

THE DIRAC OPERATOR OF A GRAPH

Oliver Knill
Harvard University

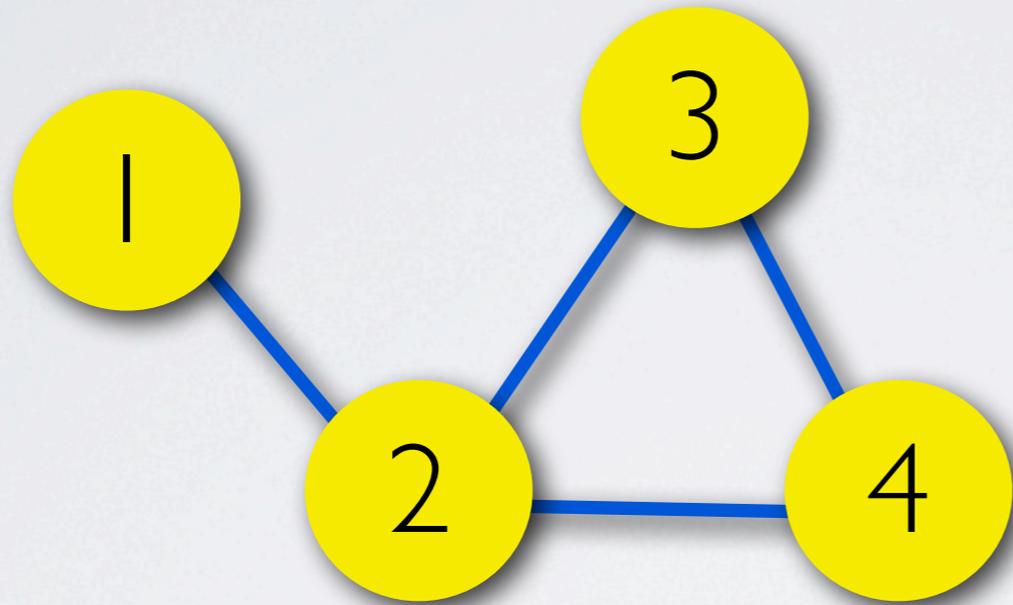
Jun 5 ILAS 2013

LEONARD EULER

1707-1783



GRAPHS



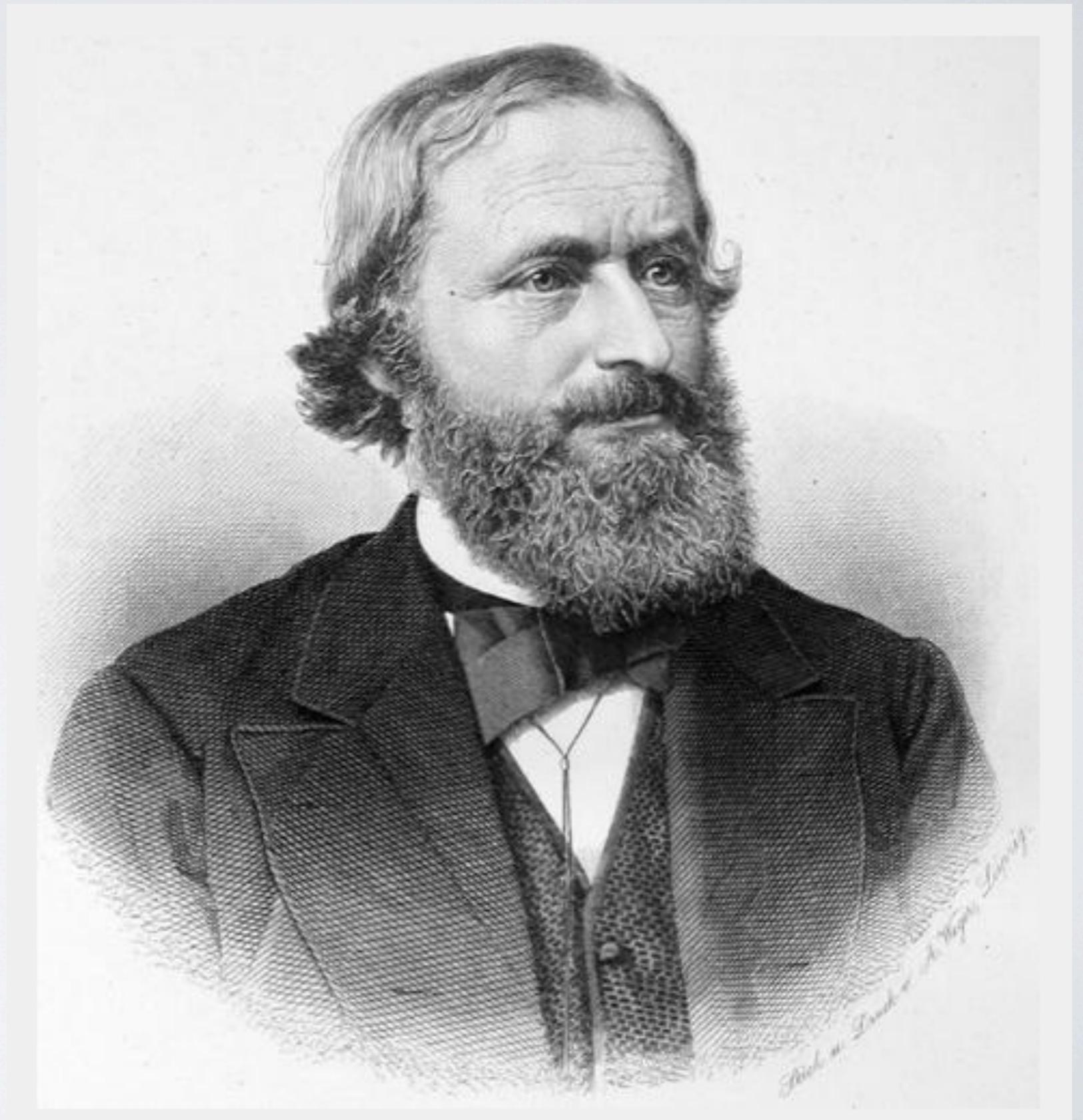
finite
simple

no loops
no multiple connections

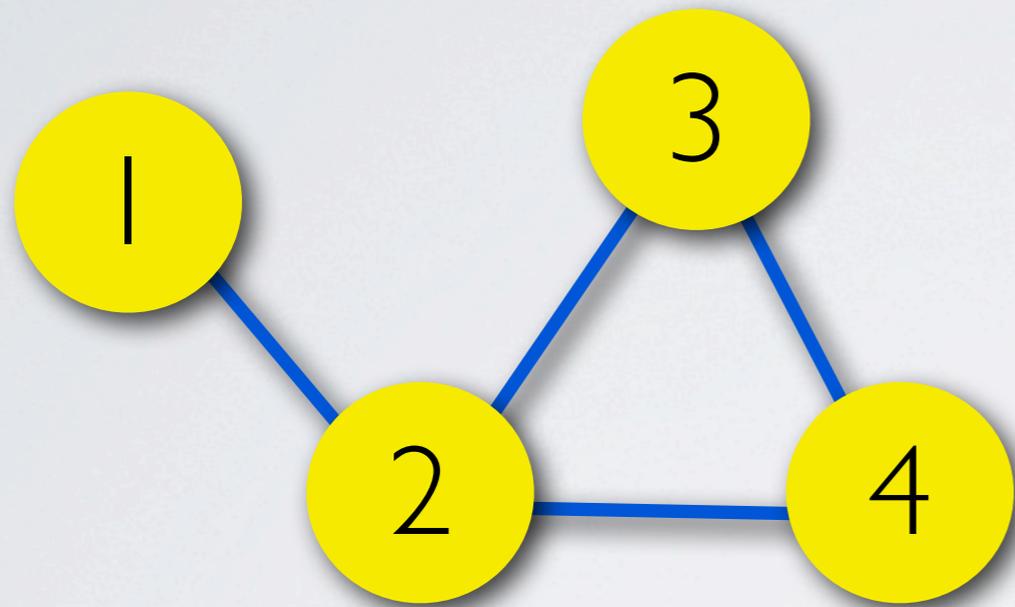
1736

GUSTAV KIRCHHOFF

1824-1887



ADJACENCY MATRIX

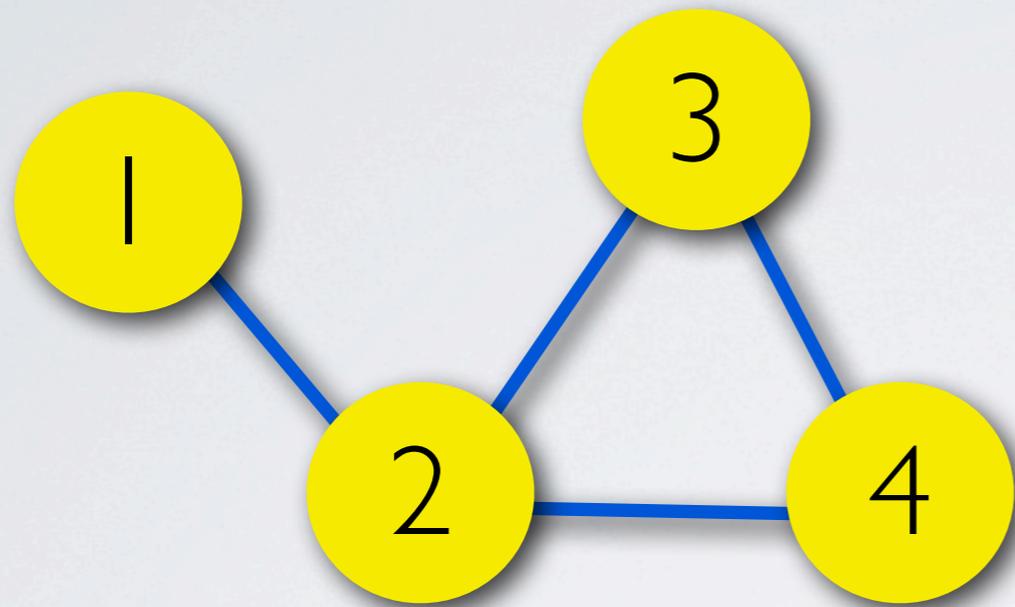


A=

0	1	0	0
1	0	1	1
0	1	0	1
0	1	1	0

1845

COMBINATORIAL LAPLACIAN



$$L = B - A =$$

B = Degree matrix

$$\sigma(L) = \{0, 1, 3, 4\} \geq 0$$

1	-1	0	0
-1	3	-1	-1
0	-1	2	-1
0	-1	-1	2

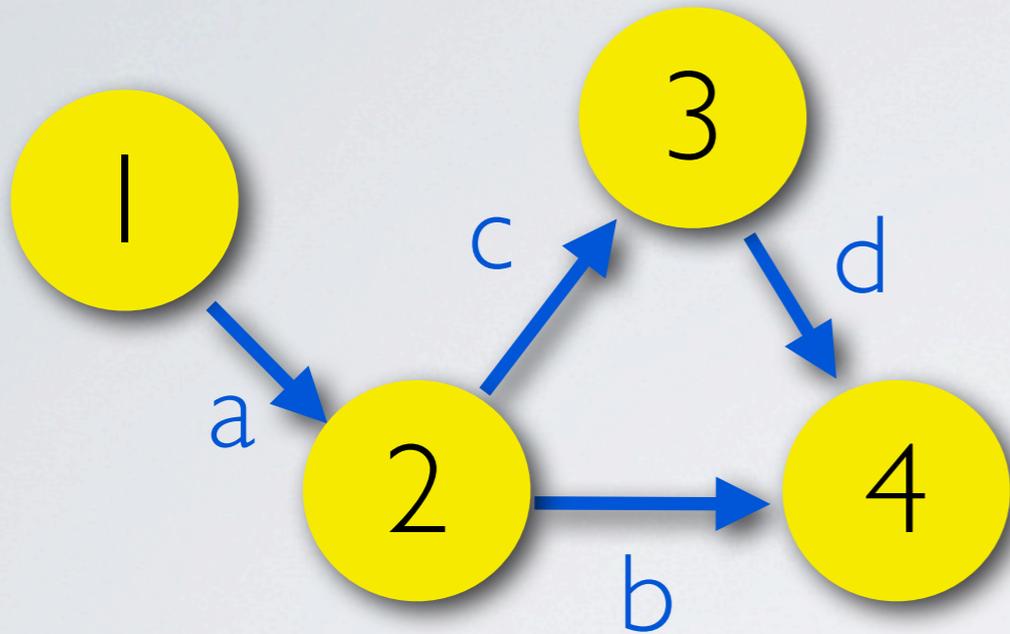
HENRI POINCARÉ

1854-1912



GRADIENT

oriented incidence matrix

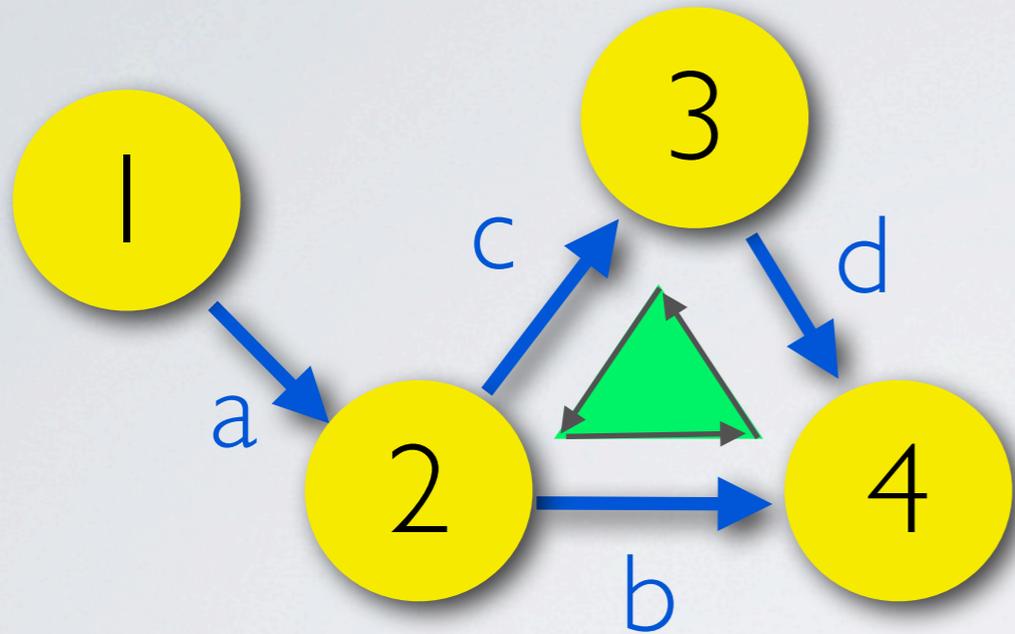


$$d_0 =$$

$df =$
gradient (f)

