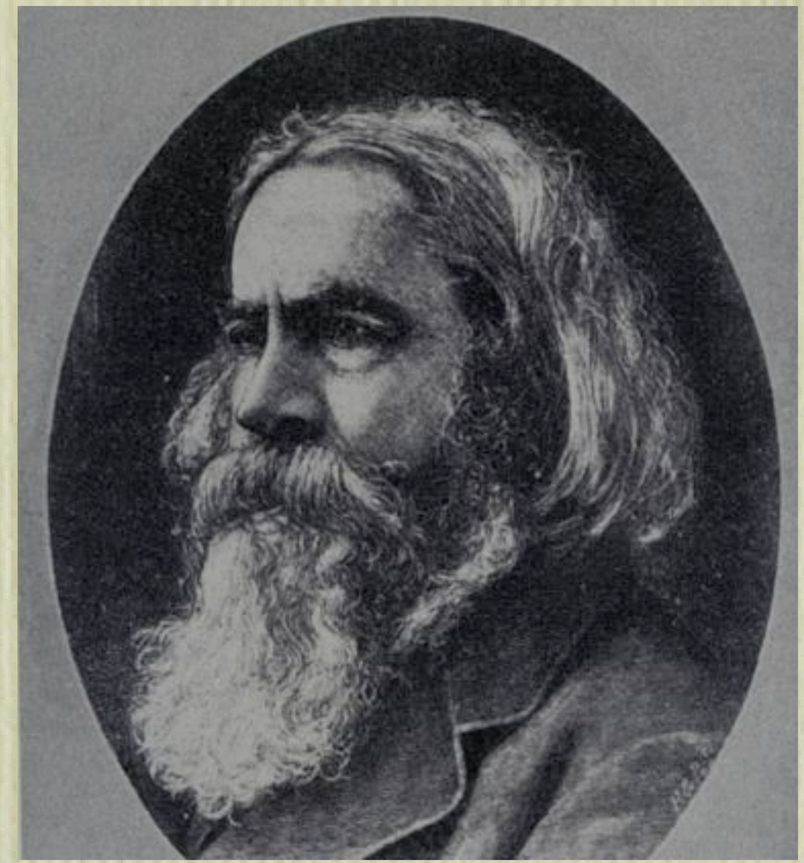
- 1596-1650
- almost perfect numbers
- 4'th and 5'th perfect number

6,28,140,270,496,672 ...

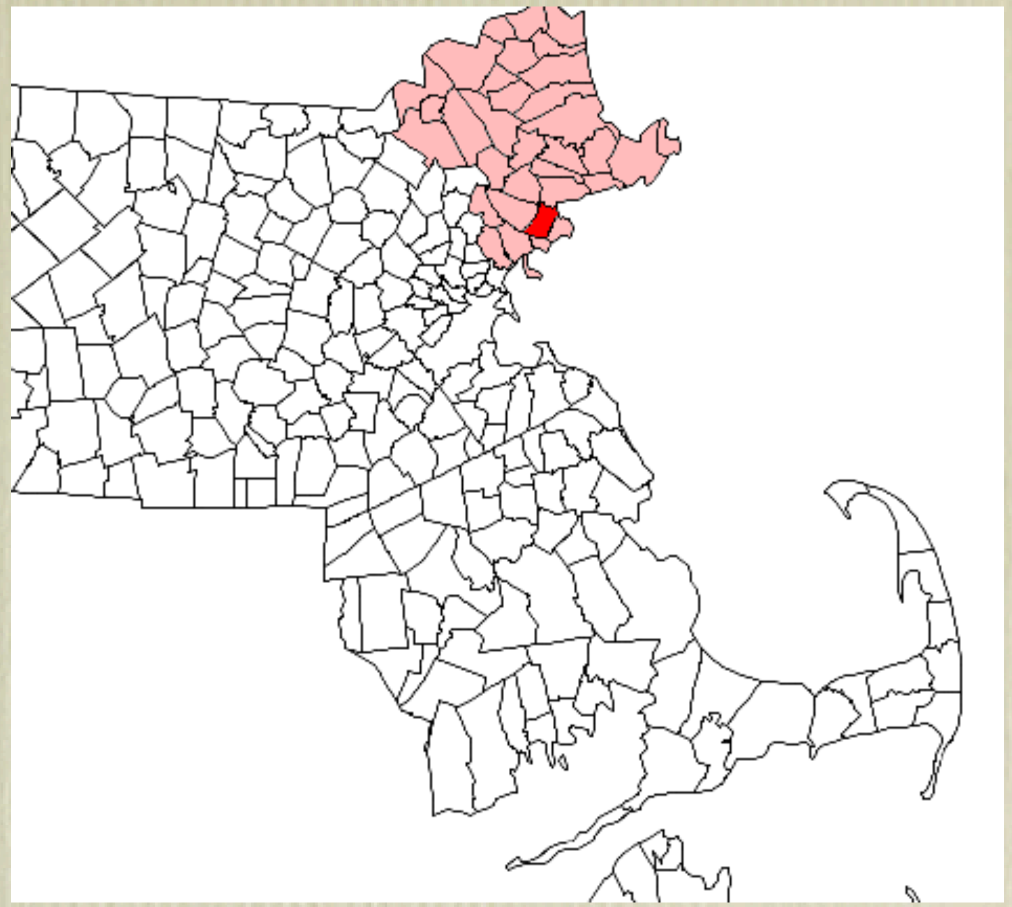
Leonard Euler



- 1707-1783
- classification of even perfect numbers
- odd perfect numbers are product of special prime and square.



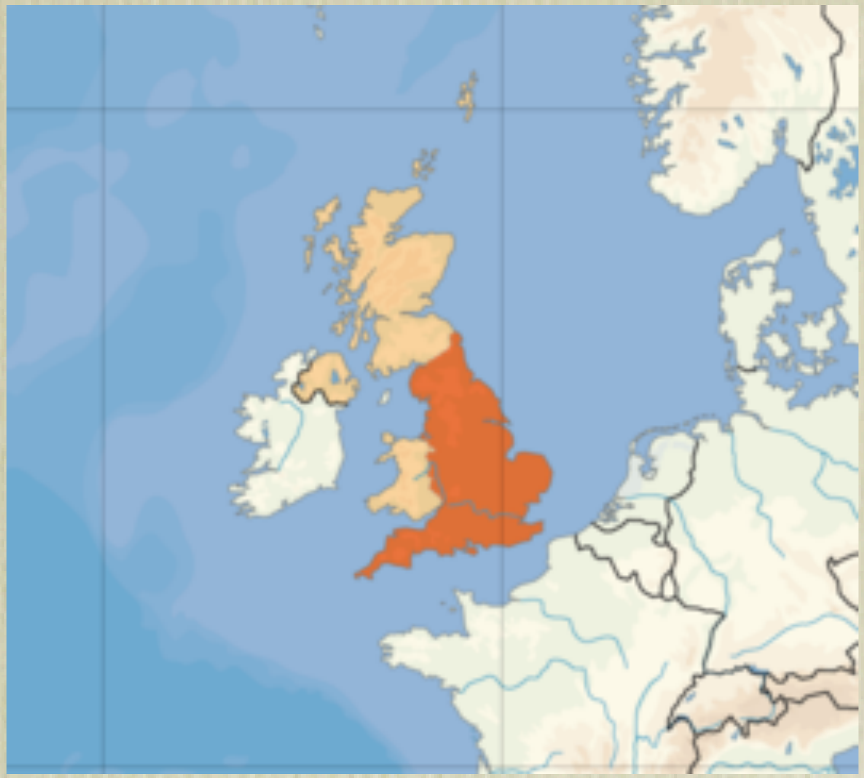
Benjamin Peirce



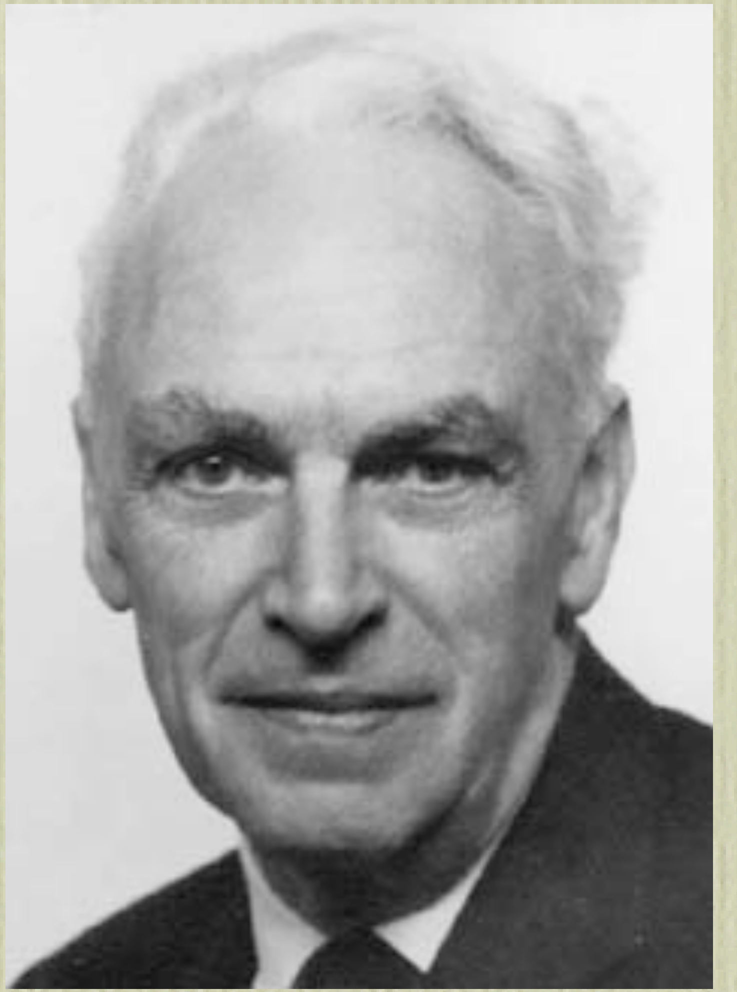
A local!

