Polyhedra and Polytopes 12/6/09, Math circle Northeastern Oliver Knill

How to see in higher dimensions

In this lecture we look at polyhedra, mostly in higher dimensions. Mathematics has no problems to deal with objects in 4 or higher dimensions. It is possible for example to give a list of all regular polytopes in higher dimensions. How can one "see" these objects? There are various possibilities. There are even books like [15, 11] for amateur mathematicians devoted entirely to this question. The geometer HTM Coxeter recommends in [3] the **axiomatic**, the **algebraic** or the **intuitive** way to conquer higher dimensional space. Here are three concrete ways to see them:

Construct and project! This works in any dimensions: we can use algebra to describe their coordinates. Data for example are by nature points in a higher dimensional space. We can build the objects in a computer, rotate then around and watch them. The vertices of a tesseract have coordinates $(\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1)$ for example. We can of course not watch the object directly in 4D, but we can project them onto our three dimensional space, which serves as a three dimensional "photographic plate". This **projection** can be done in various ways leading to different type of pictures. A popular projection is the stereographic projection $P(x, y, z, w) = -(x/(1-w), y/(1-w), z/(1-w))$. An other possibility is to slice the object at three dimensional "hyper heights" and display these slides in time. Note however that a Mathematician hardly needs to "see" an object to "work" with it. Any measurements and manipulations can be done without seeing the actual thing. The statistical notion of correlation for example can be interpreted as the cosine of an angle between two data points. Positively correlated means "acute angle", negatively correlated "obtuse angle". Angles in higher dimensional spaces have probably first been introduced by Schläfli in 1852, while studying higher-dimensional polytopes.

Use Time or Color! Using time or color to visualize the 4th dimension, works to visualize objects in 4D. This idea of using time is attributed to Alicia Boole Stott (1860-1940), the "princess of polytopia" and is a central theme in the movie "Flatland", where time is used as the third coordinate to explain 3D objects to flat-landers. An other possibility is to encode color with an other coordinate and color a point on the surface according to its hyper height. We can associate a real value with a color. A popular labeling is the "Hue" path which assigns to every real number in $[0, 1]$ a color. Is used this to produce the movies of the 6 regular $4-$ polytopes in the lecture. This is a popular way to visualize graphs of functions on the complex plane, where the graph is a two dimensional surface in 4-dimensional space. One can also show with color that any knot in four dimensions can be untied: there are no nontrivial knots in four dimensional space. It is a privilege of our three-dimensional space to be able to tie our shoes.

Live and die in it! In order to visualize surfaces in 3D, we can also treat these objects as two-dimensional manifolds and place ourselves inside this space. Similarly, when looking at a three dimensional surface 4D, we can place ourselves inside and live this 3 dimensional space. Even Abbott's "flatlanders" [1] can explore like this the 5 platonic solids. They just have to build their houses on the solid. Taking this modern point of view, allows to see 4-polytopes as three dimensional curved discrete spaces through which we can fly through. We can strap a polychoron onto the three-dimensional sphere for example and fly through the parameter domain of that sphere. Note that similarly as when a polyhedron is drawn on the unit sphere, the polyhedron on the 3D sphere is not in a flat space. This is a modern point of view advertised first by Riemann in the 19th century. Our own universe is not flat. Einstein did not envision this curved space to be embedded in some higher-dimensional space. There are ideas of physicists hinting to see our space floating as a "brane" in a higher-dimensional ambient space, but this does not help for visualization purposes and the ambient space does not need to be flat neither. In the horror movie "Cube 2", the heroes are trapped in a hypercube. They not only **"live"** higher dimensional space, they all **"die"** it.

The topic of polyhedra is traditionally a topic for entertainment or education and not mainstream mathematical research. The reasons can be many fold: the elementary nature of the topic makes it difficult for many mathematicians to take it seriously. Isn't it an ancient topic, where all questions are settled? This is of course not true. The subject got revived through sphere-packing problems in higher dimensions, combinatorics, graph theory and group theory: the symmetry groups of higher-dimensional polyhedra are interesting "beasts" like "Monsters". But open problems are everywhere: the classification of Archimedean polyhedra in 4D has only recently been cleaned up [12] (Guy and Conway had announced the classification in the 60ties) and almost nothing is known in higher dimensions about semi-regular polytopes, the analogue of Archimedian solids in higher dimensions. Conway and his coauthors write in [7]: "We have barely scratched the surface of the mathematics of symmetry. A universe awaits - Go forth." Similarly, in sphere packing, where highest dimensional packings are often realized by lattices, polyhedra play an important role. Placing as many unit spheres as possible for example around a given unit sphere is the "kissing problem". The centers of the neighboring sphere form a polyhedron.