a	-1	1	0	0
b	0	-1	0	1
c	0	-1	1	0
d	0	0	-1	1

CURL

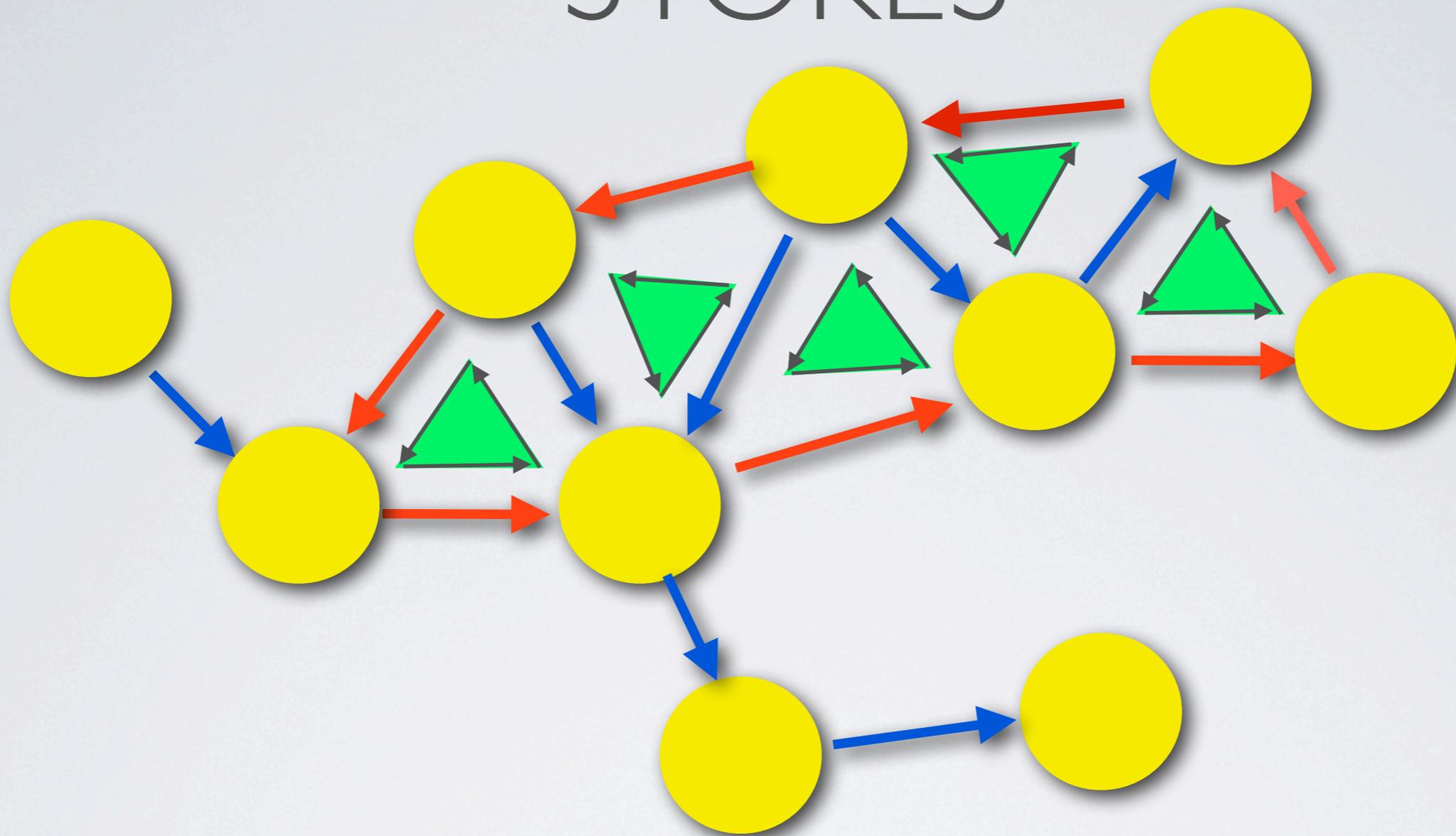


$d_1 =$

a	b	c	d
0	1	-1	-1

$$d F = \text{curl} (F)$$

STOKES



$$\int_S \text{curl}(\mathbf{F}) \, d\mathbf{S} = \int_C \mathbf{F} \, d\mathbf{r}$$

CURL OF GRAD

$$d_1 d_0 =$$

0	1	-1	-1
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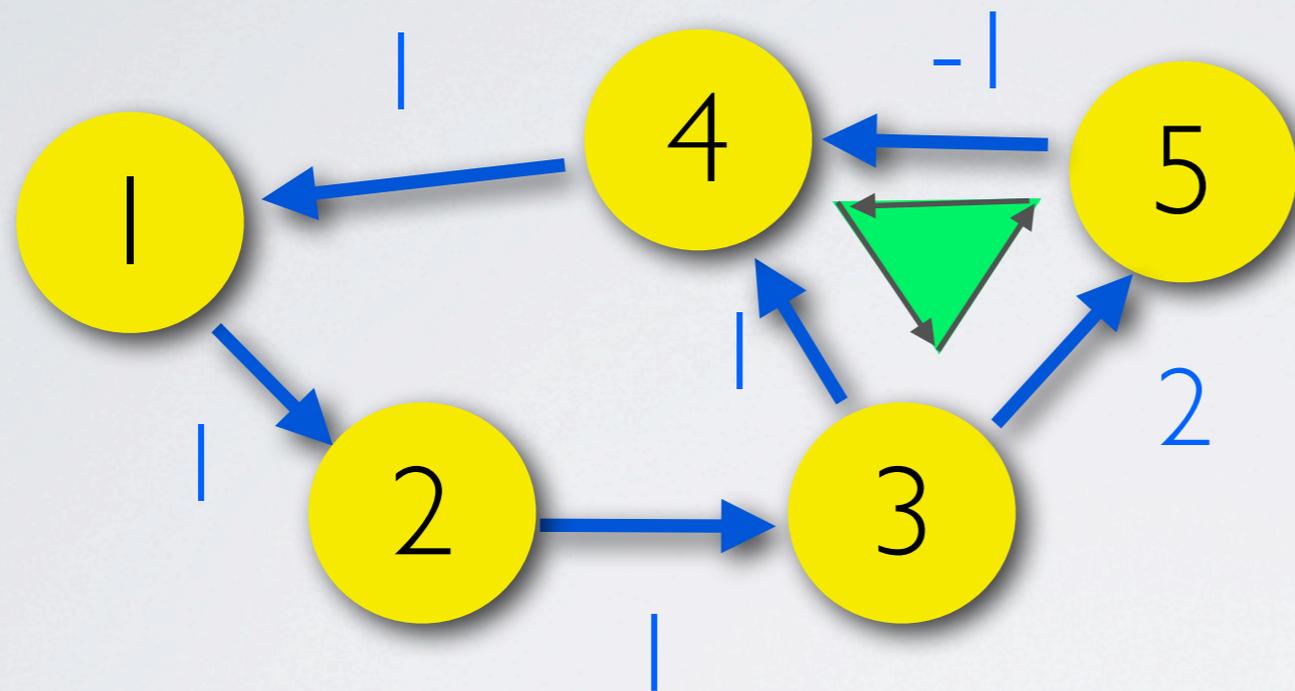
-1	1	0	0
0	-1	0	1
0	-1	1	0
0	0	-1	1