- 1809-1880
- At least 4 prime factors

James Joseph Sylvester



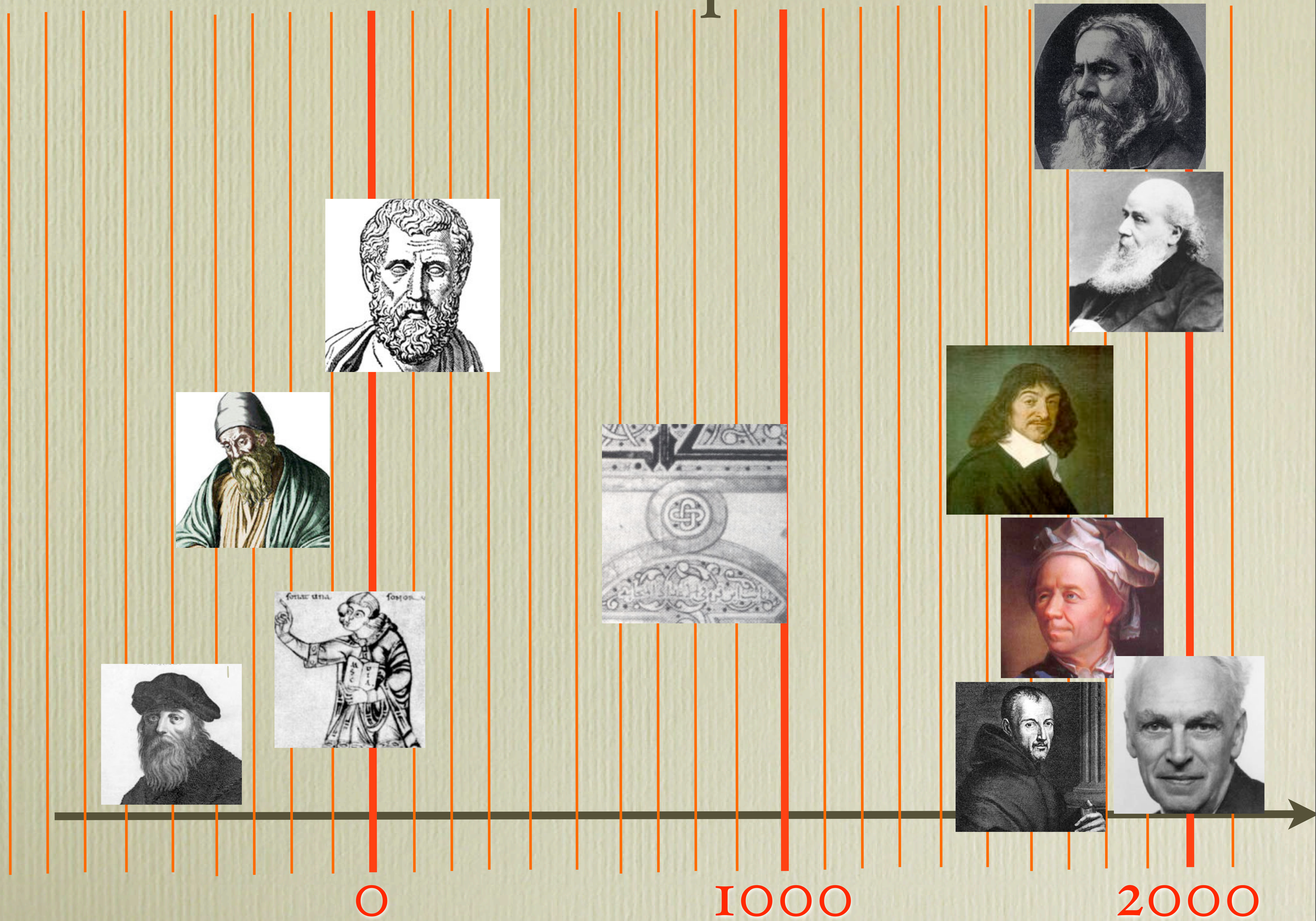
- 1814-1897
- Series of papers



Oystein Ore

- 1899-1968
- Harmonic integers

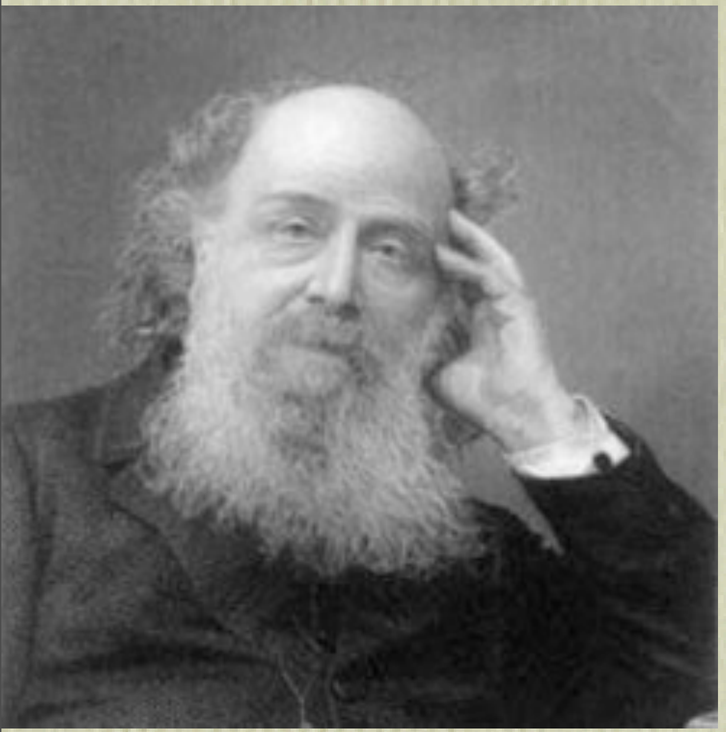
Time line comparison



Quotes

Leonard Euler:

Whether ... there are any odd perfect numbers is a most difficult question.



J. J. Sylvester

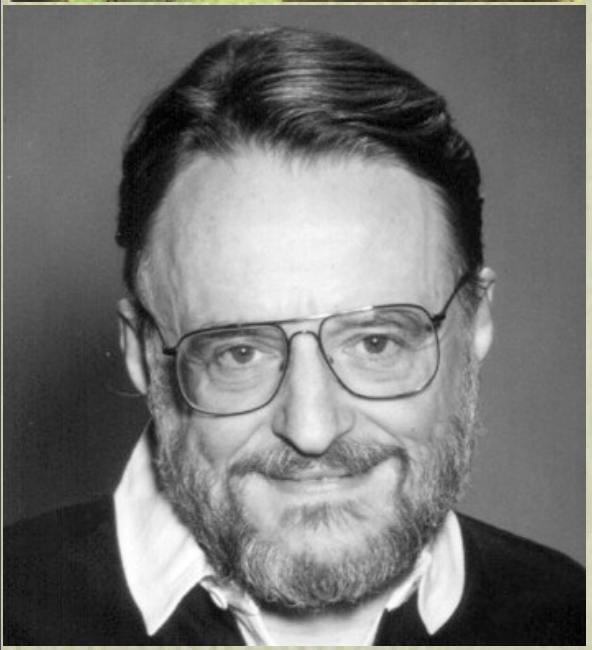
"The existence of an odd perfect number -- its escape, so to say, from the complex web of conditions which hem it in on all sides -- would be little short of a miracle."

John Edensor Littlewood

"Perfect numbers certainly never did any good but then they never did any perticular harm"

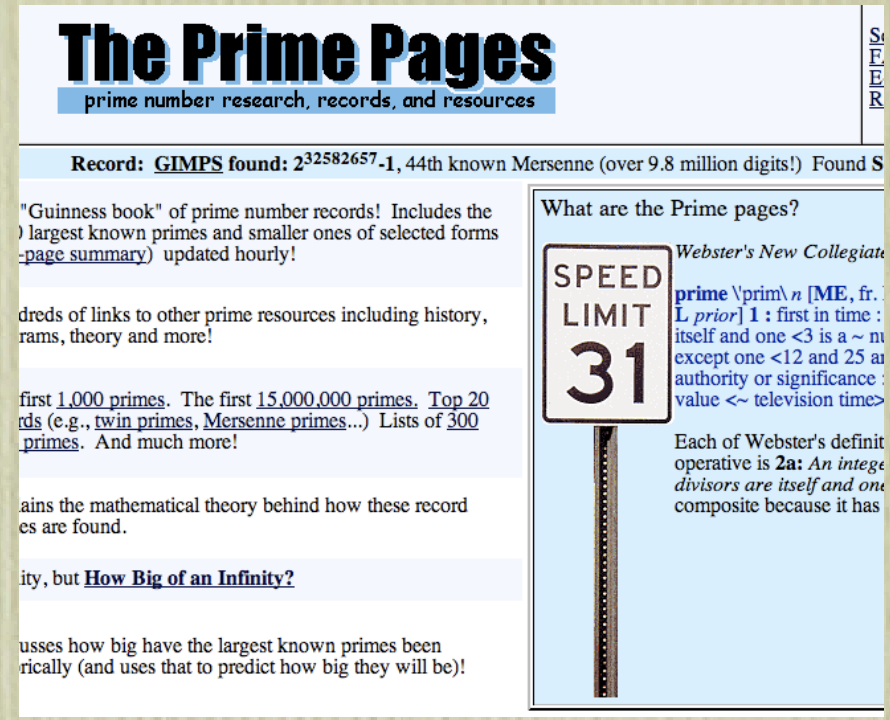
Stan Wagon

"Maybe some simple combination of a dozen or so primes in fact yield an odd perfect number!"



Conway and Guy

“There probably aren’t any!”



Chris Caldwell (University of Tennessee at Martin)

"This is probably the oldest unsolved problem in all of mathematics."

Perfect Numbers

T. M. Putnam

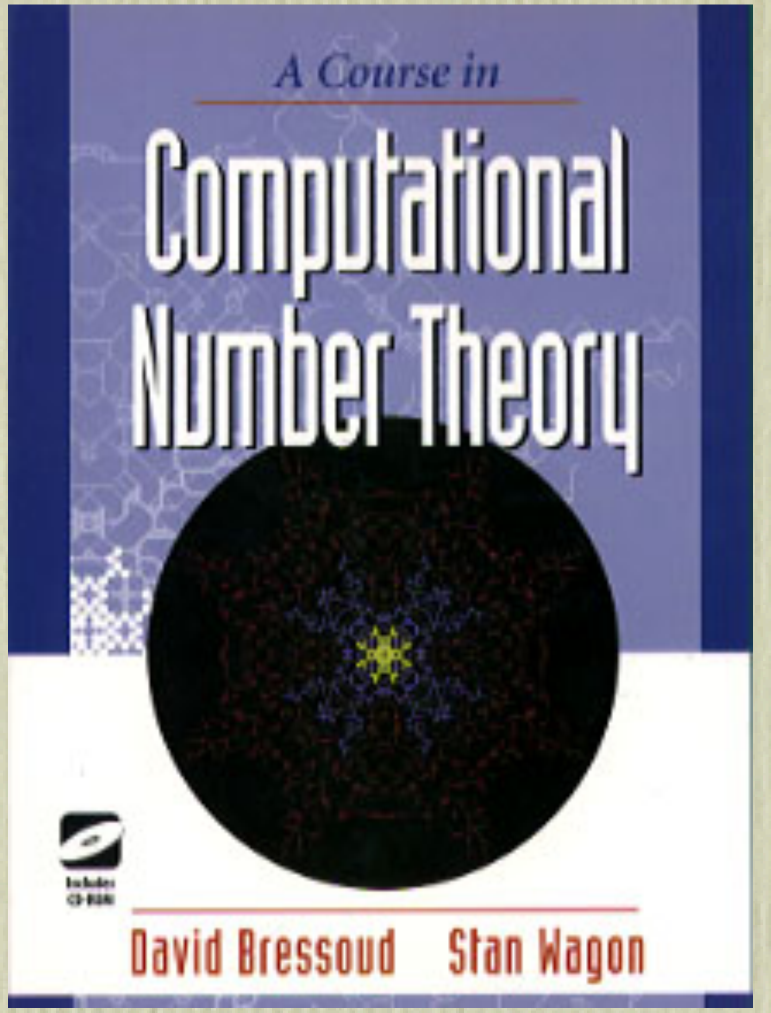
The American Mathematical Monthly, Vol. 17, No. 8/9. (Aug. - Sep., 1910), pp. 165-168.

Stable URL:
http://links.jstor.org/sici?sici=0002-9890%28191008%2F09%2917%3A8%2F9%3C165%3APN%3E2.0.CO%3B2-I

The American Mathematical Monthly is currently published by Mathematical Association of America.