Platonic solids

There are 5 **platonic solids**, two-dimensional convex polyhedra, for which all faces and all vertices are the same and every face is a regular polygon. The first mathematician who proved that the there are exactly 5 platonic solids was Theaetetus (417-369 BC). Platonic solids are also called regular 3-polytopes.

Theorem of Theaetetus: There are 5 convex regular 3-polytopes.

The proof was given in Euclid's elements: look at one of the vertices: we can take either 3,4 or 5 equilateral triangles, 3 squares or 3 regular pentagons. (6 triangles, 4 squares or 4 pentagons lead to a too large angle since each corner must have at least 3 different edges to be a polyhedron).

Name	V	E	F	V-E+F	Schläfli
tetrahedron	4	6	4	2	$\{3, 3\}$
hexahedron	8	12	6	2	$\{4, 3\}$
octahedron	6	12	8	2	$\{3, 4\}$
dodecahedron	20	30	12	2	$\{5, 3\}$
icosahedron	12	30	20	2	$\{3, 5\}$

Euler's polyhedral Formula: (Euler's Gem) For any convex polyhedron in three dimensions, the relation $V - E + F = 2$ holds.

As the story is well told in "Descartes secret notebook" [2], the formula had been discovered by Rene Descartes already (at least partially). A nice book dedicated to this formula is "Euler's gem" [14]. The proof given by Euler is easy in principle: first subdivide each of the faces into triangles. When subdividing a face, the expression $V - E + F$ do not change. Now start removing triangles. Also this process does not change $V - E + F$. We end up with a tetrahedron, for which the statement is true.
By the way, the Greek "hedr-" is cognate with the Latin "sede-" ("seat") so that for example, "dodecahedron" means "twelve-seater" ([7]). Additionally to the 5 platonic solids, there are 4 non-convex regular polyhedra, the so called **Kepler-Poinsot** polyhedra. They are regular, but not convex. Their Euler characteristic can be different:

Name	V	E	F	V-E+F	Schläfli
small stellated dodecahedron	12	30	12	-6	$\{5/2, 5\}$
great dodecahedron	12	30	12	-6	$\{5, 5/2\}$
great stellated dodecahedron	20	30	12	2	$\{5/2, 3\}$
great icosahedron	12	30	20	2	$\{3, 5/2\}$

The first two polyhedra are topologically more complex and have closed paths which can not be contracted to a point within the polyhedron. Topologists know that the Euler characteristic is $2 - 2g$, where g is the number of "holes". The first two have 4 holes.

Theorem of Cauchy: There are exactly 4 non-convex regular polyhedra.

Finally, lets look at the important notion of **duality**. A dual polyhedron can be obtained by placing vertices at the center of each face and connect those new points if their corresponding faces were adjacent. The new faces correspond then to vertices of the old graph. The dodecahedron for example is dual to the icosahedron, the octahedron is dual to the cube and the tetrahedron is self-dual.

Archimedean solids

Archimedean solids are convex polyhedra for which all vertices are isomorphic and each face is s regular polygons. There are 13 Archimedean solids and two families of prisms.
Semi-regular solids were studied first by **Archimedes** in 287 BC. Since his work is lost [3, 7] and so not much known, **Johannes Kepler** is considered the first person since antiquity to describe the whole set of thirteen in his "Harmonices Mundi" [17].

One can get 5 of them from the platonic solids by truncation. They are called "truncated Tetrahedron" (tI), "truncated Cube" (tC), "truncated Octahedron" (tO), "truncated Dodecahedron" (tD) and "truncated Icosahedron" (tI).

Theorem of Kepler: There are 13 semiregular convex 3-polytopes with tetrahedral, octahedral or icosahedral symmetry.

Name	V	E	F	V-E+F	notation
Truncated Tetrahedron	12	18	8	2	tT
Truncated Cube	24	36	14	2	tC
Truncated Octahedron	24	36	14	2	tO
Truncated Dodecahedron	60	90	32	2	tD
Truncated Icosahedron	60	90	32	2	tI
Cubeoctahedron	12	24	14	2	CO
Icosidodecahedron	30	60	32	2	ID
Rhombicuboctahedron	24	48	26	2	RCO
Rhombicosidodecahedron	60	120	62	2	RID
Great Rhombicuboctahedron	48	72	26	2	tCO
Great Icosidodecahedron	120	180	62	2	tID
Snub Cube	24	60	38	2	sC
Snub Dodecahedron	60	150	92	2	sD

We took the notation from [13], except for the last two. If one removes the symmetry condition, there is additionally the pseudo rhombicuboctahedron as well as an infinite family of **prisms** and an infinite family of **antiprisms**.