=

0	0	0	0
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$$d \circ d = 0$$

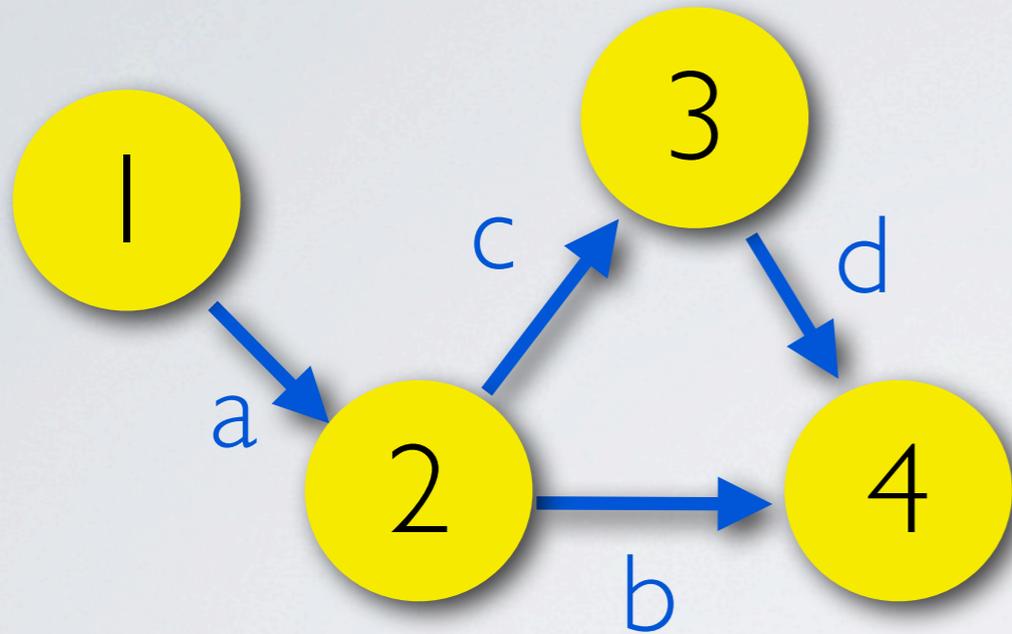
DETECT CONTRACTIBILITY



$$\text{curl}(F)=0$$

but F not equal to $\text{grad}(f)$

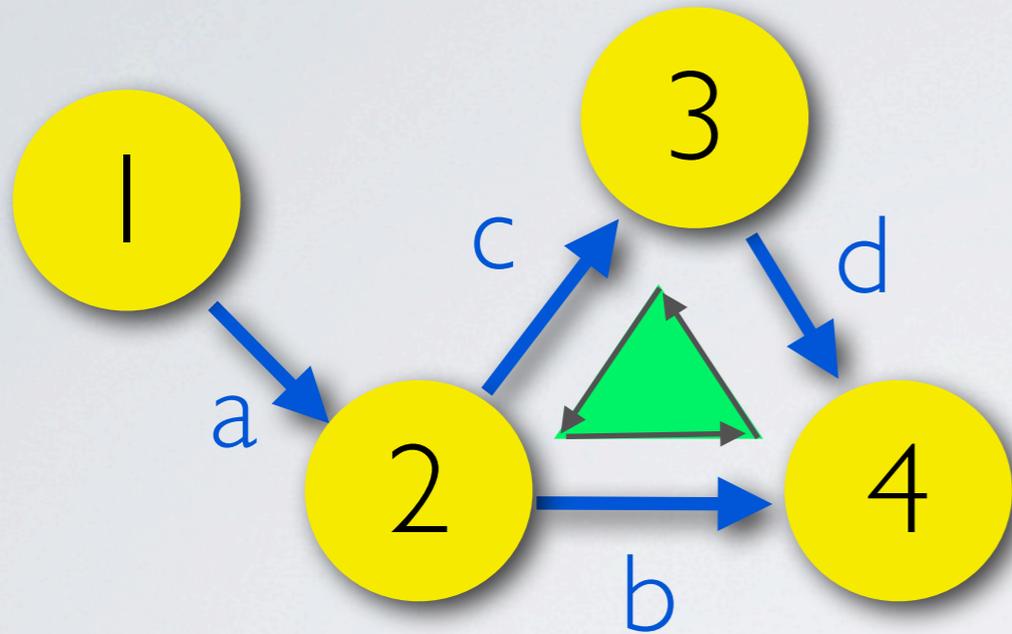
DIVERGENCE



$$d_0^* =$$

	a	b	c	d
1	-1	0	0	0
2	1	-1	-1	0
3	0	0	1	-1
4	0	1	0	1

DUAL CURL



$$d_1^* =$$

a	0
b	1
c	-1
d	-1

DIV(GRAD)=LAPLACE

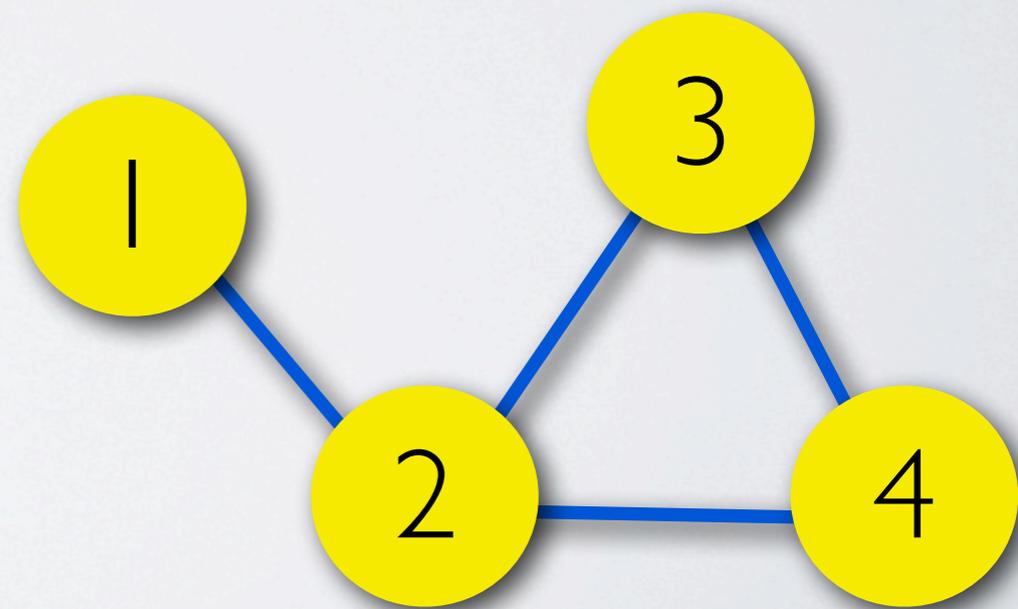
$$L_0 = d_0^* d_0 =$$

-1	0	0	0
1	-1	-1	0
0	0	1	-1
0	1	0	1

-1	1	0	0
0	-1	0	1
0	-1	1	0
0	0	-1	1

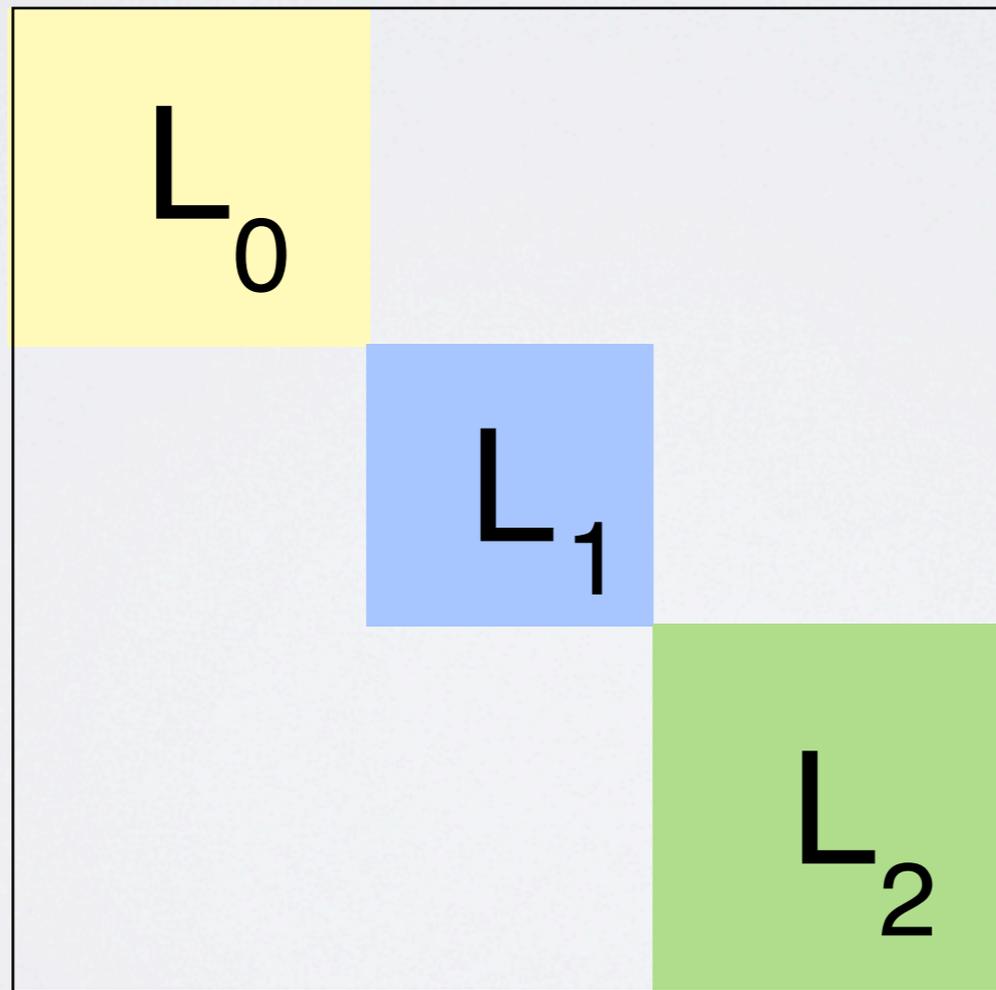
$$=$$

1	-1	0	0
-1	3	-1	-1
0	-1	2	-1
0	-1	-1	2



LAPLACE-BELTRAMI

$$L_k = d_k^* d_k + d_{k-1} d_{k-1}^*$$



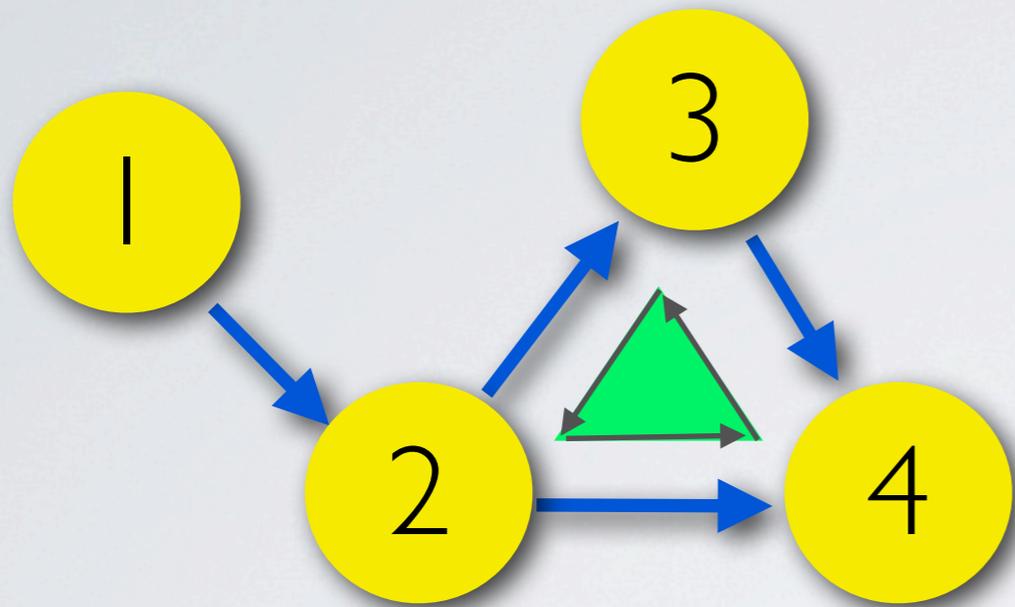
$$L = (d + d^*)^2$$

PAUL DIRAC

1902-1984



DIRAC OPERATOR



$$D = d + d^*$$

$$v_0 = 4$$

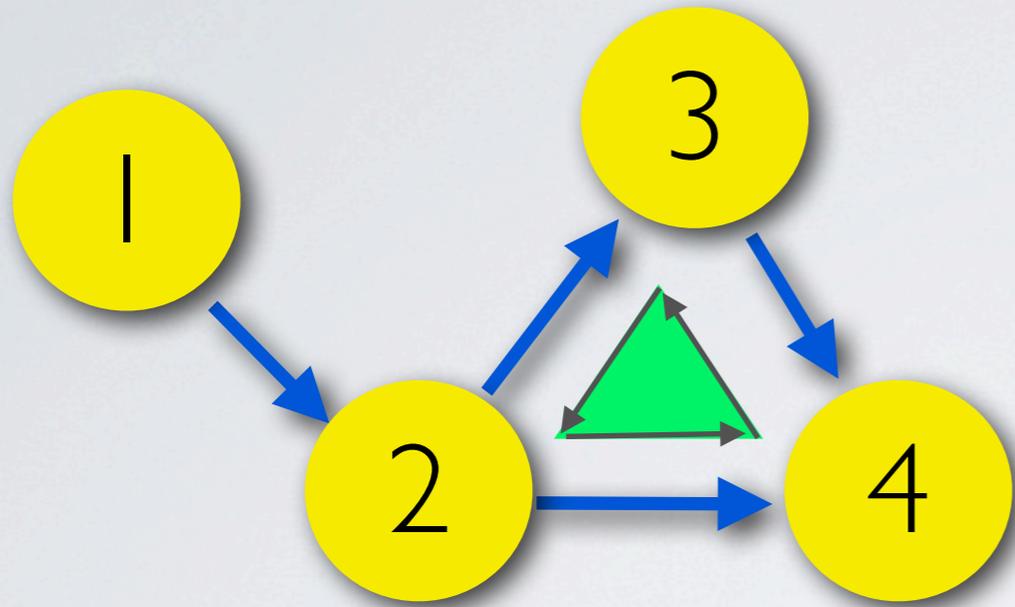
$$v_1 = 4$$

$$v_2 = 1$$

$$\chi(G) = 1$$

0	0	0	0	-1	0	0	0	0
0	0	0	0	1	-1	-1	0	0
0	0	0	0	0	0	1	-1	0
0	0	0	0	0	1	0	1	0
-1	1	0	0	0	0	0	0	0
0	-1	0	1	0	0	0	0	1
0	-1	1	0	0	0	0	0	-1
0	0	-1	1	0	0	0	0	-1
0	0	0	0	0	1	-1	-1	0