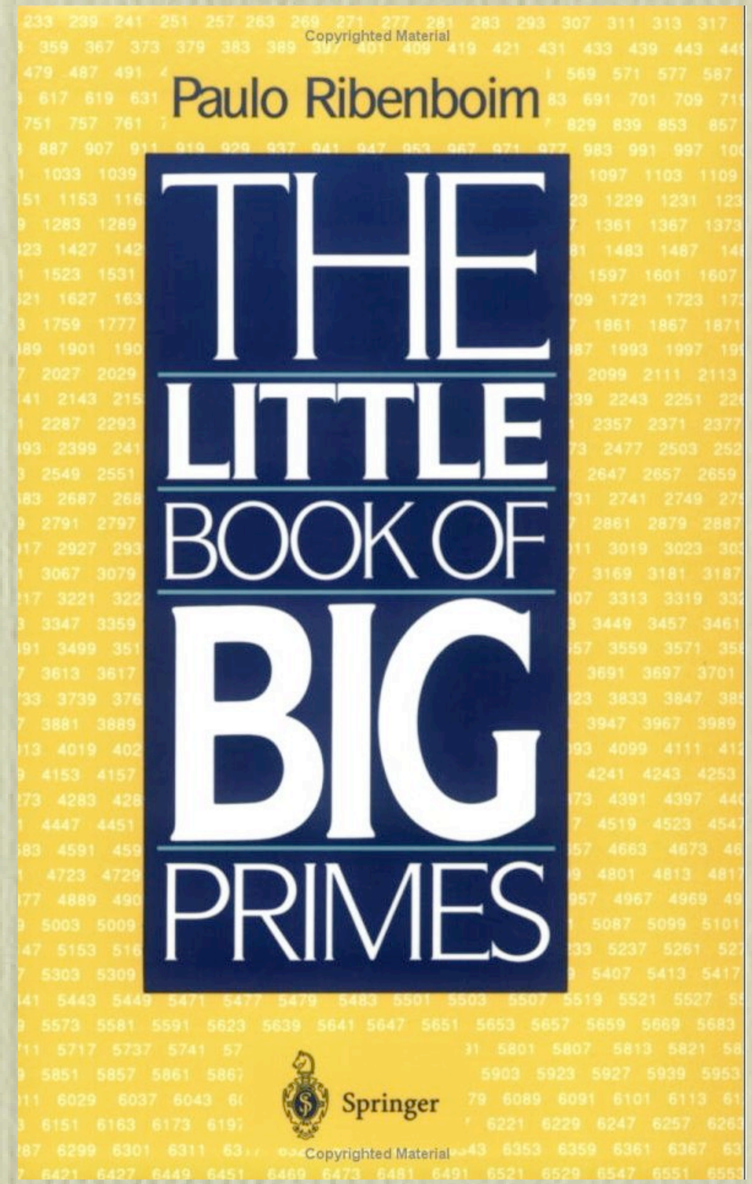
T.M. Putnam in 1910:

"It is a problem of much historic interest."



Bressoud and Wagon

"It is one of the oldest unsolved mysteries of mathematics, as it goes back to the ancient Greeks."



Ribenboim

"I think the problem stands like an unconquerable fortress."

Even perfect numbers

Theorem (Euclid-Euler)

n is an even perfect number if and only if

$$n = 2^{p-1}(2^p - 1)$$

where $2^p - 1$ is prime.

$2^p - 1$ is called a Mersenne prime.

Proof by Stan Wagon in Intelligencer:

There is a one–one correspondence between even perfect numbers and Mersenne primes.

Suppose $2^n - 1$ is prime. Then $\sigma(2^{n-1}(2^n - 1)) = \sigma(2^{n-1})\,\sigma(2^n - 1) = (1 + 2 + \ldots + 2^{n-1})\,2^n = 2 \cdot 2^{n-1}(2^n - 1)$. Conversely (this proof is due to Dickson), suppose $2^{n-1}m$ is perfect, where $n > 1$ and m is odd. Then $2^n m = \sigma(2^{n-1}m) = (2^n - 1)\sigma(m)$, whence $\sigma(m) = m + m/(2^n - 1)$. But then both m and $m/(2^n - 1)$ are integers dividing m, so the expression for $\sigma(m)$ yields that $m/(2^n - 1) = 1$ and m has no proper divisors. Thus $m = 2^n - 1$ is prime, as desired.

The currently largest known prime number has 9,808,358 digits. The Electronic Frontier Foundation offeres a 100'000 award for the first 10 million digit prime.
The GIMPS project is working on that. A new prime is now expected any moment.

Getting Started
Main page
How it works
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Prizes
Project Status
Status
Top producers
PrimeNet

GIMPS

The Great Internet Mersenne Prime Search

Finding 10 World Record Primes!

2^P-1 MAY BE PRIME!

Learning More
History
The math
Source code
Mailing list
Miscellaneous
Manual testing
Credits
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Search Status

September 2006: New Mersenne Prime!

The page summarizes the current search status for Mersenne numbers with exponents below 79,300,000. The PrimeNet server also has a stats page updated every hour.

It can easily be proven that a Mersenne prime must have a prime number as an exponent. This table summarizes the current search status of Mersenne numbers with prime exponents.

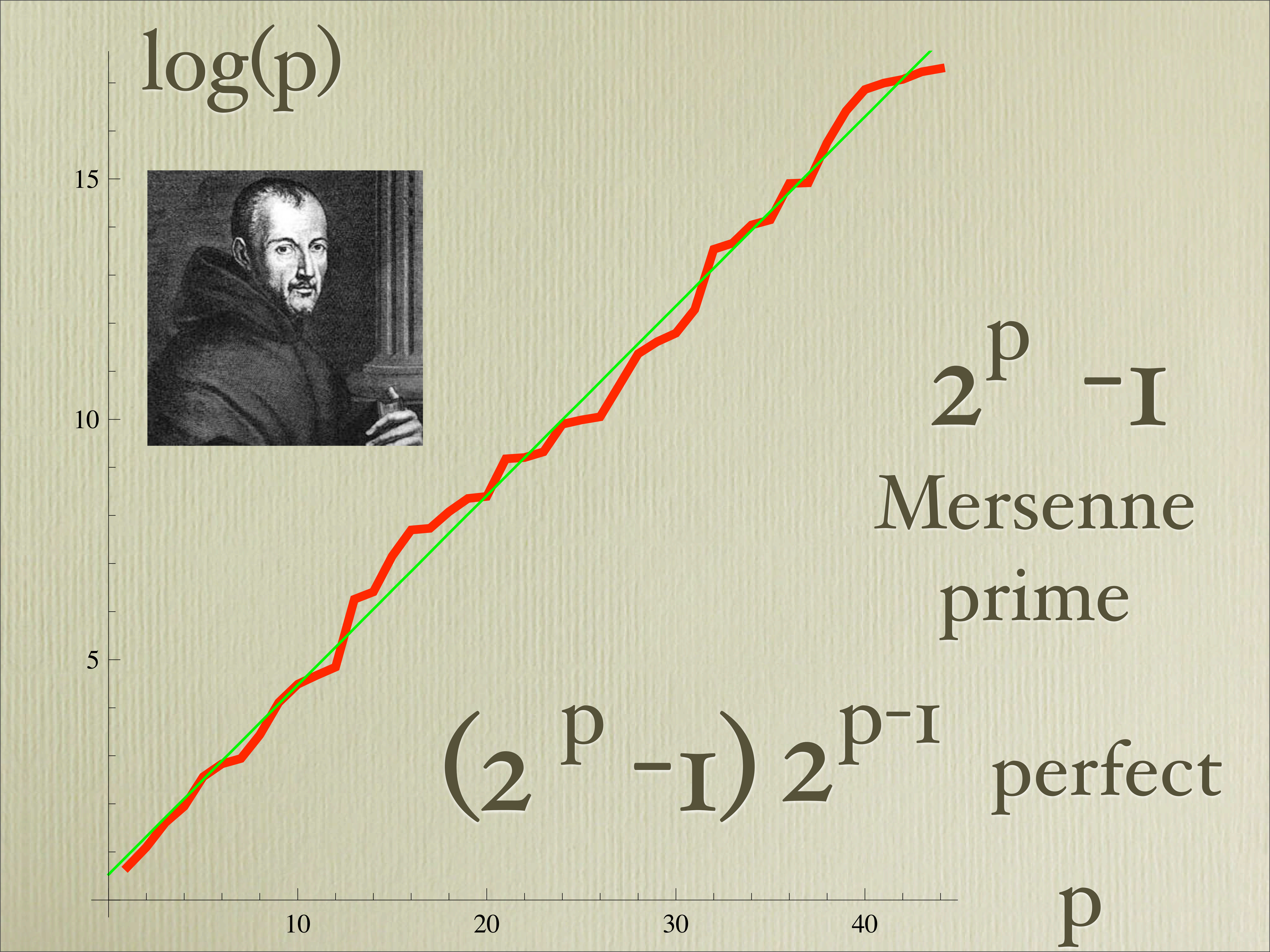
Range		Mersenne		Composite			Status Unknown	Expected New Primes	P-90* CPU Years	PII-400 Speed (sec.)	FFT** Size (in K)
Low	High	Numbers	Primes	Factored	TwoLL	OneLL					
0	15,300,000	988,851	39	627,366	361,446	0	0	0	0	N/A	Many
15,300,000	17,850,000	153,447	0	96,036	55,490	1,921	0	0.0002	0	0.536	896
17,850,000	20,400,000	152,058	0	95,591	37,998	18,464	5	0.0019	10	0.600	1024
20,400,000	25,350,000	292,031	2	184,100	2,292	105,398	239	0.01	756	0.776	1280
25,350,000	30,150,000	280,258	1	175,625	1,999	99,864	2,769	0.02	12,906	0.934	1536
30,150,000	35,100,000	285,825	2	179,010	2,228	95,470	9,115	0.04	56,703	1.113	1792
35,100,000	40,250,000	295,432	0	184,190	639	66,158	44,445	0.14	367,289	1.226	2048
40,250,000	50,000,000	553,232	0	327,241	2	387	225,602	0.56	2,911,408	1.640	2560
50,000,000	59,400,000	527,413	0	300,868	1	7	226,537	0.44	4,301,164	1.990	3072
59,400,000	69,100,000	539,666	0	303,873	1	13	235,779	0.39	6,288,396	2.380	3584
69,100,000	79,300,000	562,700	0	318,804	0	8	243,888	0.36	8,217,629	2.604	4096
	Total	4,630,913	44	2,792,704	462,096	387,690	988,379	1.96	22,156,262		
79,300,000			0		1	1					>4096

*The time it would take one 90 MHz Pentium computer to run a first-time LL test on the remaining exponents. Further factoring will reduce this estimate. One PII-400 CPU year equals 5.5 P-90 CPU years.