Name	V	E	F	V-E+F	notation
Pseudo Rhombicuboctahedron	12	24	14	2	PCO
Prism	2n	3n	n+2	2	Pn
Antiprism	2n	4n	2n+2	2	APn

The 14th semiregular Archimedean solid just mentioned was according to Lyusternik discovered by V.G. Ashkinuz. Coxeter attributes it to a J.C.P. Miller, who accidentally mis-constructed a model of the Rombicubeoctahedron. [6].

There are additionally 53 semiregular nonconvex polyhedra completing a class called **uniform polyhedra**. An example is the dodecadodecahedron.
Catalan solids are the duals to Archimedean solids. To construct them, take the center of the faces of the later to get the vertices of the dual. One can get 5 of them from platonic solids by a "kissing" construction: they are called "kis Tetrahedron" (kI), "kis Cube" (kC), "kis Octahedron" (kO), "kis Dodecahedron" (kD) and "kis Icosahedron" (kI). One could think that the name "kis" origins from the idea that each face makes a "kiss" but helas, the word "kis" comes from "times", the number of faces is multiplied.
For uniform solids, all vertices are isomorphic and a finite set of different polygons appear. No classification of uniform solids has been done in dimensions ≥ 4. In 3 dimensions, a list had been given by Coxeter and Miller in 1954. There are 75 non-prismatic uniform polyhedra. Together with 5 prisms, there are 80. A popular Mathematica package of Roman Maeder from the early Mathematica days had already presented them nicely, and posters of them were hanging in many mathematician offices of the 90ies.
Finally, lets mention that polyhedra for which there are no restrictions on the similarity of vertices are already quite general. For example, many two-dimensional polyhedra which bound a three dimensional solid can be obtained by triangulating a closed surface. So, one can start with a surface, triangulate it and get a polyhedron. One can also consider three-dimensional manifolds triangulated by tetrahedra etc, but triangulation questions can become more tricky in higher dimensions.

Polytopes

Polytopes in higher dimensions are called n-polytopes if they can be realized in n-dimensional space. Similarly as a surface like a sphere is a two-dimensional object, one should look at n-polytopes as discrete $(n-1)$-dimensional spaces. Depending on dimension, polytopes have names like:

1	2	3	4	5	6	7	8
dyad	polygon	polyhedron	polychoron	polyteron	polypeton	polyexon	polyzetton

The **regular polytopes** are the analogue of platonic solids in higher dimensions.

dimension	name	Schläfli symbols
1:	Line segment	$\{\}$
2:	Regular polygons	$\{3\},\{4\},\{5\},...$
3:	Platonic solids	$\{3,3\},\{3,4\},\{3,5\},\{4,3\},\{5,3\}$
4:	Regular 4D polytopes	$\{3,3,3\},\{4,3,3\},\{3,3,4\},\{3,4,3\},\{5,3,3\},\{3,3,5\}$
≥ 5:	Regular polytopes	$\{3,3,3,\dots,3\},\{4,3,3,\dots,3\},\{3,3,3,\dots,3,4\}$

Ludwig Schläfly found in 1852 that there are exactly six convex regular convex 4-polytopes or **polychora**. The expression "choros" is Greek for "space".

Theorem of Schläfly: There are exactly 6 regular convex 4-polytopes.

Additionally, there are 10 regular nonconvex 4-polytopes.

The Schläfli notation like $\{5,3\}$ means for example that the polyhedron is bound by pentagons and each vertex has order 3. If q regular p-gons meet at a vertex, the Schläfli symbol is $\{p,q\}$. Its dual is the polyedron $\{q,p\}$. A regular 4-polytope for which there are r regular polyhedral cells of type $\{p,q\}$ at each vertex is labeled with the symbol $\{p,q,r\}$. In general, a $\{p_1,...,p_n\}$ is the polytope which has p_n facets of type $\{p_1,...,p_{n-1}\}$ at each vertex. Here are the regular polytopes in 4 dimensions:

Polychoron	Name 1	Name 2	Schläfli	V	E	F	C	V-E+F-C
simplex	5-cell	pentachoron	3,3,3	5	10	10	5	0
tesseract	8-cell	octachoron	4,3,3	16	32	24	8	0
orthoplex	16-cell	hexadecachoron	3,3,4	8	24	32	16	0
polyoctahedron	24-cell	icositetrachoron	3,4,3	24	96	96	24	0
polydodecahedron	120-cell	hecatonicosachoron	5,3,3	600	1200	720	120	0
polytetrahedron	600-cell	hexacosichoron	3,3,5	120	720	1200	600	0

In dimensions larger then 4, there is only the n-simplex, the n-hypercube and the n-orthoplex. For example, in dimension 8, where a 8-polytope is also called a **polyzetton**, there are three regular convex polyzetta like the enneazetton (8-simplex) octacross (8-orthoplex) or octeract (8-hypercube). The books [7] (see page 387) and [15] (page 2) shed some light on the etymology:

expression	first used by	year
"simplex"	Schoute	1902
"hypercube"	Claude Bragdon	1909
"tesseract"	Charles Howad Hinton	1888
"orthoplex"	Conway/Sloan	1987
"polytope"	Alicia Boole Stott	1900

Sub-polyhedra are called vertex, edge, face, cell, (n-2) ridge, (n-1) facet n-polytope.
How can one construct the three higher dimensional polyhedra? For the simplex: Start with a tetrahedron with points A,B,C,D in the hyperplane $w=0$. Now find a forth point $(0,0,0,h)$ which has equal distance from the given points. We can inductively construct a simplex by taking a $(n-1)$-dimensional simplex and then find a point in the next dimension with the right distance. Then adjust the center.

Schläfli's formula

Schlaefli's polyhedral formula: For any convex polyhedron in four dimensions, the relation $V-E+F-C=0$ holds, where C is the number of 3 dimensional chambers.

Duality shows immediately that if the right hand side is a constant on the class of convex polyhedra, it has to be zero. The reason is duality: $V \leftrightarrow V, E \leftrightarrow F$ changes the sign of the Euler characteristic. The formula can easily be proven by dividing up three dimensional chambers into tetrahedral spaces by adding faces. This does not change the Euler characteristic. Then, we take away tetrahedra, an operation which also does not change the characteristic. Once we are finished, we get 0.

A modern proof due to Poincare writes the Euler characteristic as $b_0-b_1+b_2-b_3$, where b_i are the Betti numbers. Since $b_0=b_3, b_1=b_2$, the Euler characteristic has to be zero. Any 4-polyhedron - whether convex or not - has characteristic 0. The Euler proof still works, but the end product is not a simplex, but a polytope where one can no more remove chambers without breaking it apart or opening holes in the inside. We always have $b_0=b_2=1$ if the body is connected. And $b_2=b_3$
Uniform polytopes are the analogue of Archimedean solids in higher dimensions.

Examples are **polychora**, a uniform 4-polytopes, for which all cells are uniform polyhedra. There are 64 non-prismatic convex uniform polychora,

Tesselations and symmetries

Polyhedra matter also to understand symmetries of space. In how many different ways can one divide up Euclidean space, so that each chamber is a regular or semiregular polyhedron. One can fill three dimensional space with cubes. Is it possible to do so with an other platonic solid? No. But one can use a mixture of tetrahedra and octahedra to fill space. Additionally, there is one Archimedean solid, the truncated octahedron and one Catalan solid, the rhombic dodecahedron can be used to tessellate space. [17] These are all the 4 three-dimensional tessellations.

There are 17 plane symmetry groups or wall paper group or plane crystallographic groups There are 217 three-dimensional crystallographic space groups (crystallographers give 230 including 11 metachiral groups each having two orientations). Illustrating this would be material for an other math circle talk.

Considering polytopes in higher dimensions might look esotheric at first, since we live in three dimensional space. The subject is interesting in graph theory, since every polyhedron also defines a graph. One can look at the problem to enumerate Archimedean polyhedra as a problem in combinatorics or geometry. The classification of Archimedean polyhedra in four dimensional space, while known since 1965, is still not illustrated and explained well enough in books ([7] makes a start).

It is in **group theory** where high-dimensional polytops matter too. The symmetry group of a polyhedron or polytope is the group generated by all reflections at hyperplanes, which leave the polytope invariant. Symmetry group of polyhedra are **Coxeter groups**. The simplest one is the group of all symmetries of the regular n-gon (which is a 2-polytope). There are two reflections which generate the symmetries, a reflection a at a line through two opposite vertices, and a reflection b at a line perpendicular to two opposite edges. These reflections satisfy the relation $a^2 = b^2 = (ab)^n$. The group is the **dihedral group**.