LAPLACE-BELTRAMI MATRIX

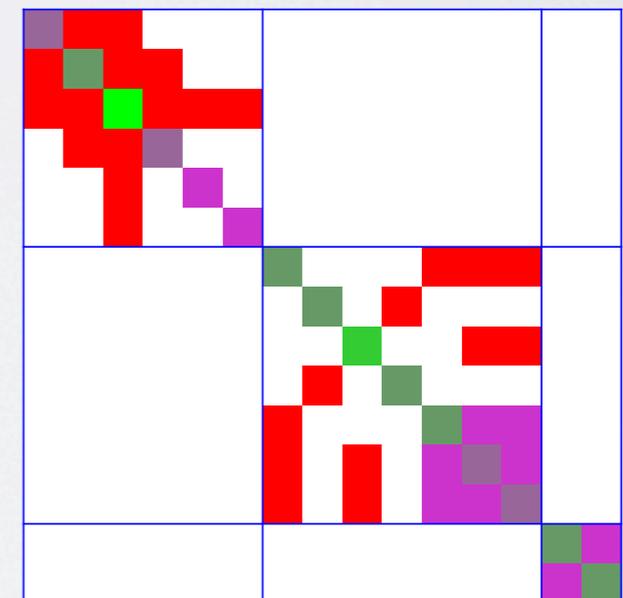
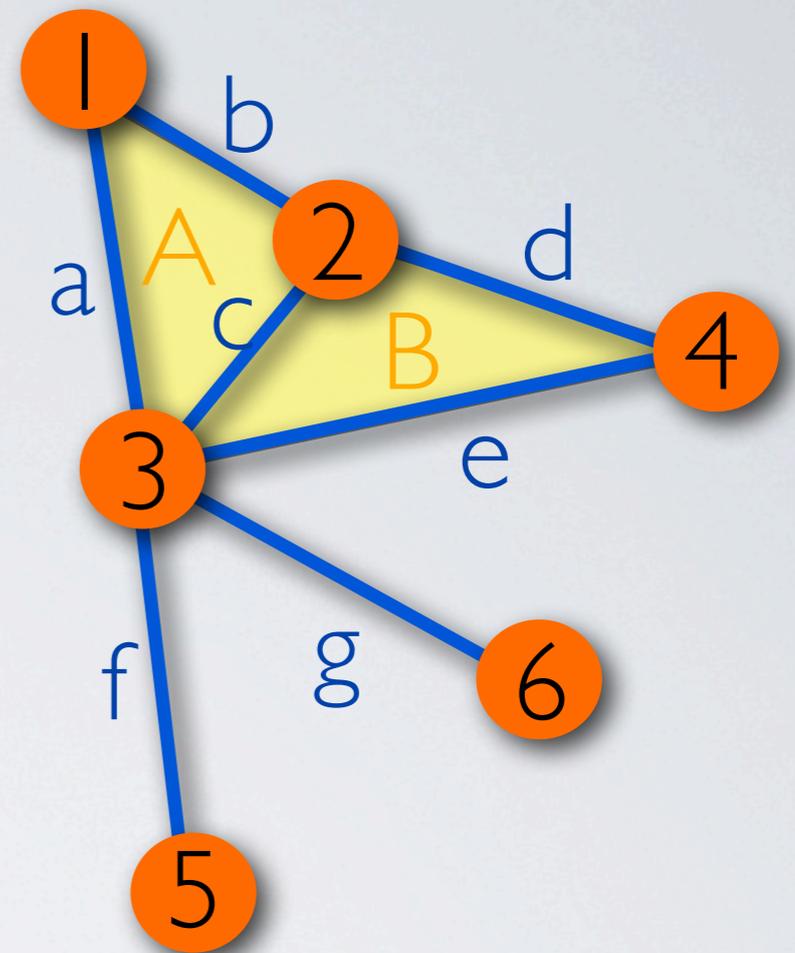
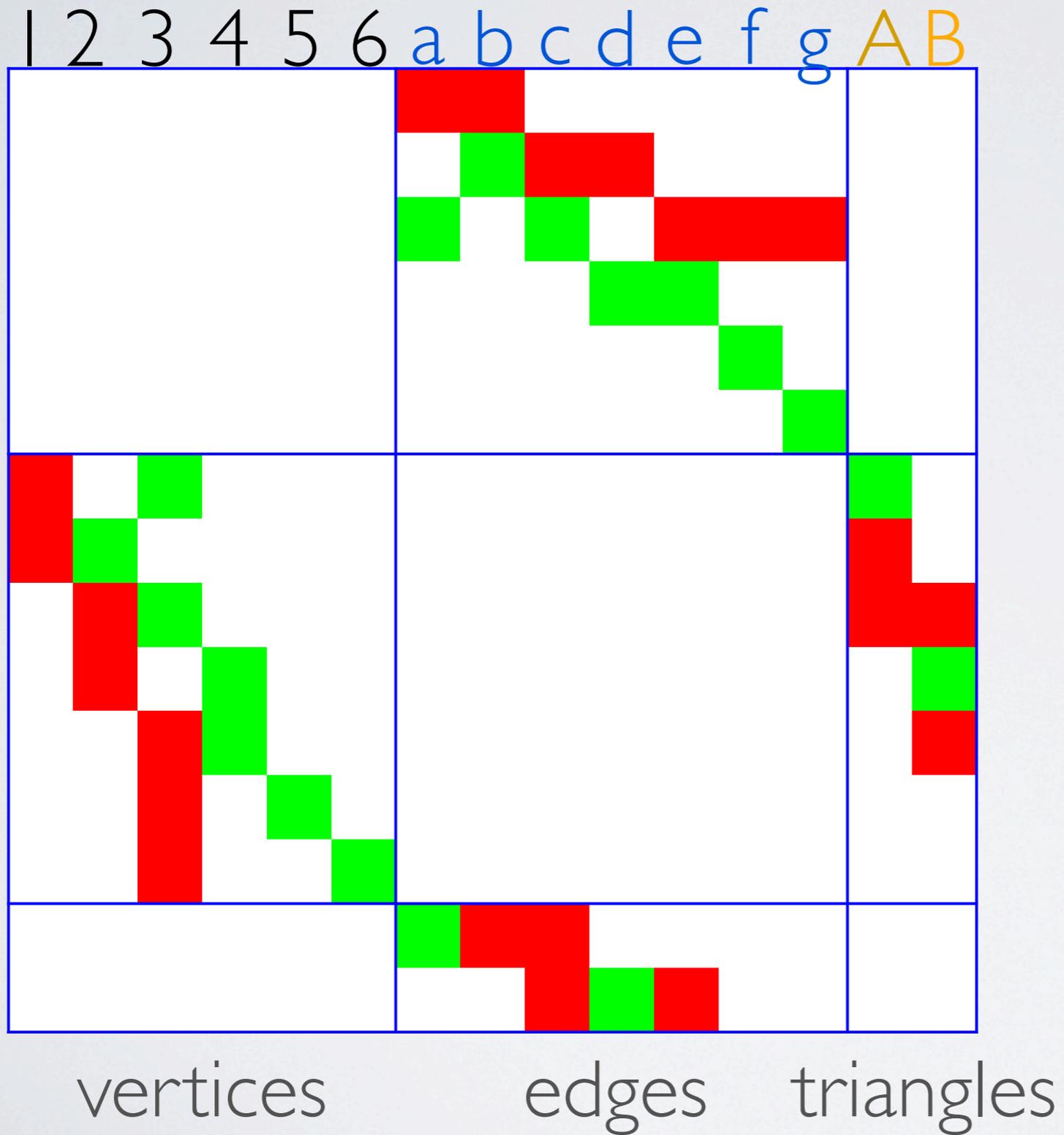


$$L = (d + d)^2 =$$

1	-1	0	0	0	0	0	0	0	0
-1	3	-1	-1	0	0	0	0	0	0
0	-1	2	-1	0	0	0	0	0	0
0	-1	-1	2	0	0	0	0	0	0
0	0	0	0	2	-1	-1	0	0	0
0	0	0	0	-1	3	0	0	0	0
0	0	0	0	-1	0	3	0	0	0
0	0	0	0	0	0	0	3	0	0
0	0	0	0	0	0	0	0	0	3

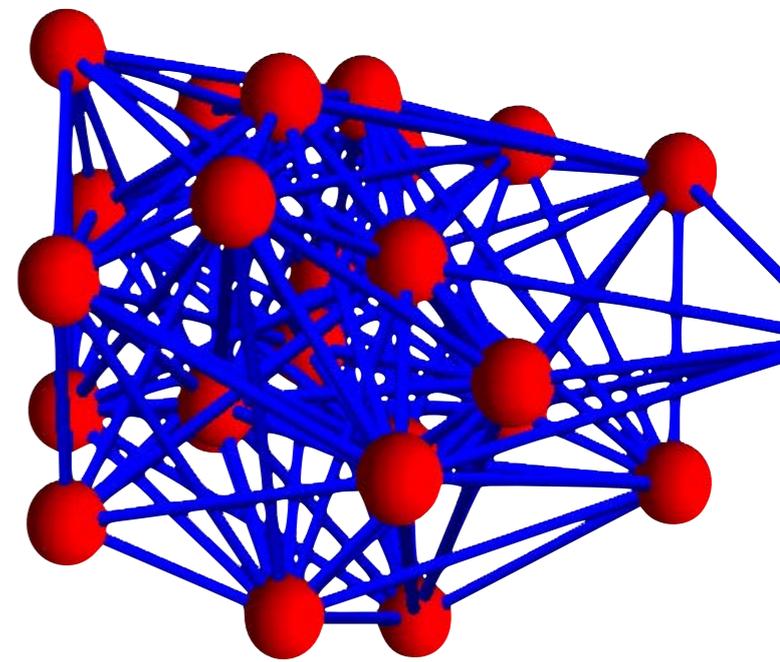
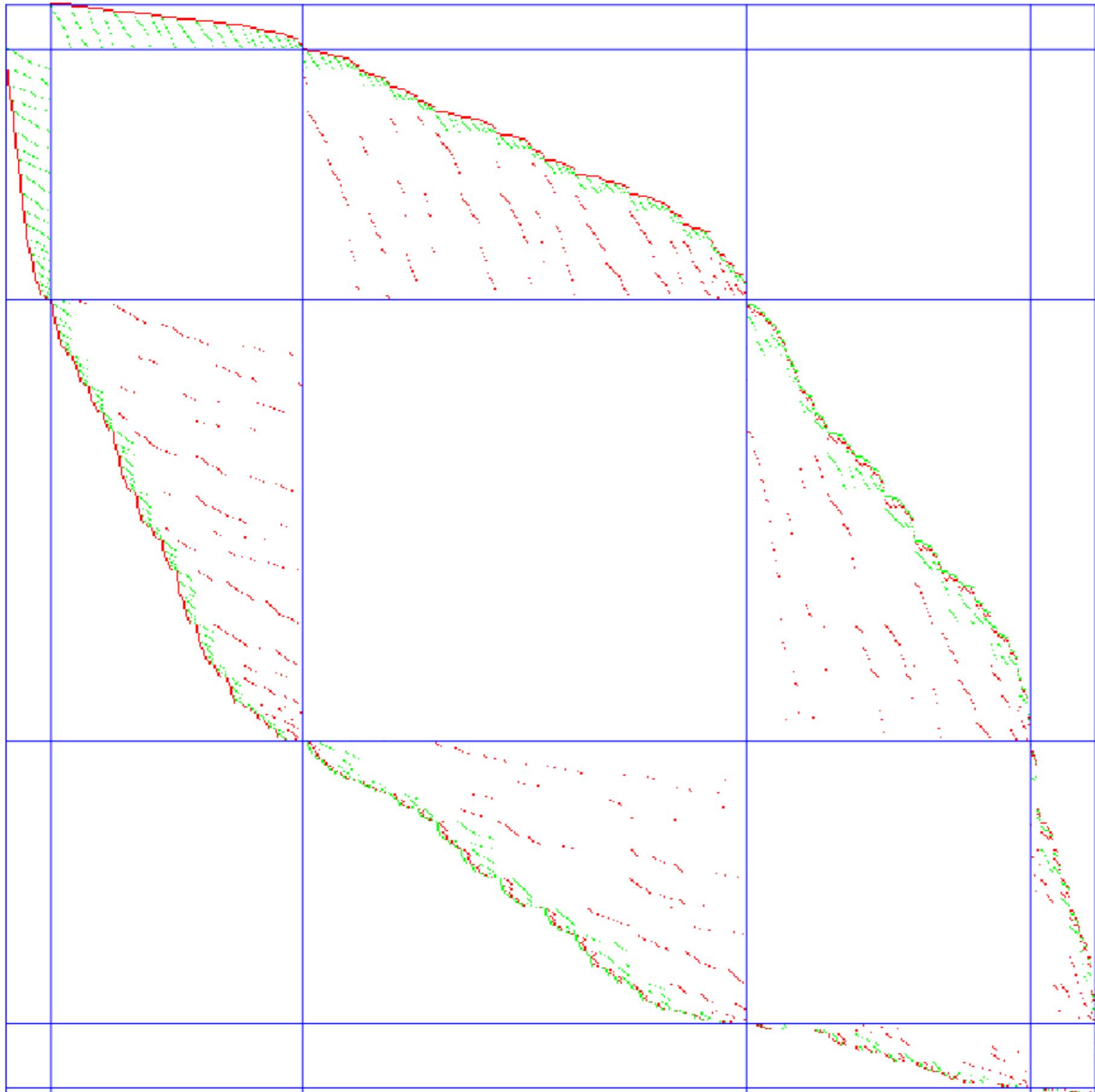
is block diagonal.

AGAIN:



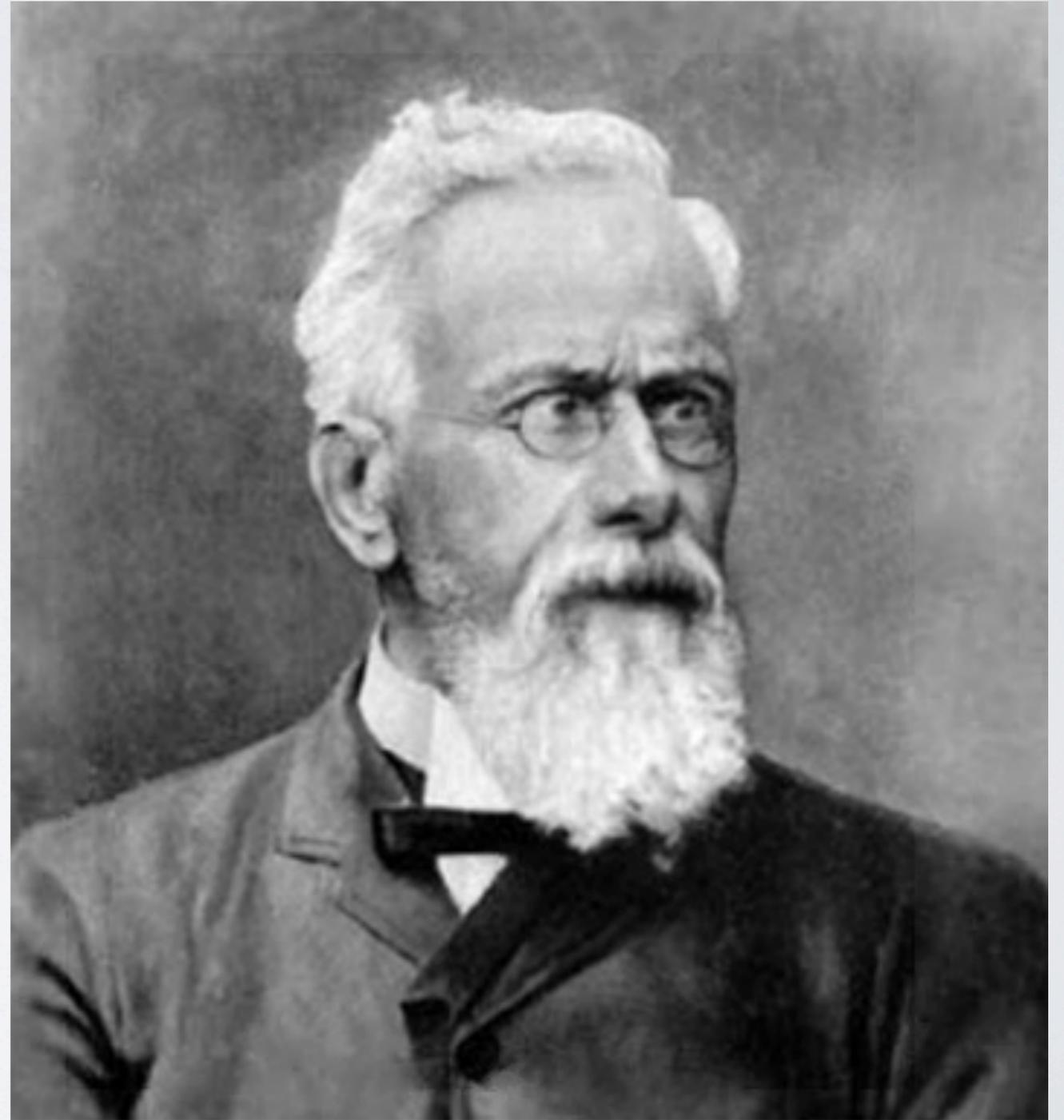
$$D^2 = L$$

RANDOM 25 VERTEX GRAPH

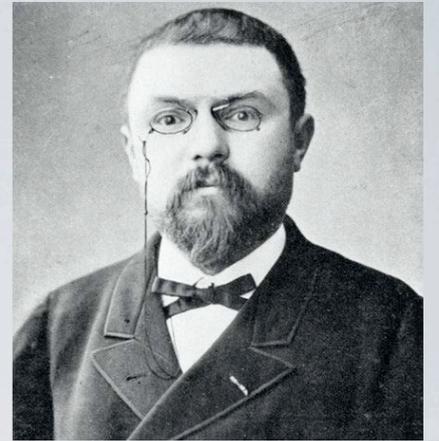


ENRICO BETTI

1823-1892



COHOMOLOGY



$$H^k(G) = \ker(d_k) / \text{im}(d_{k-1})$$

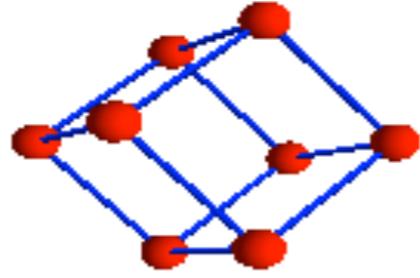
k'th cohomology group

$$b_k(G) = \dim(H^k(G)) \text{ Betti number}$$

$$b_0 - b_1 + b_2 \dots = \chi(G)$$

cohomological Euler characteristic

CUBE

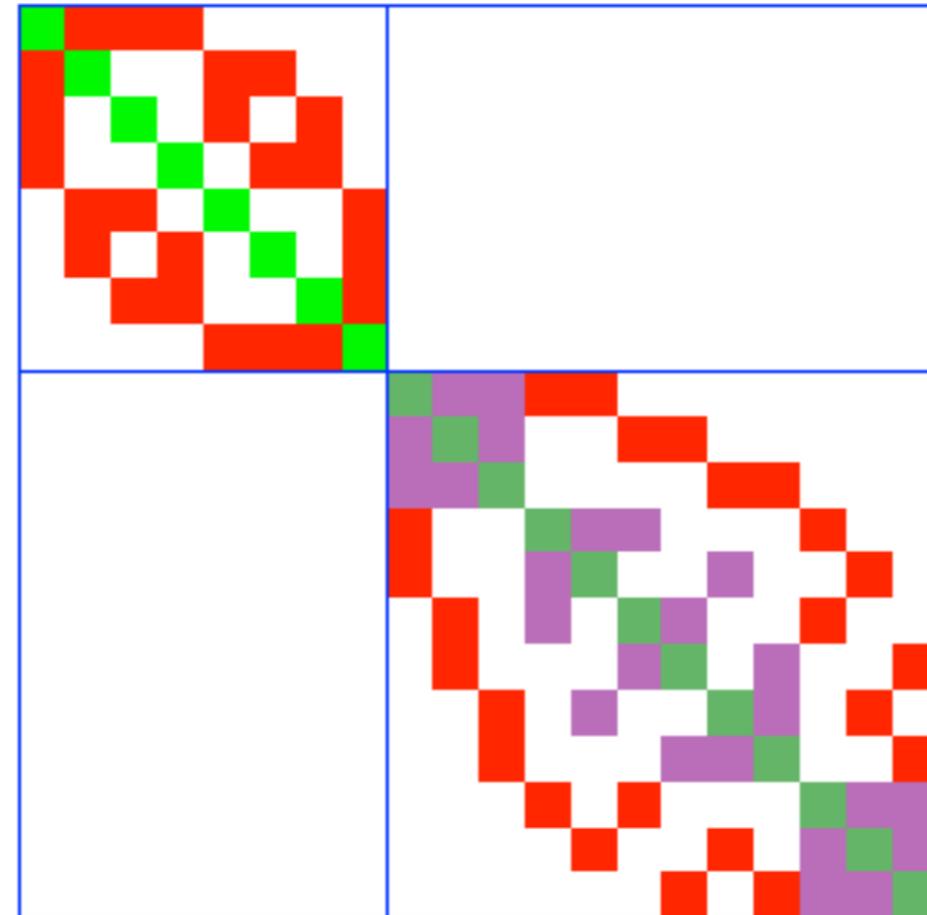
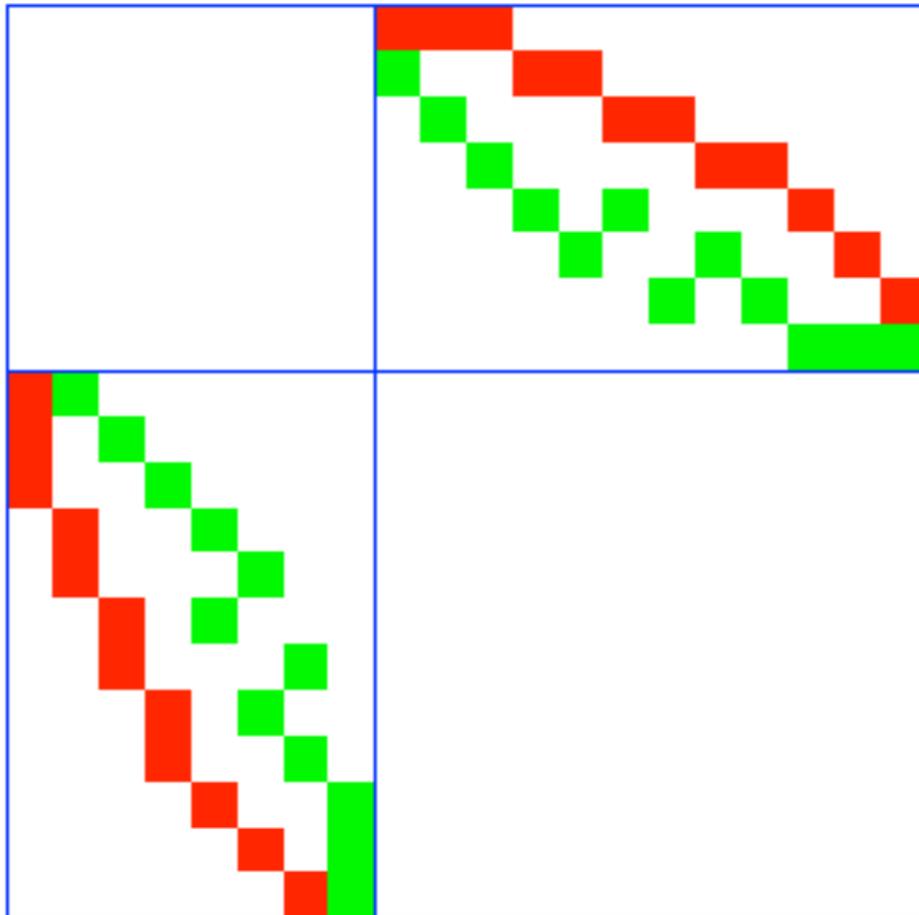


$$v_0 = 8$$

$$v_1 = 12$$

$$\chi = -4$$

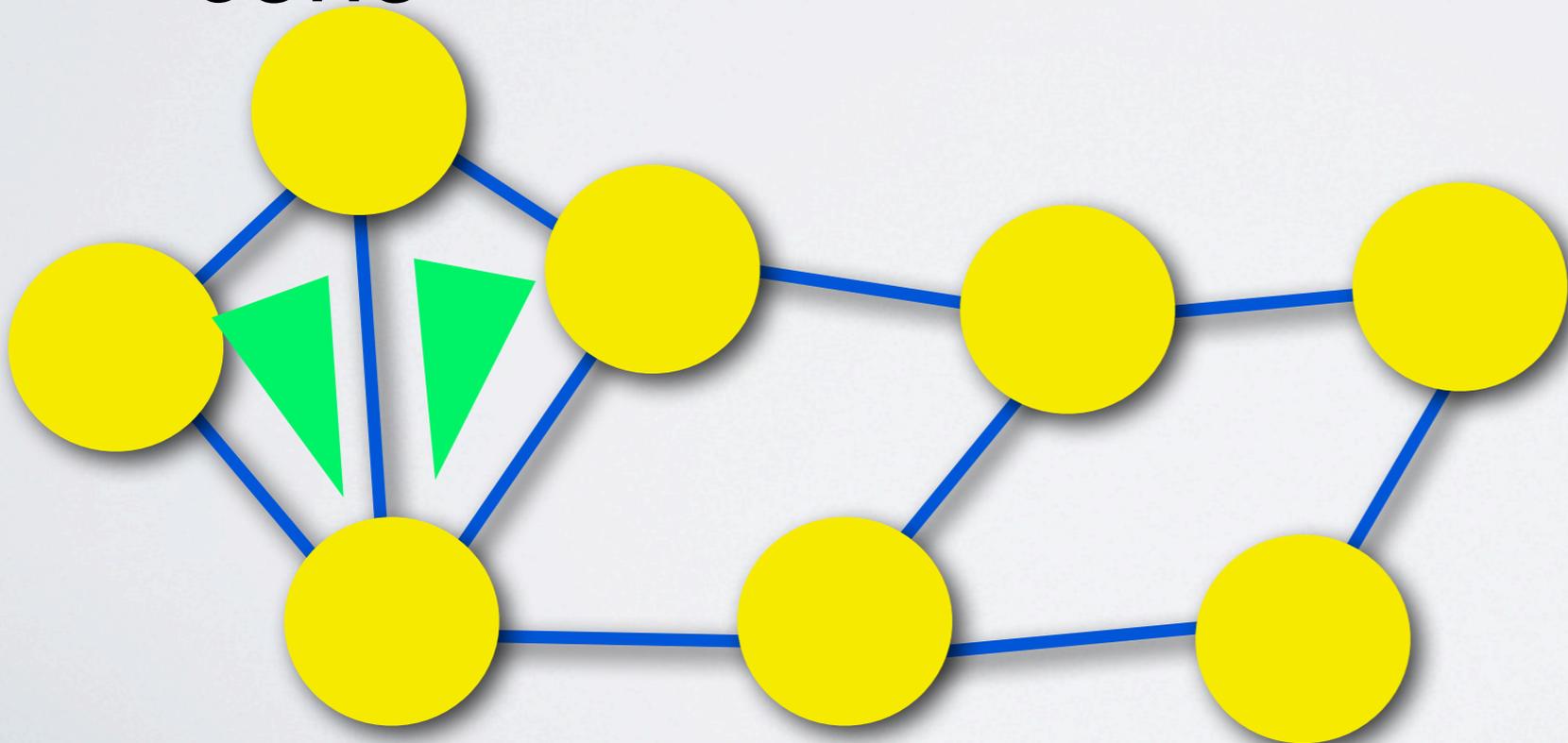
$$\dim = 1.$$



EULER CHARACTERISTIC

$$\chi_{\text{simp}}(G) = v_0 - v_1 + v_2 - \dots = 8 - 11 + 2$$

$$\chi_{\text{coho}}(G) = b_0 - b_1 + b_2 - \dots = 1 - 2$$

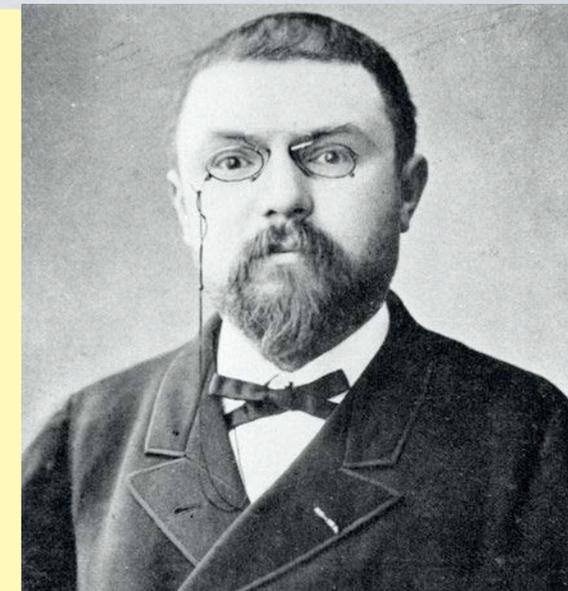


Example: no tetrahedra, then
 $v - e + f = 1 - g$

EULER-POINCARÉ



$$\chi_{\text{simp}}(G) = \chi_{\text{coho}}(G)$$



Proof. C_m space of m forms

$Z_m = \ker(d_m)$ space of m cocycles

$R_m = \text{ran}(d_m)$ space of m coboundaries

$z_m = v_m - r_m$ rank-nullity theorem

$b_m = z_m - r_{m-1}$ definition of cohomology

$$\sum_m (-1)^m (v_m - b_m) = \sum_m (-1)^m (r_m - r_{m-1}) = 0$$

QED

WILLIAM HODGE

1903-1975



HODGE THEOREM

$$H^m(G) \cong \text{Ker}(L_m)$$

Proof:

$Lf = 0$ is equivalent to $df = d^*f = 0$

$$\langle f, Lf \rangle = \langle d^*f, d^*f \rangle + \langle df, df \rangle$$

$$\text{im}(d) + \text{im}(d^*) + \text{ker}(L) = \mathbb{R}^n$$

$\langle dg, d^*h \rangle = 0$. Also $\text{ker}(L)$ and $\text{im}(L)$ are perp.