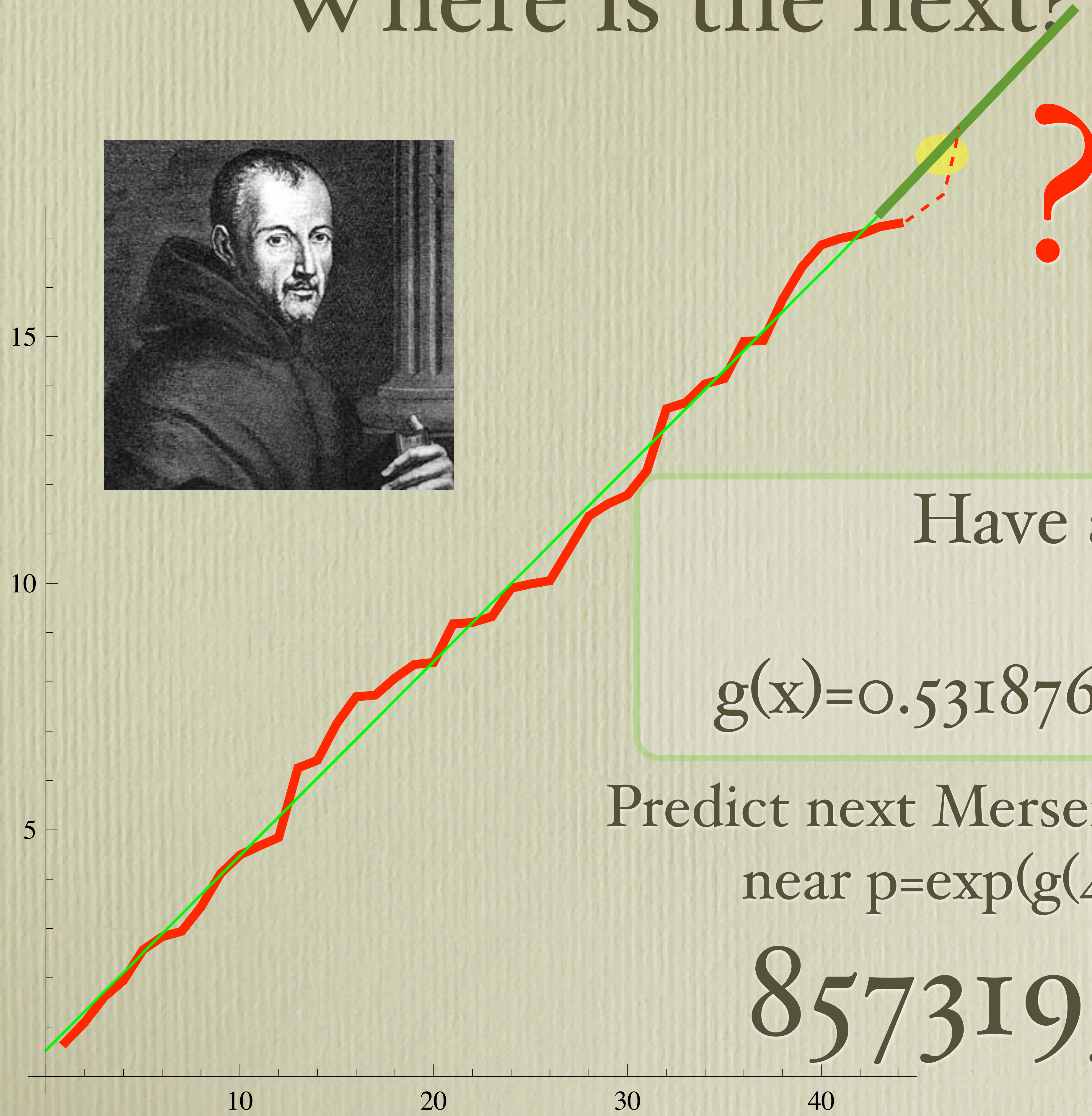

100’000 dollars

Why not try?

Lets see what a linear extrapolation from previous data tells us on where the next prime will be.


log(p)
15
10
5
10
20
30
40
p
2^p -1
Mersenne prime
(2^p -1) 2^(p-1) perfect

Where is the next?



Have a line:

g(x)=0.531876+ 0.394198 x

Predict next Mersenne prime near p=exp(g(45)) =

85731939 p

How large do we expect the next Mersenne prime to be?

Predict next Mersenne prime with p near exp(g(45)) = 85731939

Next has 25.8 Million digits

85731913 is closest prime

How does GIMPS search for Mersenne primes?

Lucas - Lehmer test:

Define $S(0)=4$, $S(n+1) = S(n)^2-2$

$M(n) = 2^n-1$ is prime

$\Updownarrow$

$M(n)$ divides $S(n)$

A mystery!
(with an easy proof)

All the Mersenne primes the sum of odd consecutive cubes. See Pickovers book: wonders of numbers/

$$28 = 1 + 3^3$$

$$496 = 1 + 3^3 + 5^3 + 7^3$$

etc. There is a very easy explanation for that. Can you find it?

Odd perfect numbers

Theorem (Euler)

If X is an odd perfect number then

$$x = p\, q^2$$

where p is a prime and q is an integer.

p is called the special prime.

What else is known?

- n must have at least 300 digits.
- n must have a prime power factor with at least 12 digits and a prime factor with at least 8 digits.
- n gives rest 1 when divided by 12 or rest 9 when divided by 36. (this is easy to prove)
- if there are k distinct prime factors then $n<4^{4^k}$

Ore harmonic numbers

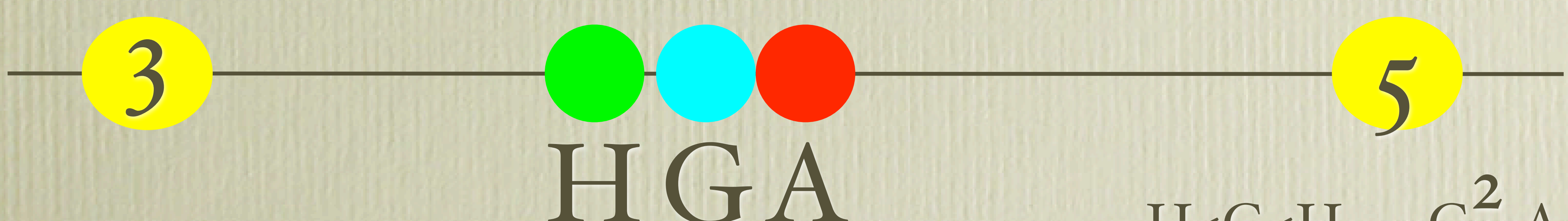
Harmonic mean

$$H(a,b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

$$A(a,b) = (a+b)/2$$

$$G(a,b) = \sqrt{a\,b}$$

Example: $H(3,5) = \dfrac{2}{\frac{1}{3} + \frac{1}{5}} = 3\ 3/4$



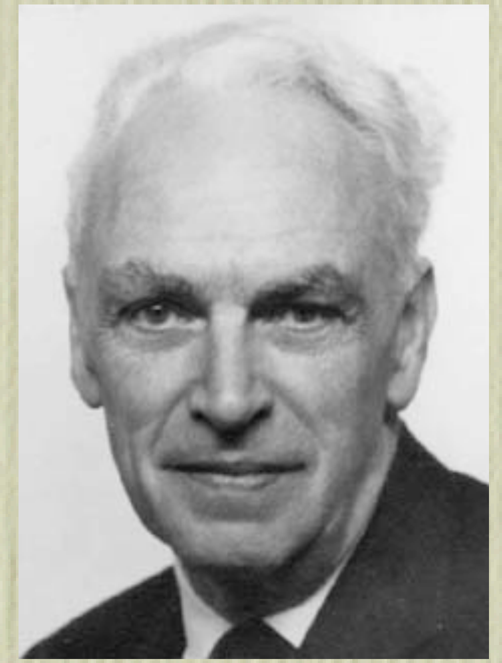
$H<G<H$ $\quad G^2 = A\,H$

Ore harmonic number

The harmonic mean of all the divisors is an integer.

Example: 140 . The harmonic mean of its proper divisors is 5.

1,2,4,5,7,10,14,20,28,45,70,140.



Theorem (Ore)

Every perfect number is harmonic

Proof: $AH = n$

H harmonic mean
A arithmetic mean

$$H = \frac{k}{\frac{1}{d_1} + \dots + \frac{1}{d_k}} \qquad A = \frac{d_1 + \dots + d_k}{k} = 2n/k$$

(perfect)

Use: $\frac{n}{d_1} + \dots + \frac{n}{d_k} = d_1 + \dots + d_k$ (n/d is again a divisor)

So $(2n/k)\, H = n$ which implies $H = k/2$

But k, the number of divisors of n is always even, if n is not a square or 2 times a square.

Squares or 2 times a square can not be perfect. (We know the even perfect numbers and odd squares satisfy $\sigma(n)$ odd.)

The Ulam spiral

905	788	679	578	485	484	483	482	481	480	479	478	477	476	475	474	473	472	471	470	469	468	467	466	465	464	463	552	649	754	867	988
906	789	680	579	486	401	400	399	398	397	396	395	394	393	392	391	390	389	388	387	386	385	384	383	382	381	462	551	648	753	866	987
907	790	681	580	487	402	325	324	323	322	321	320	319	318	317	316	315	314	313	312	311	310	309	308	307	380	461	550	647	752	865	986
908	791	682	581	488	403	326	257	256	255	254	253	252	251	250	249	248	247	246	245	244	243	242	241	306	379	460	549	646	751	864	985
909	792	683	582	489	404	327	258	197	196	195	194	193	192	191	190	189	188	187	186	185	184	183	240	305	378	459	548	645	750	863	984
910	793	684	583	490	405	328	259	198	145	144	143	142	141	140	139	138	137	136	135	134	133	182	239	304	377	458	547	644	749	862	983
911	794	685	584	491	406	329	260	199	146	101	100	99	98	97	96	95	94	93	92	91	132	181	238	303	376	457	546	643	748	861	982
912	795	686	585	492	407	330	261	200	147	102	65	64	63	62	61	60	59	58	57	90	131	180	237	302	375	456	545	642	747	860	981
913	796	687	586	493	408	331	262	201	148	103	66	37	36	35	34	33	32	31	56	89	130	179	236	301	374	455	544	641	746	859	980
914	797	688	587	494	409	332	263	202	149	104	67	38	17	16	15	14	13	30	55	88	129	178	235	300	373	454	543	640	745	858	979
915	798	689	588	495	410	333	264	203	150	105	68	39	18	5	4	3	12	29	54	87	128	177	234	299	372	453	542	639	744	857	978
916	799	690	589	496	411	334	265	204	151	106	69	40	19	6	1	2	11	28	53	86	127	176	233	298	371	452	541	638	743	856	977
917	800	691	590	497	412	335	266	205	152	107	70	41	20	7	8	9	10	27	52	85	126	175	232	297	370	451	540	637	742	855	976
918	801	692	591	498	413	336	267	206	153	108	71	42	21	22	23	24	25	26	51	84	125	174	231	296	369	450	539	636	741	854	975
919	802	693	592	499	414	337	268	207	154	109	72	43	44	45	46	47	48	49	50	83	124	173	230	295	368	449	538	635	740	853	974
920	803	694	593	500	415	338	269	208	155	110	73	74	75	76	77	78	79	80	81	82	123	172	229	294	367	448	537	634	739	852	973
921	804	695	594	501	416	339	270	209	156	111	112	113	114	115	116	117	118	119	120	121	122	171	228	293	366	447	536	633	738	851	972
922	805	696	595	502	417	340	271	210	157	158	159	160	161	162	163	164	165	166	167	168	169	170	227	292	365	446	535	632	737	850	971
923	806	697	596	503	418	341	272	211	212	213	214	215	216	217	218	219	220	221	222	223	224	225	226	291	364	445	534	631	736	849	970
924	807	698	597	504	419	342	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	363	444	533	630	735	848	969
925	808	699	598	505	420	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360	361	362	443	532	629	734	847	968
926	809	700	599	506	421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439	440	441	442	531	628	733	846	967
927	810	701	600	507	508	509	510	511	512	513	514	515	516	517	518	519	520	521	522	523	524	525	526	527	528	529	530	627	732	845	966