The famous **Monster group** in group theory is the symmetry group of a 196'883 dimensional polytope [7].

Finally, there is a topological interest in polytopes if they are viewed as discrete versions of manifolds. Much geometry done in the continuum correspond to results for these discrete structures. Topologists have chosen to use an other terminology however and talk about simplicial complexes.

People

The first mathematicians working on polyhedra was Theaetetus (417-369 BC). The picture below shows Plato (428-348) by whom we know about Theatetus. His work is known through Euclid of Alexandria (365-275 BC) who covered polyhedra in "book 13" of his "Elements". Archimedes (287-212 BC) was the first to work on semiregular solids.
After a long dark area for geometry, Johannes Kepler (1571-1630) classified all Archimedean solids. His name is also linked to Louis Poinsot (1777-1859) by the Kepler-Poinsot polyhedra. Catalan (1814-1894) got his name associated with the duals of Archimedean solids.

The Swiss mathematician **Ludwig Schläfli** (1814-1895) is **the** pioneer of higher dimensional polytopes. He classified them in all dimensions and was also the first to discover Euler's formula in arbitrary dimension. (Cauchy had a particular formula in 4 dimensions[14].) Schläfli grew up in Burgdorf near Bern in Switzerland and is considered one of the three architects of multidimensional geometry, together with the British **Arthur Cayley** (1821-1895) and **Bernhard Riemann** (1826-1866). He was one of the first in 1852 to define distances in higher dimensional space. His magnum opus "Theory of Continuous Manifolds" appeared in 1901, years after his death. Schläfli got appreciated only through [3] by the mathematics community, Schläfli was also one of the first mathematician also who preview relativity theory by combining space and time and consider the ideas of Riemann for this space-time [9]. Even specialized books like [5] do not cite Schläfli and mention him only shortly in the introduction.

Finally lets mention three more mathematicians: The amateur mathematician Alicia Boole Stott (1860-1940) mentioned in the introduction was a Irish women, who never studied mathematics but taught herself to see the 4th dimension and collaborated with the geometer P.H. Shoute. HSM Coxeter (1907-2003)'s books and work revived the interest in polytopes. John Conway (1937-) helped the subject back to some interest, both with his work as well as his books.

Literature

Here are some more references from my personal book library, I consulted to prepare for this talk:

The still today best resource for higher-dimensional polytopes is [3]. The history is very well outlined in [4]. For teachers, [16] and [10] are valuable. I like the booklet [17] and a classical book [8] written by an architect.

A newer book, which contains a lot of information also about polytopes is [7]. This gorgeous book also contains information on semiregular polytopes in higher dimensions, a topic which has been investigated first seriously in 1912 by [13]. Work on semiregular polytopes in higher dimensions. About Euler's formula see [14].

For the slideshow, I used movie clips from:

- **The elegant Universe**, NOVA, presented by Brian Green and others, 2003, DVD,
- **Flatland**, A journey of many dimensions, 2007, DVD, by Martin Sheen, Kristen Bell, Michael York, Tony Hale and Joe Estevez
- **Dimensions**, 2008, DVD Video
- **Phantom Tollbooth**, Video, 1070
- **Hyperspace**, From the birth of the universe to the end of our word, with Sam Neill, BBC Video, 2001.
- **Cube 2: Hypercube**, 2002, DVD Video
- **The Time Machine**, 2002, DVD Video
- **Time line**, 2003, DVD Video
- **Back to the future**, 1985, DVD Video

References

[1] Edwin Abbott. *Flatland: A romance of Many Dimensions.* Basil Blackwell, Oxford, 1884.

[2] Amir D. Aczel. *Descartes's secrete notebook, a true tale of Mathematics, Mysticism and the Quest to Understand the Universe.* Broadway Books, 2005.

[3] H.S.M. Coxeter. *Regular Polytopes.* Dover Publications, New York, 1973.

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[14] David S. Richeson. *Euler's Gem*. Princeton University Press, Princeton, NJ, 2008. The polyhedron formula and the birth of topology.

[15] Rudy Rucker. *Geometry, Relativity and the Fourth Dimension*. Dover Books on Physics. Dover Publications, Inc, New York, 1977.

[16] Marjorie Senechal and George Fleck, editors. *Shaping space*, Design Science Collection, Boston, MA, 1988. Birkhäuser Boston Inc. A polyhedral approach, A Pro Scientia Viva Title.

[17] Daud Sutton. *Platonic and Archimedean Solids*. Walker and Company, New York, 2002.