$dg=0$ implies $g=d f+h$

can match cohomology class $[g]$ - harmonic form

PDE ANALOGUES ON GRAPHS



$$Lu=j$$

Poisson



$$u' = -Lu$$

Heat



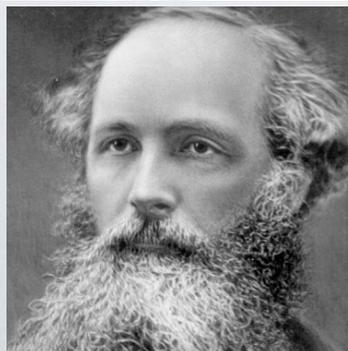
$$u' = iLu$$

Schrödinger



$$u'' = -Lu$$

Wave



$$dF=0, d^*F=j$$

Maxwell

HEAT EQUATION SOLUTION

$$u' = -L u$$

$$u(t) = e^{-tL} u(0)$$

Heat kernel

HEAT KERNEL

$$\exp(-Lt)$$

projection onto harmonics

WAVE EQUATION SOLUTION

$$u'' = -L u$$

$$u'' = -D^2 u$$

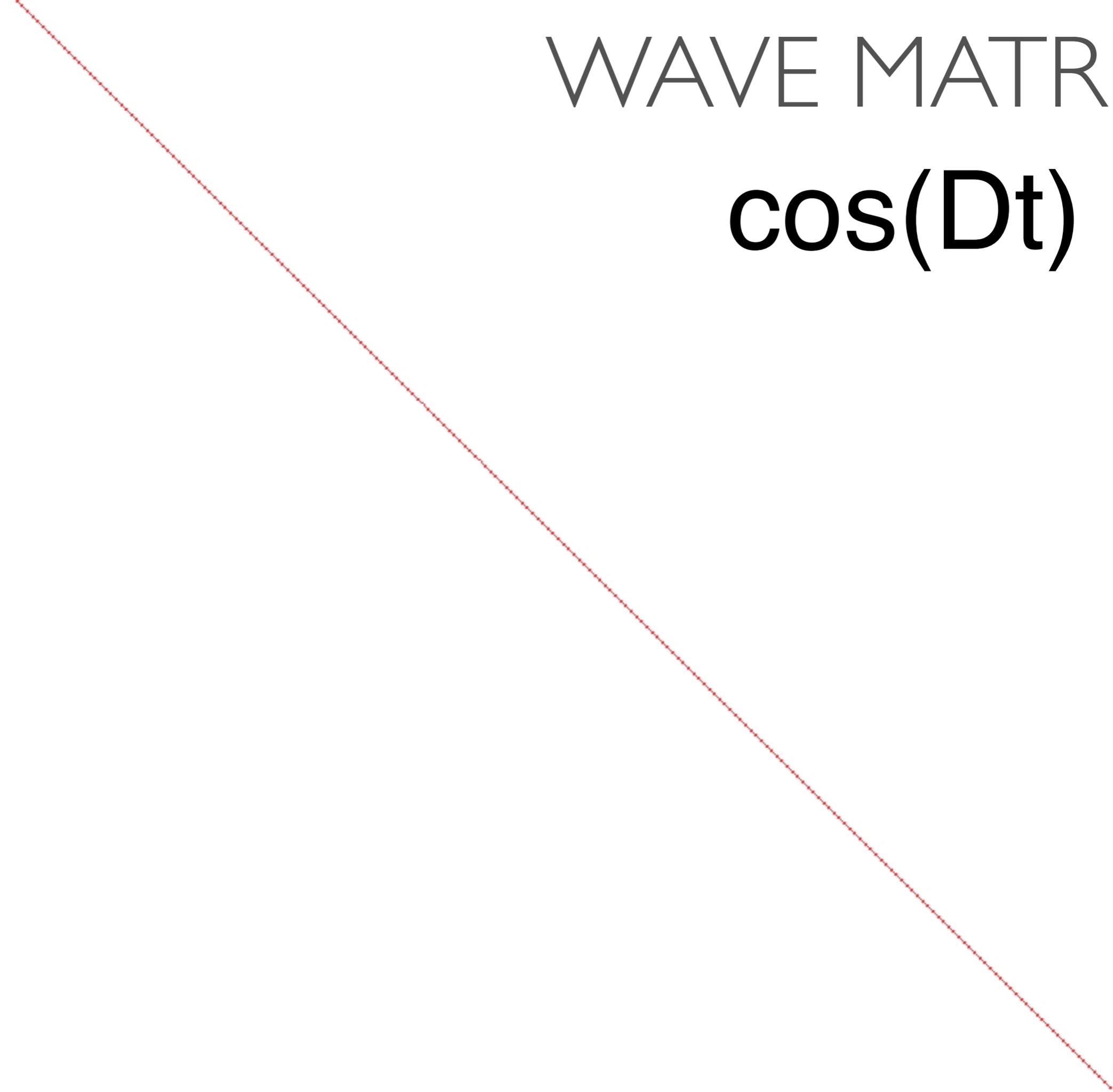
$$(\partial + i D)(\partial - i D) u = 0$$

$$u(t) = \cos(Dt)u(0) + \sin(Dt)D^{-1}u'(0)$$

d'Alembert type solution

WAVE MATRIX

$$\cos(Dt)$$

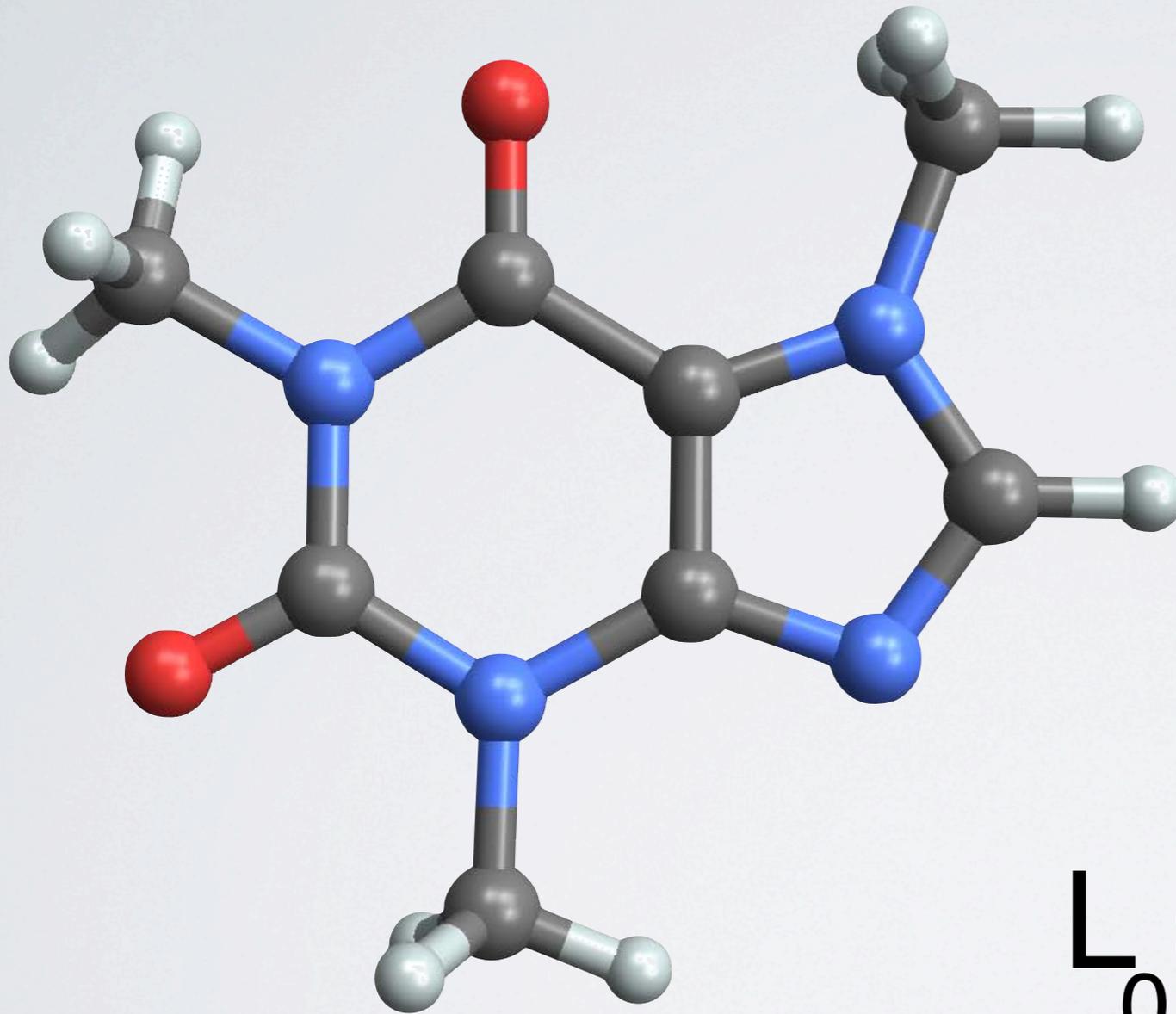


ERICH HUECKEL

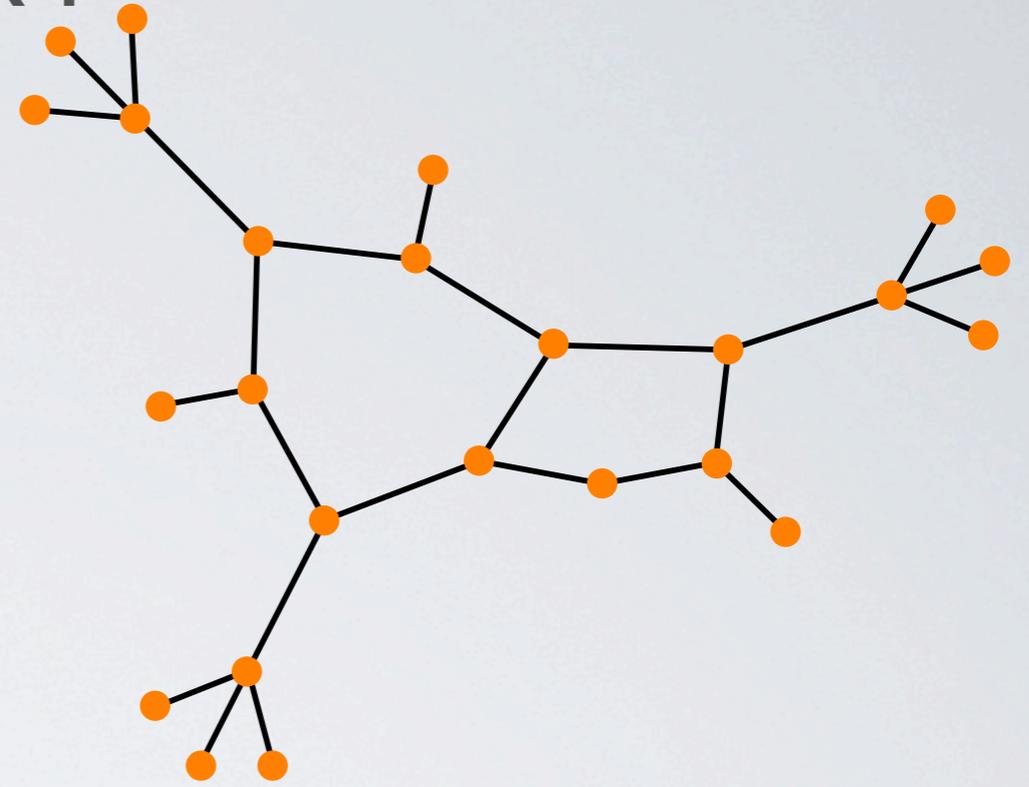
1896-1980



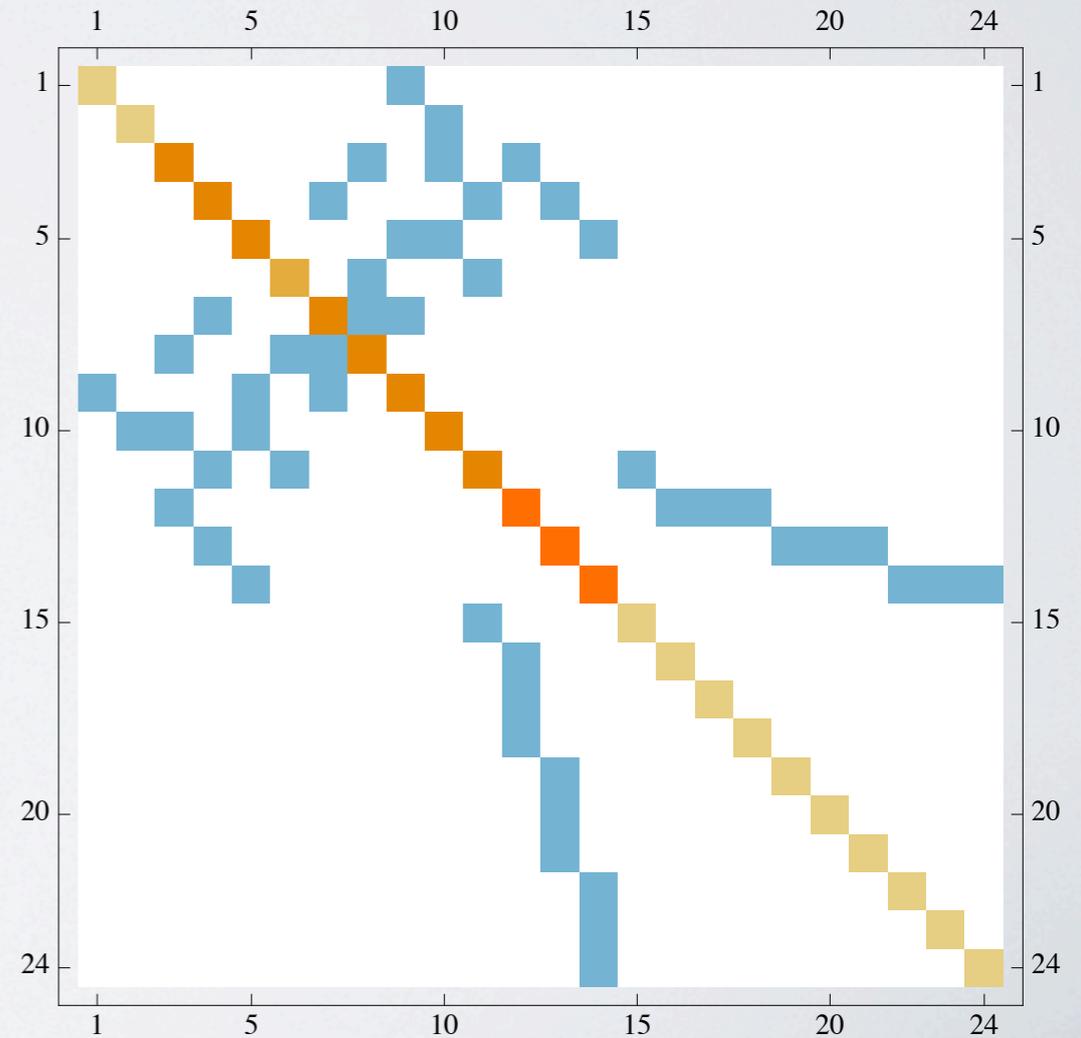
HUECKEL THEORY



caffeine



$$L_0 =$$



ED WITTEN

1951-



SYMMETRY

λ eigenvalue \Rightarrow $-\lambda$ eigenvalue

$$P = \begin{array}{|c|c|} \hline 1 & \\ \hline & -1 \\ \hline & & 1 \\ \hline \end{array}$$