198	145	144	143	142	141	140	139	138	137	136	135	134	133	182	239
199	146	101	100	99	98	97	96	95	94	93	92	91	132	181	238
200	147	102	65	64	63	62	61	60	59	58	57	90	131	180	237
201	148	103	66	37	36	35	34	33	32	31	56	89	130	179	236
202	149	104	67	38	17	16	15	14	13	30	55	88	129	178	235
203	150	105	68	39	18	5	4	3	12	29	54	87	128	177	234
204	151	106	69	40	19	6	1	2	11	28	53	86	127	176	233
205	152	107	70	41	20	7	8	9	10	27	52	85	126	175	232
206	153	108	71	42	21	22	23	24	25	26	51	84	125	174	231
207	154	109	72	43	44	45	46	47	48	49	50	83	124	173	230
208	155	110	73	74	75	76	77	78	79	80	81	82	123	172	229
209	156	111	112	113	114	115	116	117	118	119	120	121	122	171	228

Ulam prime spiral

199	146	101	100	99	98	97	96	95	94	93	92	91	132	181	238
200	147	102	65	64	63	62	61	60	59	58	57	90	131	180	237
201	148	103	66	37	36	35	34	33	32	31	56	89	130	179	236
202	149	104	67	38	17	16	15	14	13	30	55	88	129	178	235
203	150	105	68	39	18	5	4	3	12	29	54	87	128	177	234
204	151	106	69	40	19	6	1	2	11	28	53	86	127	176	233
205	152	107	70	41	20	7	8	9	10	27	52	85	126	175	232
206	153	108	71	42	21	22	23	24	25	26	51	84	125	174	231
207	154	109	72	43	44	45	46	47	48	49	50	83	124	173	230
208	155	110	73	74	75	76	77	78	79	80	81	82	123	172	229
209	156	111	112	113	114	115	116	117	118	119	120	121	122	171	228
210	157	158	159	160	161	162	163	164	165	166	167	168	169	170	227

44	1771	1606	1449	1300	1159	1026	901	900	899	898	897	896	895	894	893	892	891	890	889	888	887	886	885	884	883	882	881	880	879	878	877	876	875	874	873	872	871	992	1121	1258	1403	1556	1717	1886
45	1772	1607	1450	1301	1160	1027	902	785	784	783	782	781	780	779	778	777	776	775	774	773	772	771	770	769	768	767	766	765	764	763	762	761	760	759	758	757	870	991	1120	1257	1402	1555	1716	1885
46	1773	1608	1451	1302	1161	1028	903	786	677	676	675	674	673	672	671	670	669	668	667	666	665	664	663	662	661	660	659	658	657	656	655	654	653	652	651	756	869	990	1119	1256	1401	1554	1715	1884
47	1774	1609	1452	1303	1162	1029	904	787	678	577	576	575	574	573	572	571	570	569	568	567	566	565	564	563	562	561	560	559	558	557	556	555	554	553	650	755	868	989	1118	1255	1400	1553	1714	1883
48	1775	1610	1453	1304	1163	1030	905	788	679	578	485	484	483	482	481	480	479	478	477	476	475	474	473	472	471	470	469	468	467	466	465	464	463	552	649	754	867	988	1117	1254	1399	1552	1713	1882
49	1776	1611	1454	1305	1164	1031	906	789	680	579	486	401	400	399	398	397	396	395	394	393	392	391	390	389	388	387	386	385	384	383	382	381	462	551	648	753	866	987	1116	1253	1398	1551	1712	1881
50	1777	1612	1455	1306	1165	1032	907	790	681	580	487	402	325	324	323	322	321	320	319	318	317	316	315	314	313	312	311	310	309	308	307	380	461	550	647	752	865	986	1115	1252	1397	1550	1711	1880
51	1778	1613	1456	1307	1166	1033	908	791	682	581	488	403	326	257	256	255	254	253	252	251	250	249	248	247	246	245	244	243	242	241	306	379	460	549	646	751	864	985	1114	1251	1396	1549	1710	1879
52	1779	1614	1457	1308	1167	1034	909	792	683	582	489	404	327	258	197	196	195	194	193	192	191	190	189	188	187	186	185	184	183	240	305	378	459	548	645	750	863	984	1113	1250	1395	1548	1709	1878
53	1780	1615	1458	1309	1168	1035	910	793	684	583	490	405	328	259	198	145	144	143	142	141	140	139	138	137	136	135	134	133	182	239	304	377	458	547	644	749	862	983	1112	1249	1394	1547	1708	1877
54	1781	1616	1459	1310	1169	1036	911	794	685	584	491	406	329	260	199	146	101	100	99	98	97	96	95	94	93	92	91	132	181	238	303	376	457	546	643	748	861	982	1111	1248	1393	1546	1707	1876
55	1782	1617	1460	1311	1170	1037	912	795	686	585	492	407	330	261	200	147	102	65	64	63	62	61	60	59	58	57	90	131	180	237	302	375	456	545	642	747	860	981	1110	1247	1392	1545	1706	1875
56	1783	1618	1461	1312	1171	1038	913	796	687	586	493	408	331	262	201	148	103	66	37	36	35	34	33	32	31	56	89	130	179	236	301	374	455	544	641	746	859	980	1109	1246	1391	1544	1705	1874
57	1784	1619	1462	1313	1172	1039	914	797	688	587	494	409	332	263	202	149	104	67	38	17	16	15	14	13	30	55	88	129	178	235	300	373	454	543	640	745	858	979	1108	1245	1390	1543	1704	1873
58	1785	1620	1463	1314	1173	1040	915	798	689	588	495	410	333	264	203	150	105	68	39	18	5	4	3	12	29	54	87	128	177	234	299	372	453	542	639	744	857	978	1107	1244	1389	1542	1703	1872
59	1786	1621	1464	1315	1174	1041	916	799	690	589	496	411	334	265	204	151	106	69	40	19	6	1	2	11	28	53	86	127	176	233	298	371	452	541	638	743	856	977	1106	1243	1388	1541	1702	1871
60	1787	1622	1465	1316	1175	1042	917	800	691	590	497	412	335	266	205	152	107	70	41	20	7	8	9	10	27	52	85	126	175	232	297	370	451	540	637	742	855	976	1105	1242	1387	1540	1701	1870
61	1788	1623	1466	1317	1176	1043	918	801	692	591	498	413	336	267	206	153	108	71	42	21	22	23	24	25	26	51	84	125	174	231	296	369	450	539	636	741	854	975	1104	1241	1386	1539	1700	1869
62	1789	1624	1467	1318	1177	1044	919	802	693	592	499	414	337	268	207	154	109	72	43	44	45	46	47	48	49	50	83	124	173	230	295	368	449	538	635	740	853	974	1103	1240	1385	1538	1699	1868
63	1790	1625	1468	1319	1178	1045	920	803	694	593	500	415	338	269	208	155	110	73	74	75	76	77	78	79	80	81	82	123	172	229	294	367	448	537	634	739	852	973	1102	1239	1384	1537	1698	1867
64	1791	1626	1469	1320	1179	1046	921	804	695	594	501	416	339	270	209	156	111	112	113	114	115	116	117	118	119	120	121	122	171	228	293	366	447	536	633	738	851	972	1101	1238	1383	1536	1697	1866
65	1792	1627	1470	1321	1180	1047	922	805	696	595	502	417	340	271	210	157	158	159	160	161	162	163	164	165	166	167	168	169	170	227	292	365	446	535	632	737	850	971	1100	1237	1382	1535	1696	1865
66	1793	1628	1471	1322	1181	1048	923	806	697	596	503	418	341	272	211	212	213	214	215	216	217	218	219	220	221	222	223	224	225	226	291	364	445	534	631	736	849	970	1099	1236	1381	1534	1695	1864
67	1794	1629	1472	1323	1182	1049	924	807	698	597	504	419	342	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	363	444	533	630	735	848	969	1098	1235	1380	1533	1694	1863
68	1795	1630	1473	1324	1183	1050	925	808	699	598	505	420	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360	361	362	443	532	629	734	847	968	1097	1234	1379	1532	1693	1862
69	1796	1631	1474	1325	1184	1051	926	809	700	599	506	421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439	440	441	442	531	628	733	846	967	1096	1233	1378	1531	1692	1861
70	1797	1632	1475	1326	1185	1052	927	810	701	600	507	508	509	510	511	512	513	514	515	516	517	518	519	520	521	522	523	524	525	526	527	528	529	530	627	732	845	966	1095	1232	1377	1530	1691	1860
71	1798	1633	1476	1327	1186	1053	928	811	702	601	602	603	604	605	606	607	608	609	610	611	612	613	614	615	616	617	618	619	620	621	622	623	624	625	626	731	844	965	1094	1231	1376	1529	1690	1859
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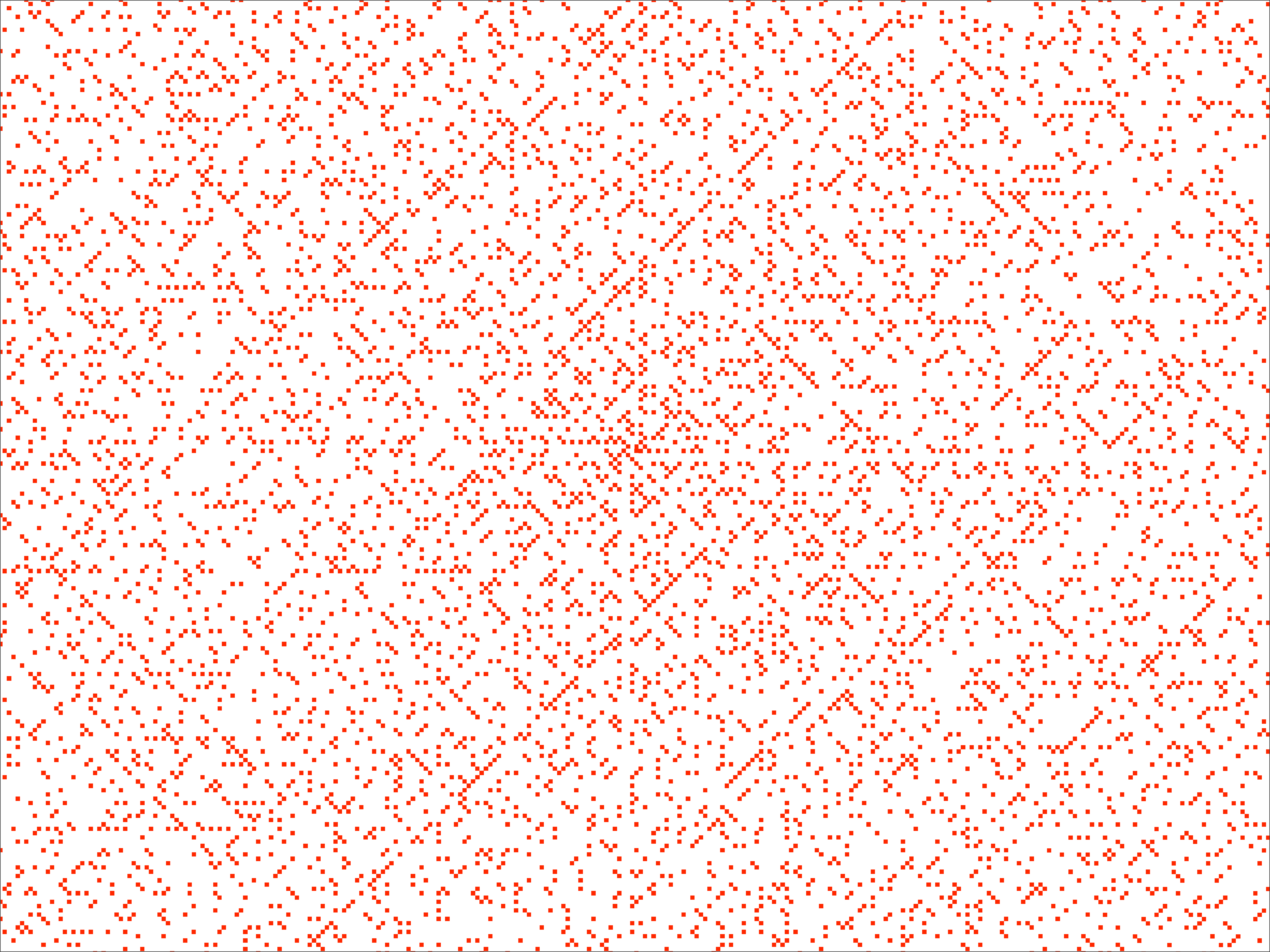
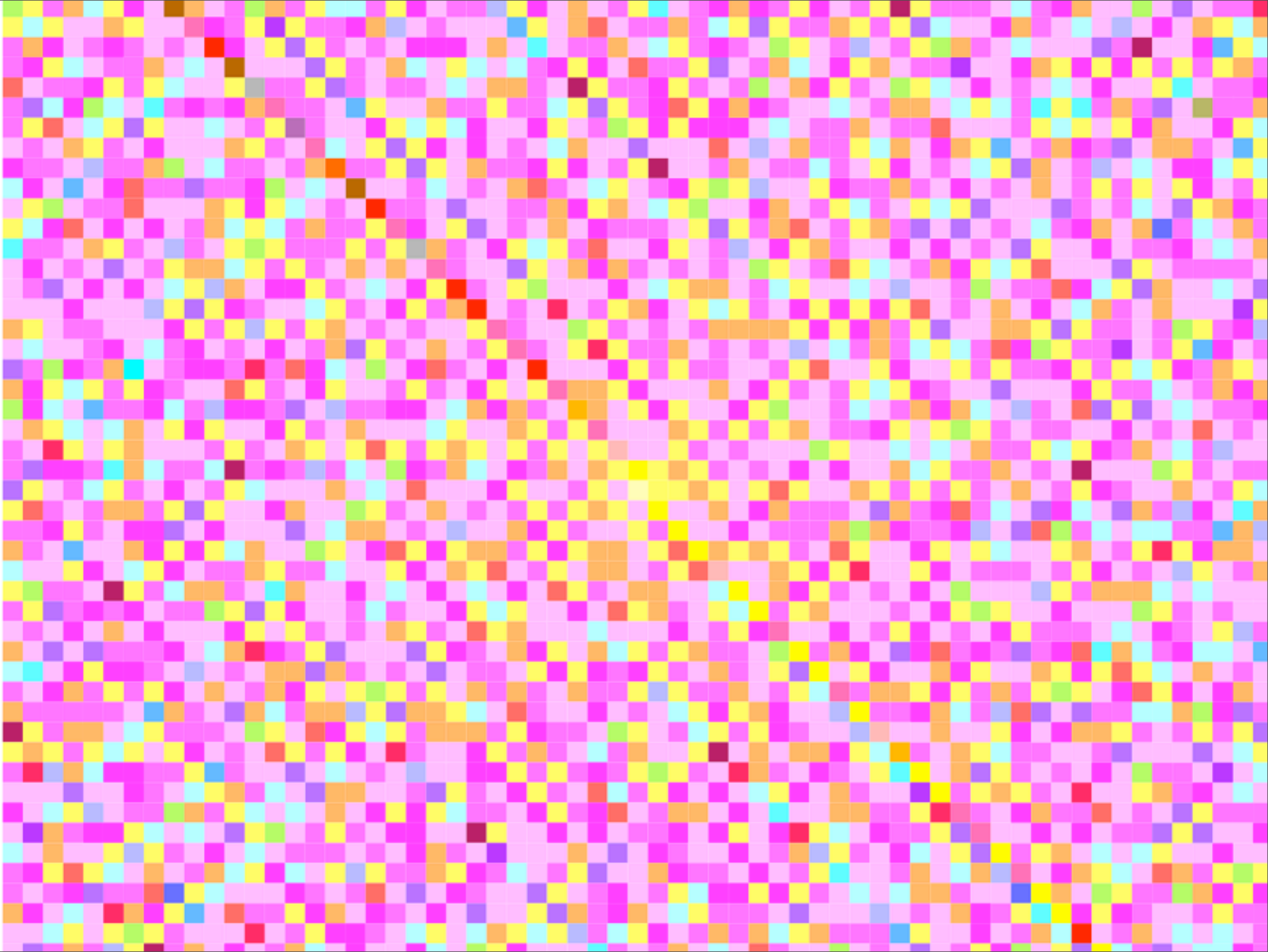
73	1800	1635	1478	1329	1188	1055	930	813	814	815	816	817	818	819	820	821	822	823	824	825	826	827	828	829	830	831	832	833	834	835	836	837	838	839	840	841	842	963	1092	1229	1374	1527	1688	1857
74	1801	1636	1479	1330	1189	1056	931	932	933	934	935	936	937	938	939	940	941	942	943	944	945	946	947	948	949	950	951	952	953	954	955	956	957	958	959	960	961	962	1091	1228	1373	1526	1687	1856
75	1802	1637	1480	1331	1190	1057	1058	1059	1060	1061	1062	1063	1064	1065	1066	1067	1068	1069	1070	1071	1072	1073	1074	1075	1076	1077	1078	1079	1080	1081	1082	1083	1084	1085	1086	1087	1088	1089	1090	1227	1372	1525	1686	1855
76	1803	1638	1481	1332	1191	1192	1193	1194	1195	1196	1197	1198	1199	1200	1201	1202	1203	1204	1205	1206	1207	1208	1209	1210	1211	1212	1213	1214	1215	1216	1217	1218	1219	1220	1221	1222	1223	1224	1225	1226	1371	1524	1685	1854
77	1804	1639	1482	1333	1334	1335	1336	1337	1338	1339	1340	1341	1342	1343	1344	1345	1346	1347	1348	1349	1350	1351	1352	1353	1354	1355	1356	1357	1358	1359	1360	1361	1362	1363	1364	1365	1366	1367	1368	1369	1370	1523	1684	1853