$$PD + DP = 0$$

$$P^2 = I, D^2 = L$$

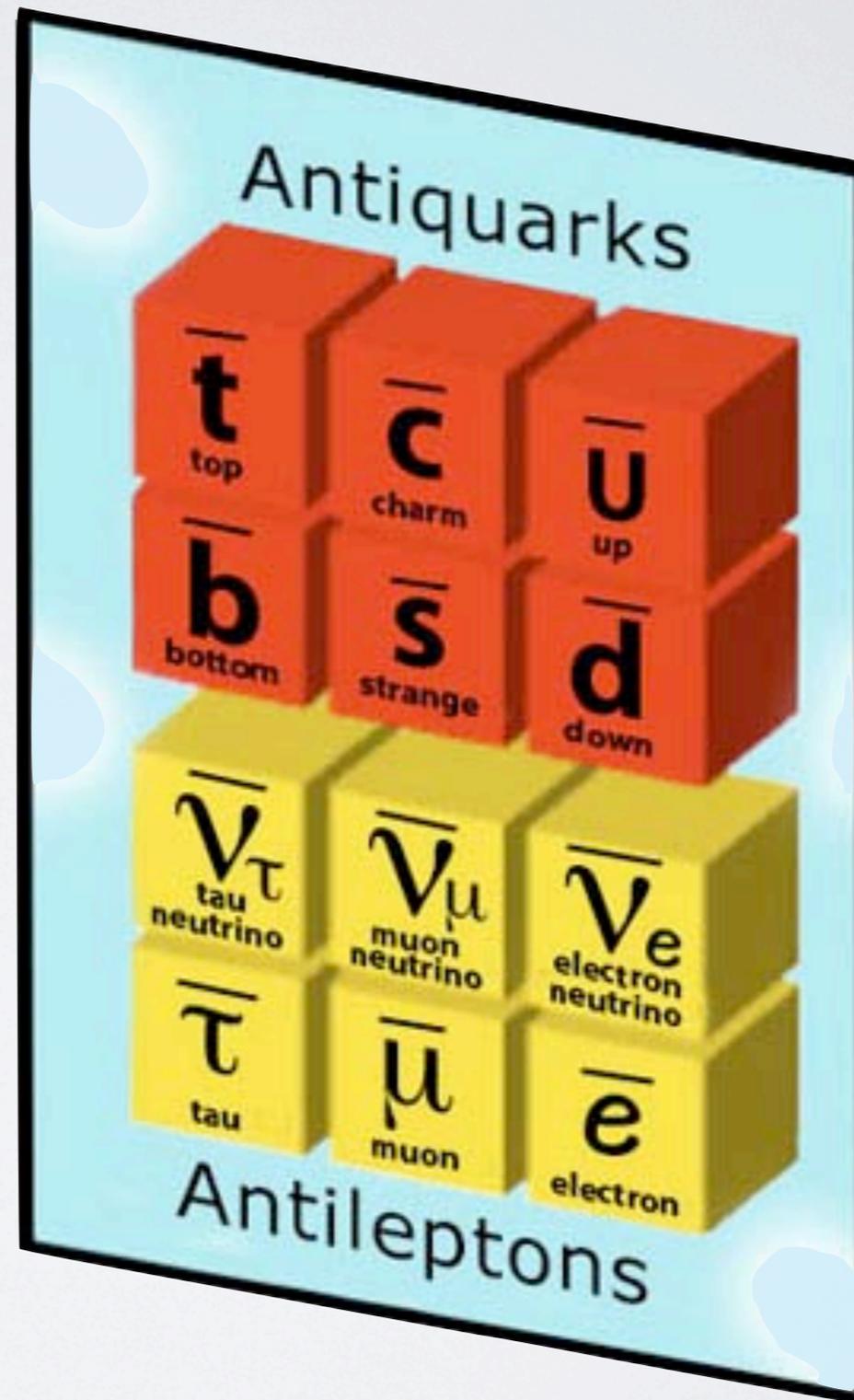
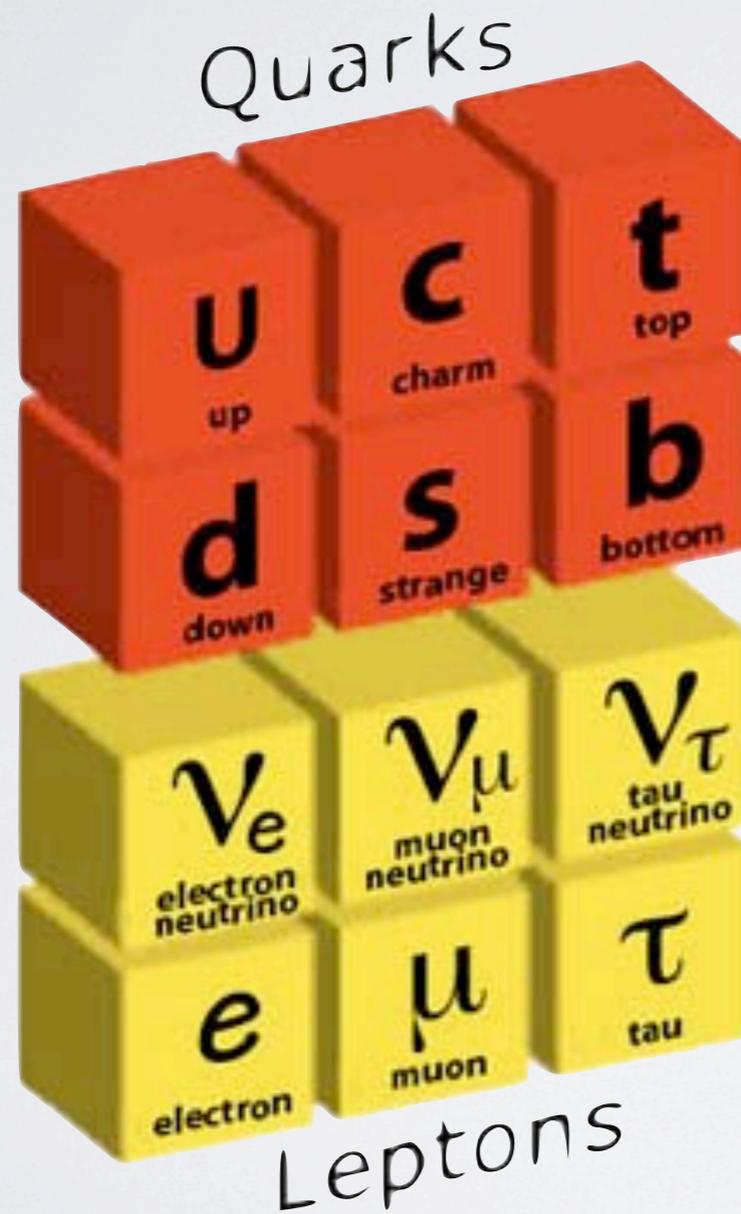
Proof: $Df = \lambda f$