Color number of divisors




number of divisors gives color

the greatest
prophecies of
all time...
Protected
by the faith
of one man.
THE CODE
SPIRACY
DVD-CODE CONSPIRACY, THE
AC
N3
164805831
DVD

Catalan-Dickson conjecture

Define $T(x) = \sigma(x)-x$, which is the sum over all proper factors of x. Define also $T(0)=0$.

Catalan-Dickson Conjecture: all orbits of T are bounded.

30 42 54 66 78 90 144 259 45 33 15 9 3 1

about 1 percent of orbits do not terminate in the available limit of time

Fixed points: perfect numbers

period 2 points: amicable pairs

primes are in the domain of attraction of 1,

unbounded orbit?

904	787	678	577	576	575	574	573	572	571	570	569	568	567	566	565	564	563	562	561	560	559	558	557	556	555	554	553	650	755	868	989
905	788	679	578	485	484	483	482	481	480	479	478	477	476	475	474	473	472	471	470	469	468	467	466	465	464	463	552	649	754	867	988
906	789	680	579	486	401	400	399	398	397	396	395	394	393	392	391	390	389	388	387	386	385	384	383	382	381	462	551	648	753	866	987
907	790	681	580	487	402	325	324	323	322	321	320	319	318	317	316	315	314	313	312	311	310	309	308	307	380	461	550	647	752	865	986
908	791	682	581	488	403	326	257	256	255	254	253	252	251	250	249	248	247	246	245	244	243	242	241	306	379	460	549	646	751	864	985
909	792	683	582	489	404	327	258	197	196	195	194	193	192	191	190	189	188	187	186	185	184	183	240	305	378	459	548	645	750	863	984
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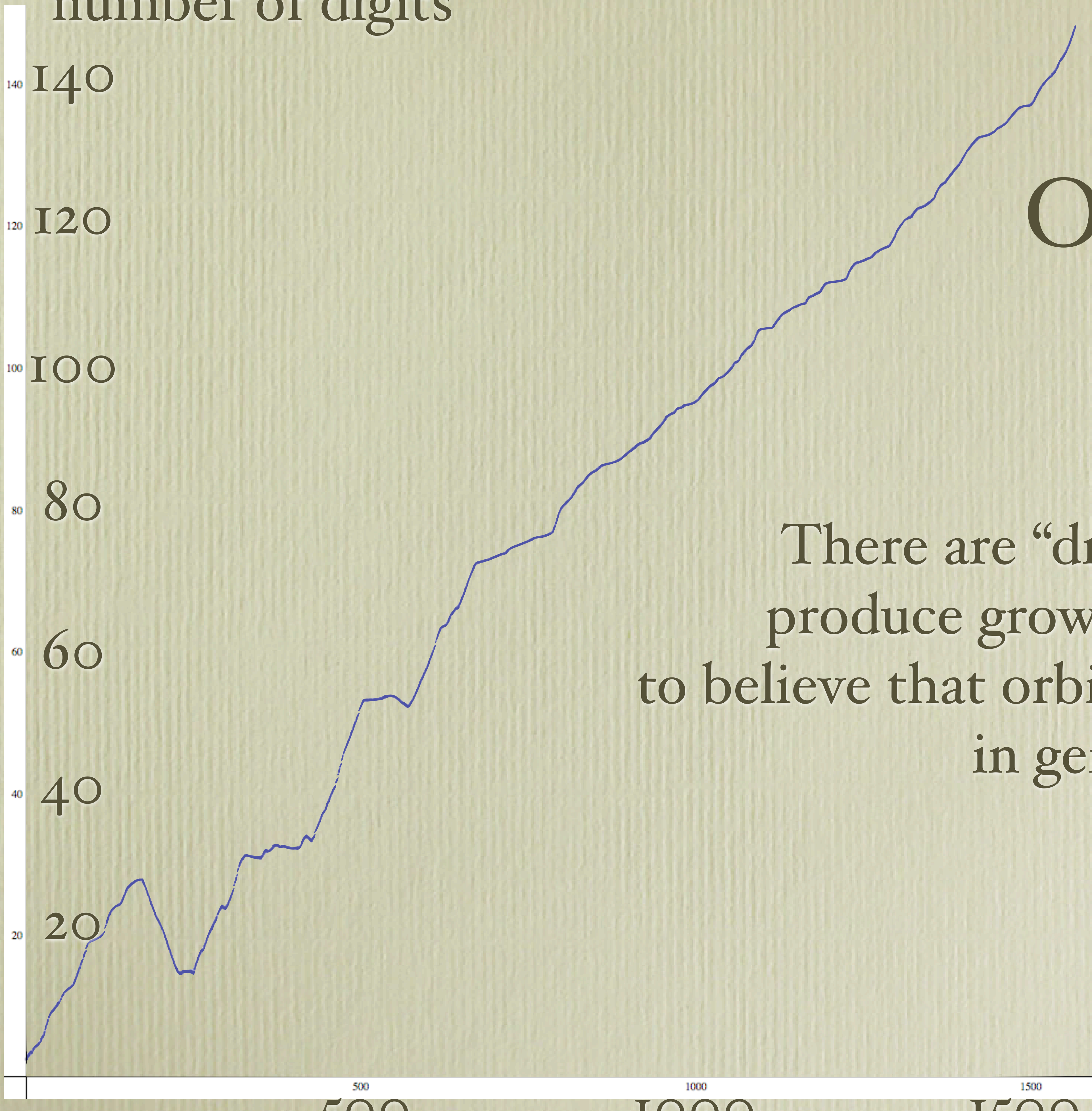
911	794	685	584	491	406	329	260	199	146	101	100	99	98	97	96	95	94	93	92	91	132	181	238	303	376	457	546	643	748	861	982
912	795	686	585	492	407	330	261	200	147	102	65	64	63	62	61	60	59	58	57	90	131	180	237	302	375	456	545	642	747	860	981
913	796	687	586	493	408	331	262	201	148	103	66	37	36	35	34	33	32	31	56	89	130	179	236	301	374	455	544	641	746	859	980
914	797	688	587	494	409	332	263	202	149	104	67	38	17	16	15	14	13	30	55	88	129	178	235	300	373	454	543	640	745	858	979
915	798	689	588	495	410	333	264	203	150	105	68	39	18	5	4	3	12	29	54	87	128	177	234	299	372	453	542	639	744	857	978
916	799	690	589	496	411	334	265	204	151	106	69	40	19	6	1	2	11	28	53	86	127	176	233	298	371	452	541	638	743	856	977
917	800	691	590	497	412	335	266	205	152	107	70	41	20	7	8	9	10	27	52	85	126	175	232	297	370	451	540	637	742	855	976
918	801	692	591	498	413	336	267	206	153	108	71	42	21	22	23	24	25	26	51	84	125	174	231	296	369	450	539	636	741	854	975
919	802	693	592	499	414	337	268	207	154	109	72	43	44	45	46	47	48	49	50	83	124	173	230	295	368	449	538	635	740	853	974
920	803	694	593	500	415	338	269	208	155	110	73	74	75	76	77	78	79	80	81	82	123	172	229	294	367	448	537	634	739	852	973
921	804	695	594	501	416	339	270	209	156	111	112	113	114	115	116	117	118	119	120	121	122	171	228	293	366	447	536	633	738	851	972
922	805	696	595	502	417	340	271	210	157	158	159	160	161	162	163	164	165	166	167	168	169	170	227	292	365	446	535	632	737	850	971
923	806	697	596	503	418	341	272	211	212	213	214	215	216	217	218	219	220	221	222	223	224	225	226	291	364	445	534	631	736	849	970
924	807	698	597	504	419	342	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	363	444	533	630	735	848	969
925	808	699	598	505	420	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360	361	362	443	532	629	734	847	968
926	809	700	599	506	421	422	423	424	425	426	427	428	429	430	431	432	433	434	435	436	437	438	439	440	441	442	531	628	733	846	967
927	810	701	600	507	508	509	510	511	512	513	514	515	516	517	518	519	520	521	522	523	524	525	526	527	528	529	530	627	732	845	966
928	811	702	601	602	603	604	605	606	607	608	609	610	611	612	613	614	615	616	617	618	619	620	621	622	623	624	625	626	731	844	965

Orbit of 276



There are “drivers” which produce growth. Guy tends to believe that orbits are not bounded in general.

Amicable pairs

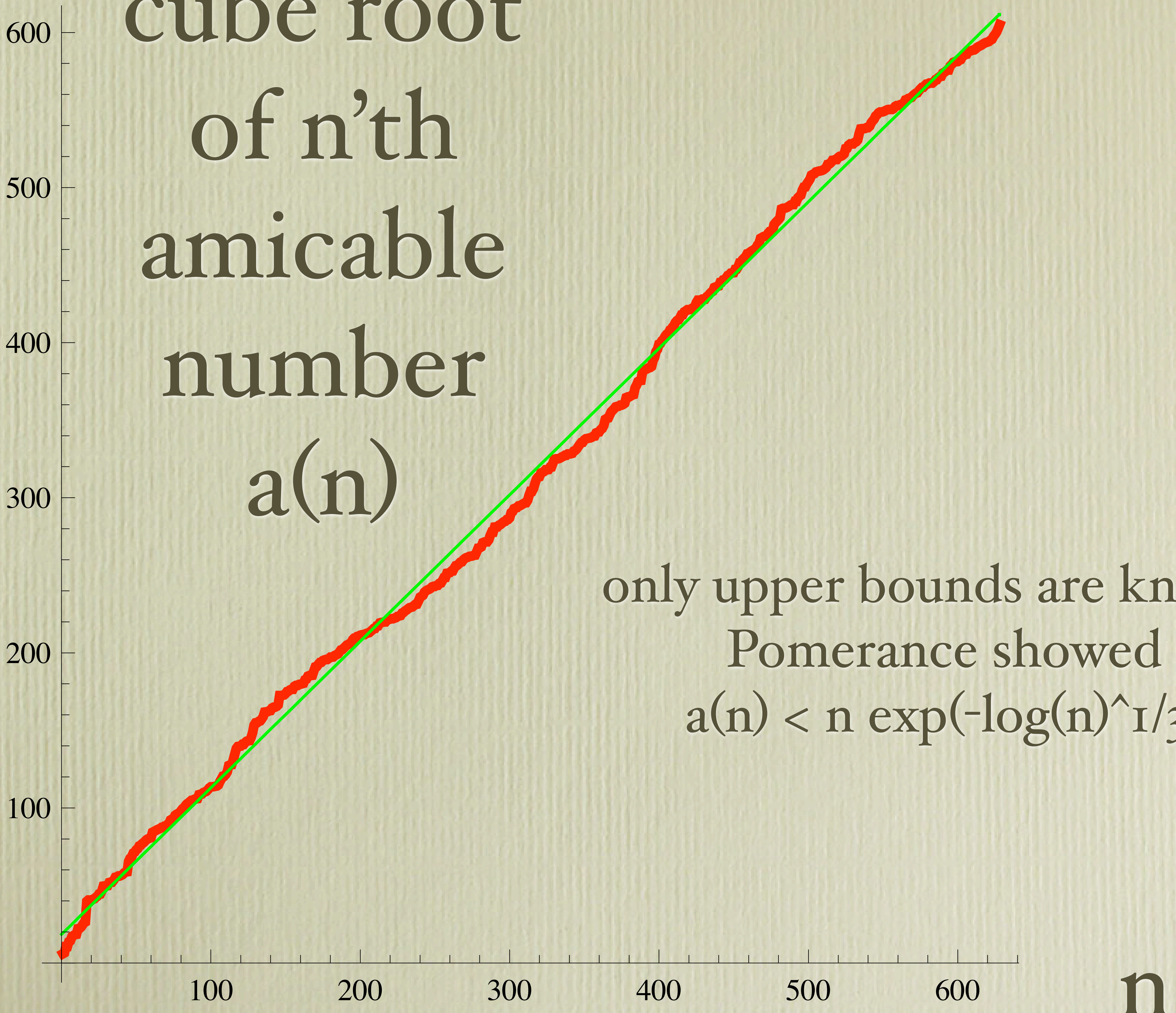
here are
the first few:

220	284
284	220
1184	1210
1210	1184
2620	2924
2924	2620
5020	5564
5564	5020
6232	6368
6368	6232

Thabit

Figure 1.5. The cover of Thabit's book on amicable numbers (by courtesy of Guedj [85])

cube root of n'th amicable number a(n)



only upper bounds are known.
Pomerance showed
a(n) < n exp(-log(n)^1/3)

The large law of small numbers

Richard Guy's law of small numbers:

You can't tell by looking at a few examples.

Superficial similarities spawn spurious statements.

Capricious coincidences cause careless conjectures.

Early exceptions eclipse eventual essentials.

Initial irregularities inhibit incisive intuition.

Example 1:

31,331, 3331,33331,333331,3333331 ,
etc are all prime

Example 2:

$2^{2^n}+1$ are all prime

F(0) = 3
F(1) = 5
F(2)= 17
F(3) =257
F(4)=65537

F(6)=18446744073709551617
=274177*67280421310721

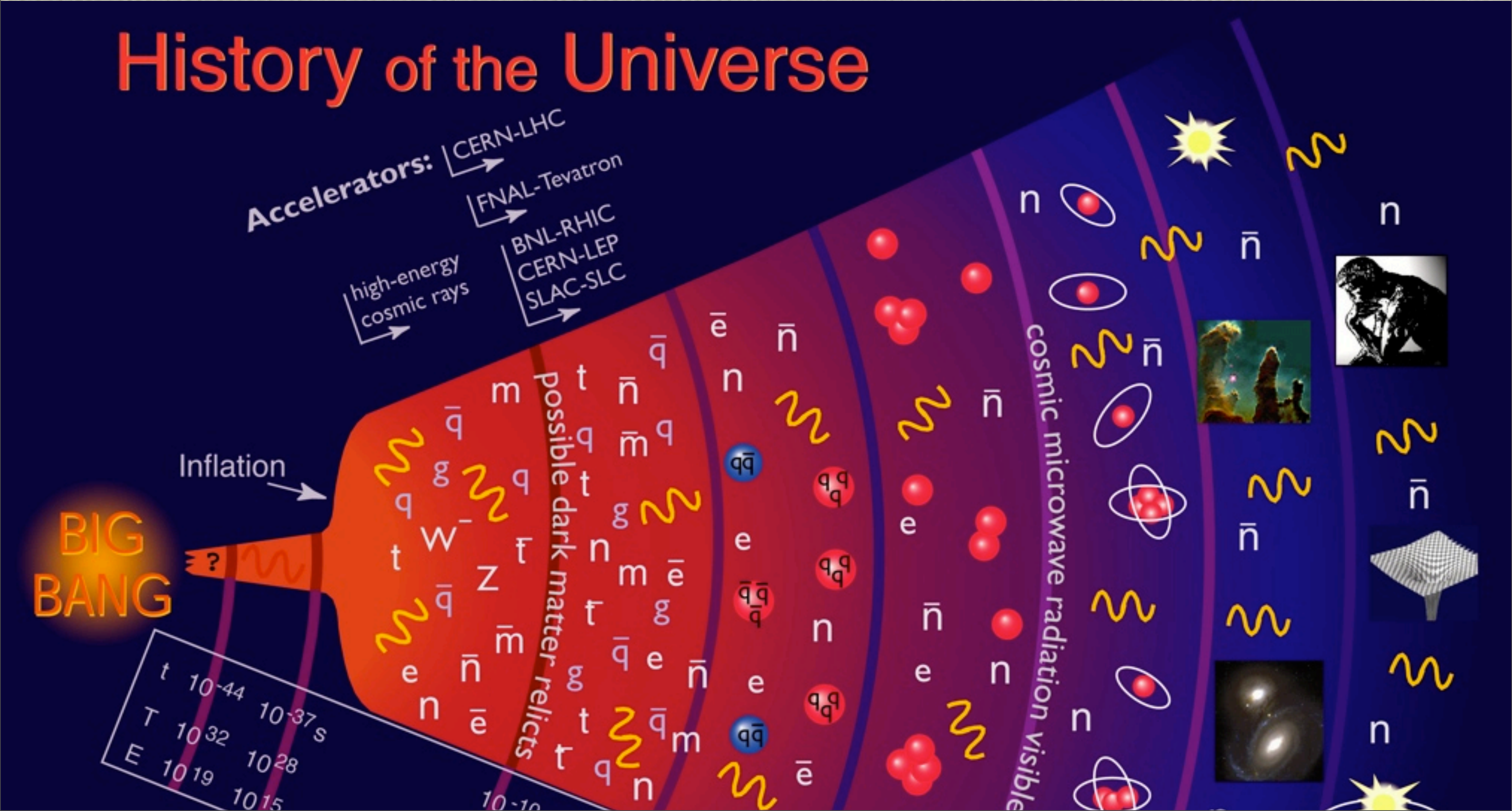
F(7)=340282366920938463463374607431768211457
=5704689200685129054721*5964958912749721

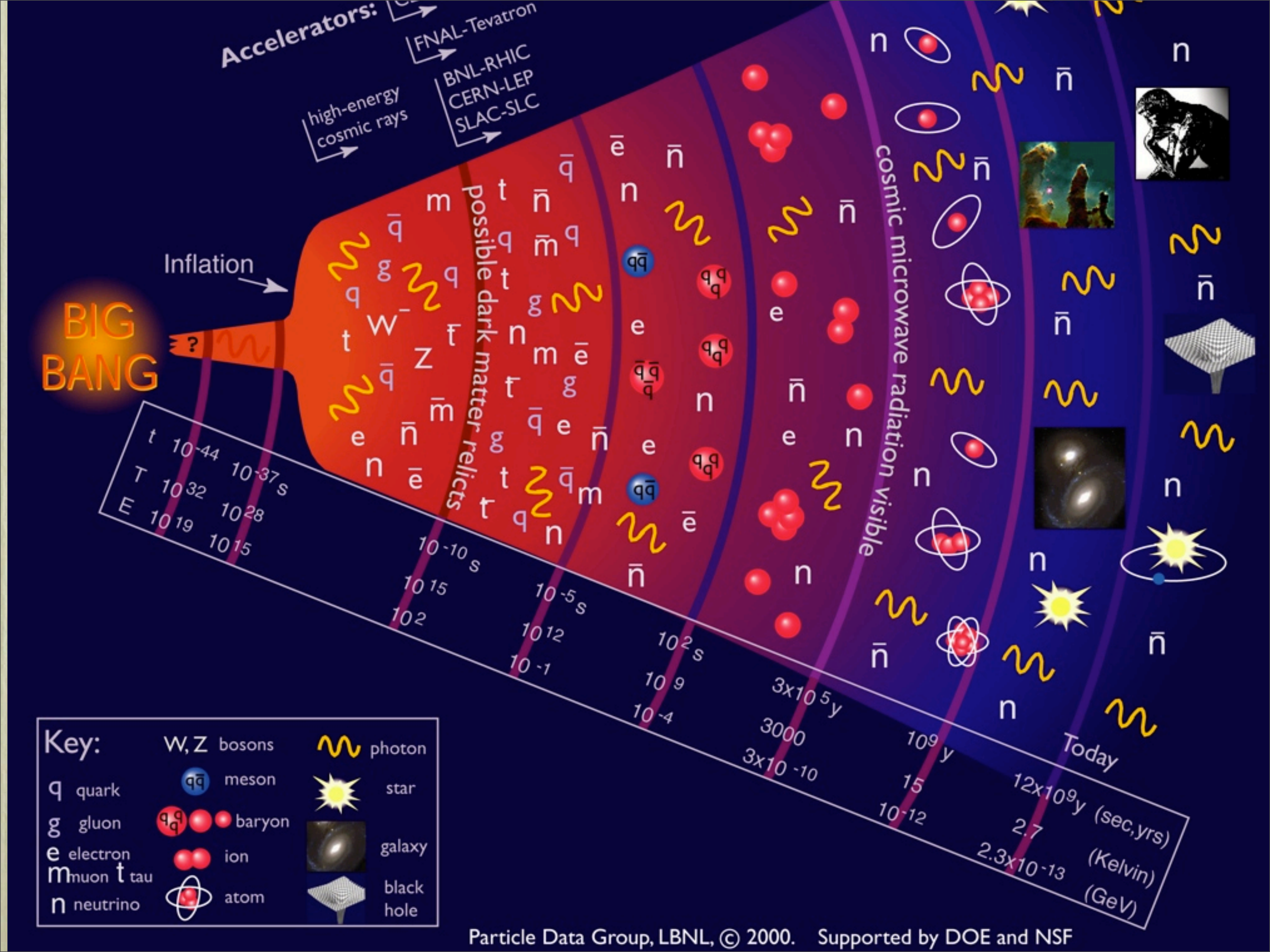
Example 3:

$$\gcd(n^{17}+9, (n+1)^{17}+9)=1$$

First counter example:

n=8424432925592889329288197322308900672459420460792433 note: this is a number with 51 digits. Our universe is estimated to be 10 seconds old. even if we could check 10 numbers per second, we would never get through.




Accelerators:
FNAL-Tevatron
BNL-RHIC
CERN-LEP
SLAC-SLC
high-energy cosmic rays
Inflation
BIG BANG
?
possible dark matter relicts
cosmic microwave radiation visible
t 10-44 10-37 s
T 10 32 10 28
E 10 19 10 15
10 -10 s
10 15
10 2
10 -5 s
10 12
10 -1
10 2 s
10 9
10 -4
3x10 5 y
3000
3x10 -10
10 9 y
15
10 -12
Today
12x10 9 y (sec,yrs)
2.7 (Kelvin)
2.3x10 -13 (GeV)
Key:
q quark
g gluon
e electron
m muon t tau
n neutrino
W, Z bosons
meson
baryon
ion
atom
photon
star
galaxy
black hole
Particle Data Group, LBNL, © 2000. Supported by DOE and NSF

Example 4: A open problem by Guy: (a case for the law?)

Is for prime p, the number

$$2^p - 1$$

square free?

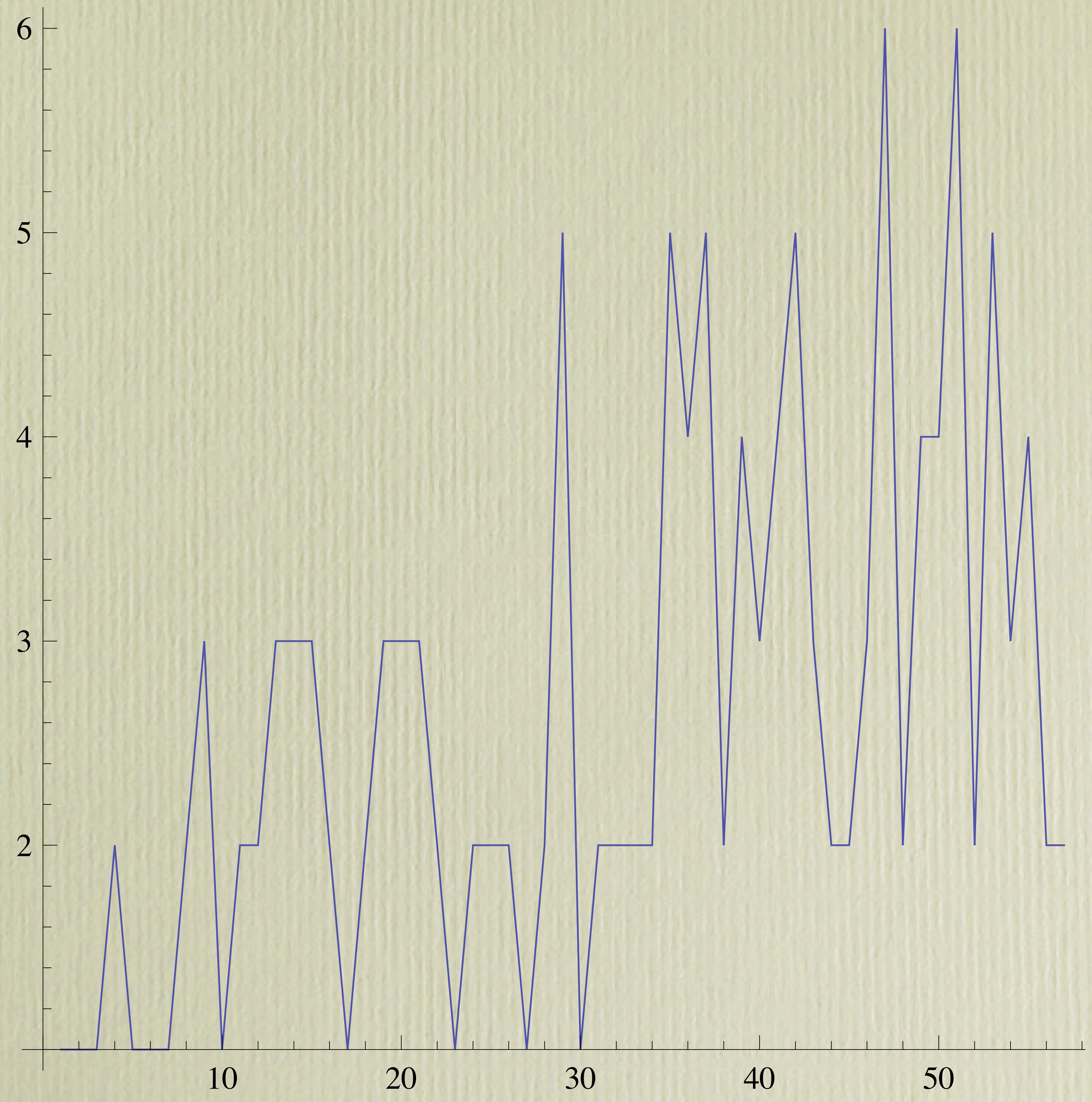
Examples:

$$2^{37} - 1 = 223 * 16318177$$

$$2^{37} - 1 = 1504073 * 20492753 * 59833457464970183 * 467795120187583723534280000348743236593$$

Its true of course for Mersenne primes ...

Number of prime factors of $2^p - 1$



Example 5:

All amicable pairs are either both even or both odd.

This is an open problem.

12285 is the first odd pair. When looking at the first few examples, one could have conjectured that all amicable pairs are both even.

Example 6:

Even perfect numbers have a decimal expansion ending by 6 or 8

6
28
496
8128
33550336
8589869056
137438691328

2305843008139952128

2658455991569831744654692615953842176

191561942608236107294793378084303638130997321548169216

13164036458569648337239753460458722910223472318386943117783728128

14474011154664524427946373126085988481573677491474835889066354349131199152128

235627234572673470657895489967099049884775478583926007101430275975063372831786222397303655396026005613602555664625032701750528925780432155433824984287771524270103944969186640286445341280338314397902368386240331714359223566432197031017207131635274872987474006478019395871659364010874193756490579185494921605556469 76

1410537837067120690632079580860631898814867435147156678388386759999548677426523801141041933290376902515619505687098293271640877243663700871167312681593136524874506524398058772962072974467232951666582288469268077866528701889208678794514783645693139220603706950647360735723786951764730552668262532848863837150729743244638353000531384294602965751433680655707595373281 28

A warning to the end:

The subject of perfect numbers is not a good subject to focus its research on. It is tasty and like eating chocolate not very healthy. Successful researchers in that field use it as a topic of many topics in mathematics or as an illustration for other mathematics. I used it here as an illustration of the language of dynamical systems theory.

Also,
nobody would be as foolish as admitting that they are trying to settle the oldest problem of mathematics. The risk is too big to be treated as a crackpot. I myself swear that I never attempted to work on the conjecture ...

The end.