$$\lambda Pf = P \lambda f$$

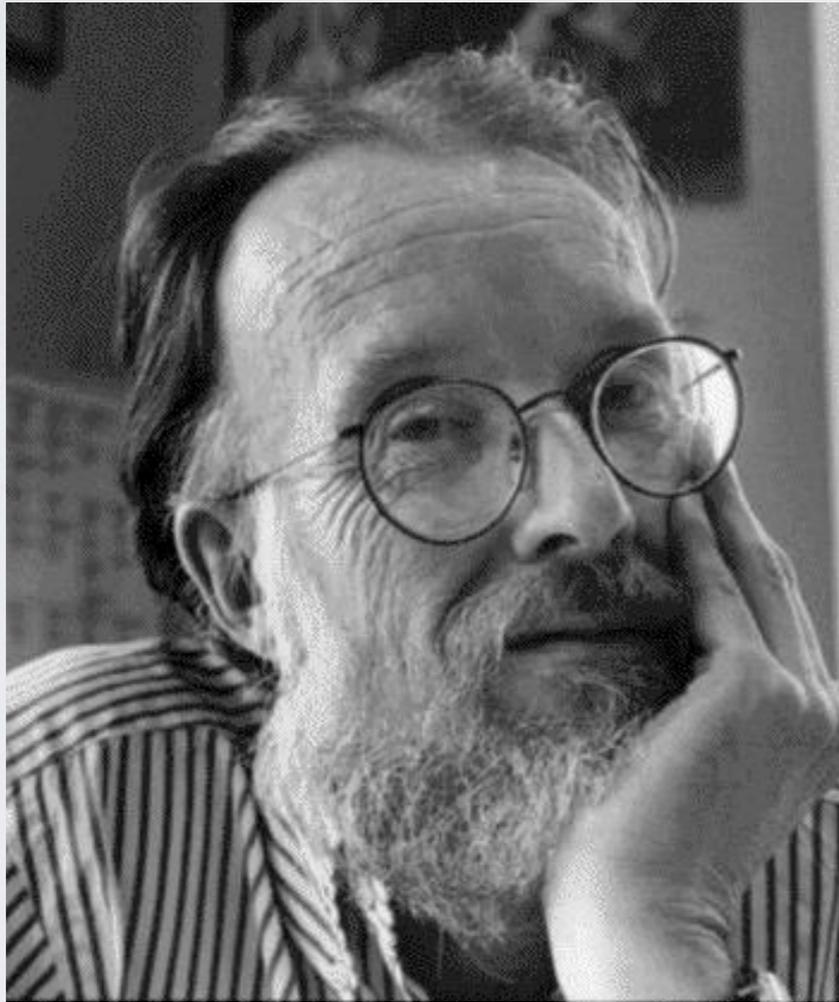
$$= PDf = -DPf$$

Super symmetry

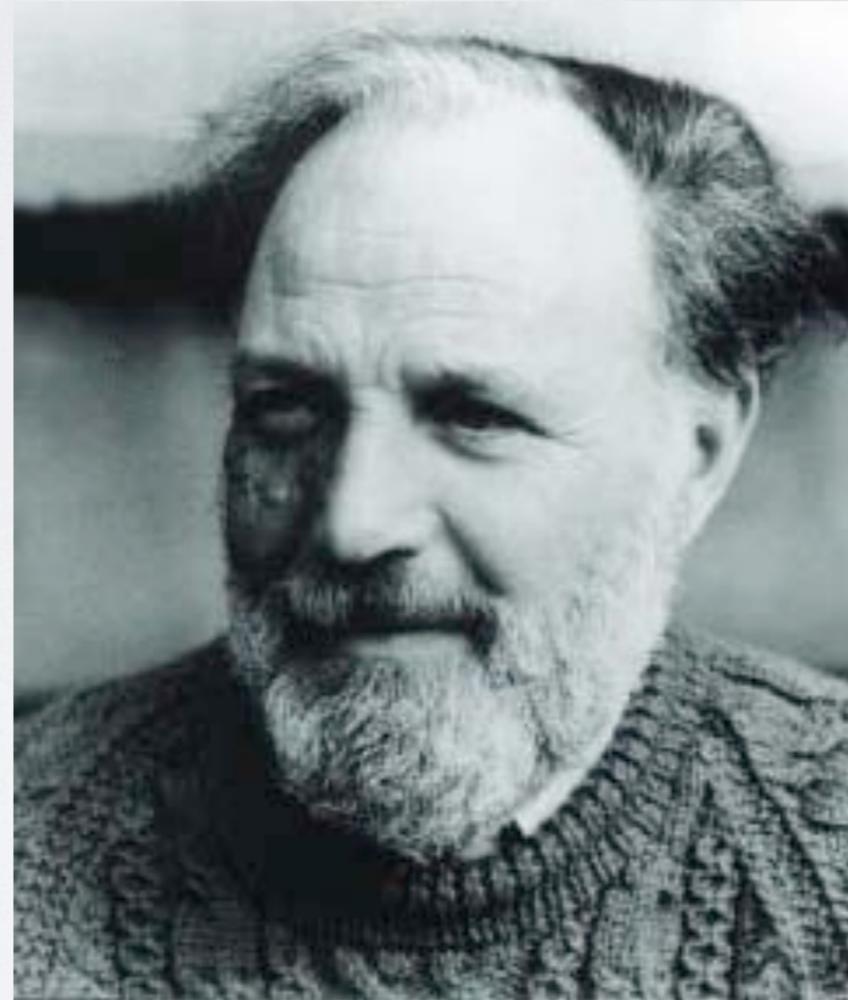
ANTI MATTER



MC KEAN - SINGER



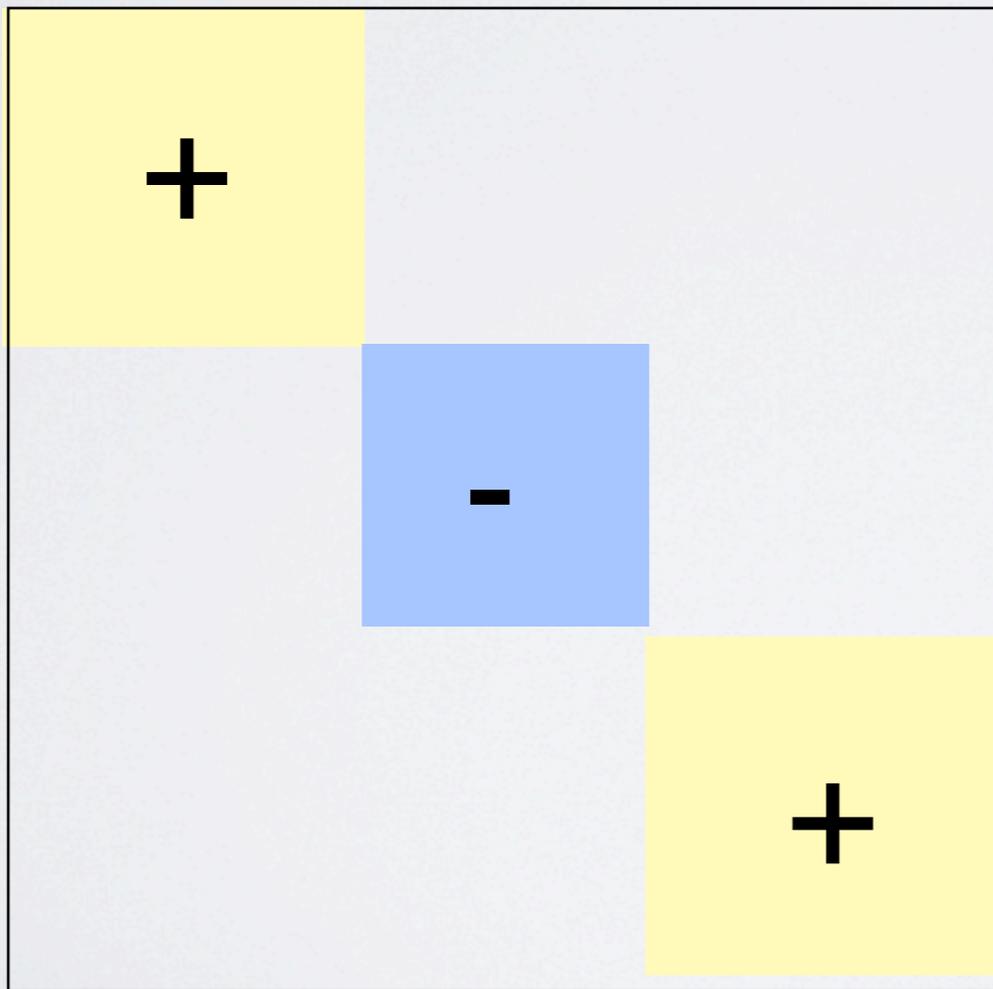
1930-



1924-

SUPER TRACE

$$\text{str}(A) = \text{tr}(P A)$$



Example:

$$\text{str}(1) = \text{tr}(P) = \chi(G)$$

MCKEAN SINGER

$$\text{str}(\exp(-t L)) = \chi(G)$$

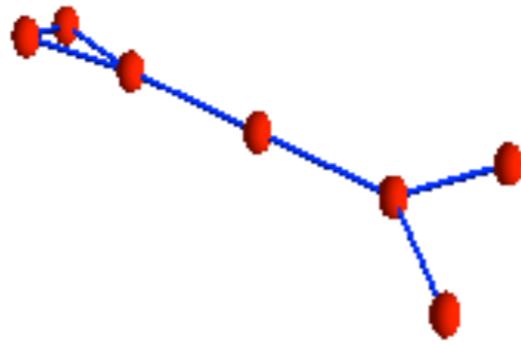
Proof. Each Boson f matches a Fermion Df

$$Lf = \lambda f$$

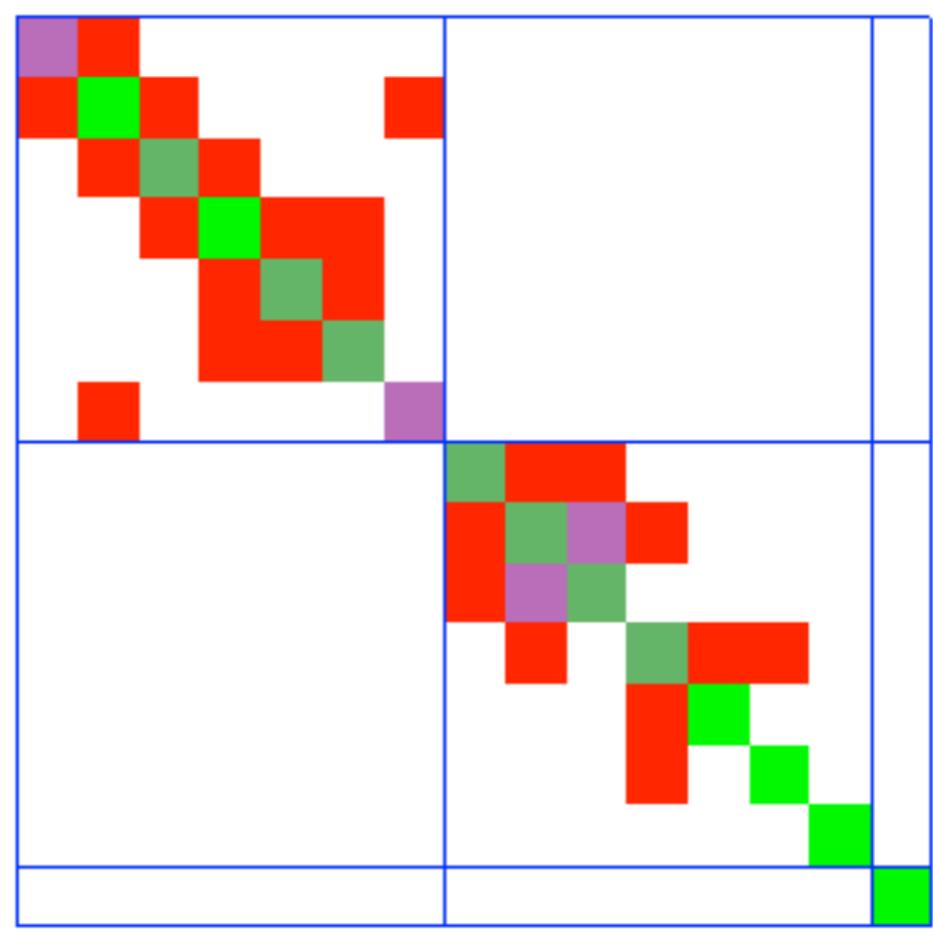
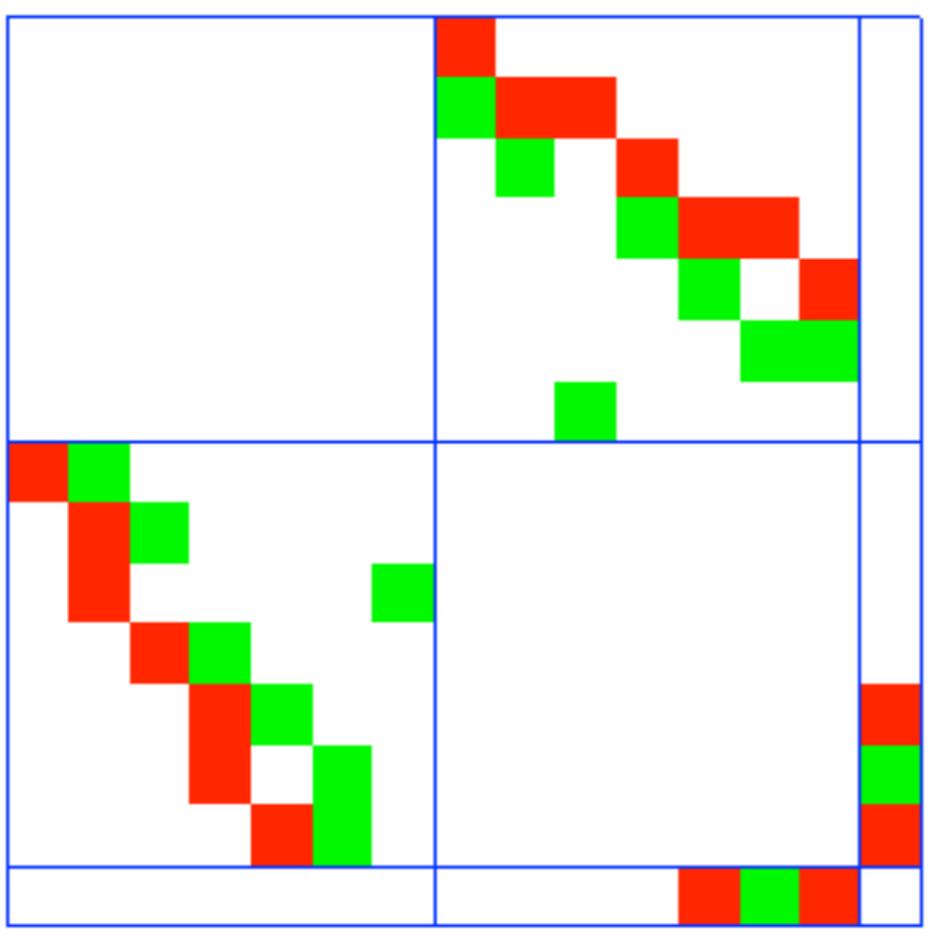
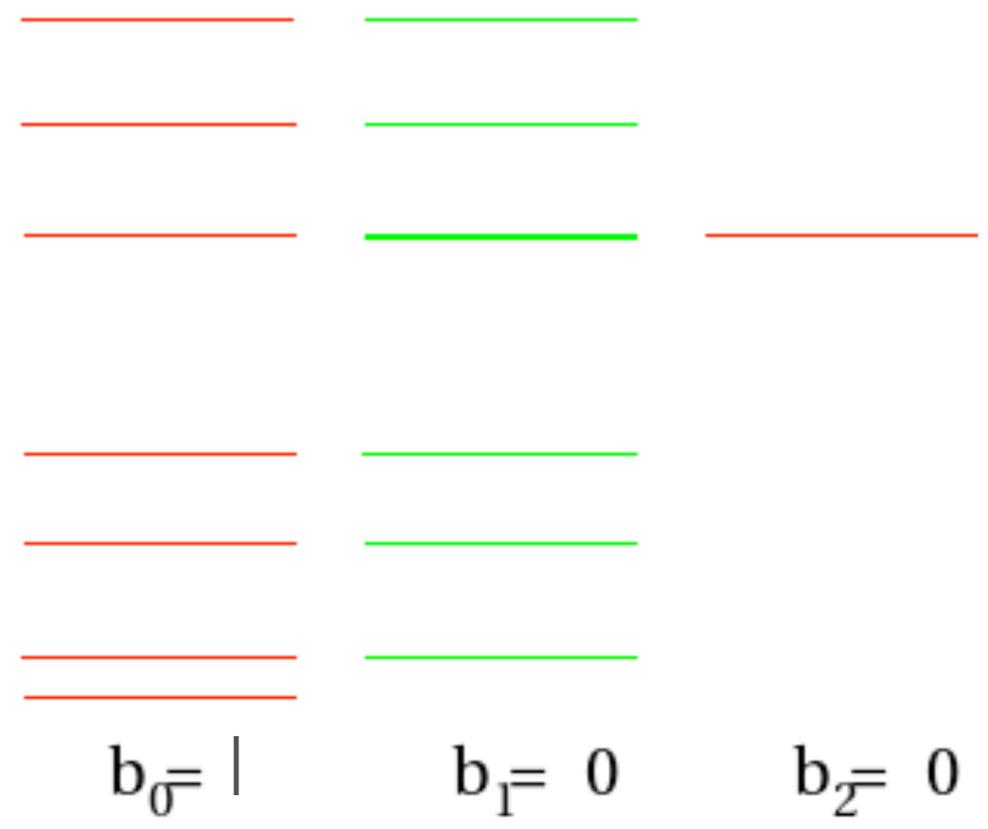
$$L Df = D Lf = D \lambda f = \lambda Df$$

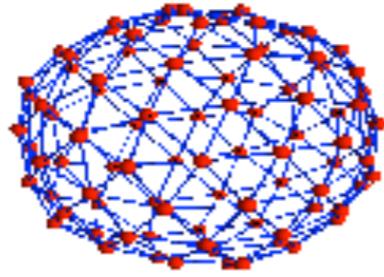
$$\text{str}(L^n) = 0 \text{ for } n > 0$$

$$\text{str}(\exp(-t L)) = \text{str}(1) = \chi(G)$$



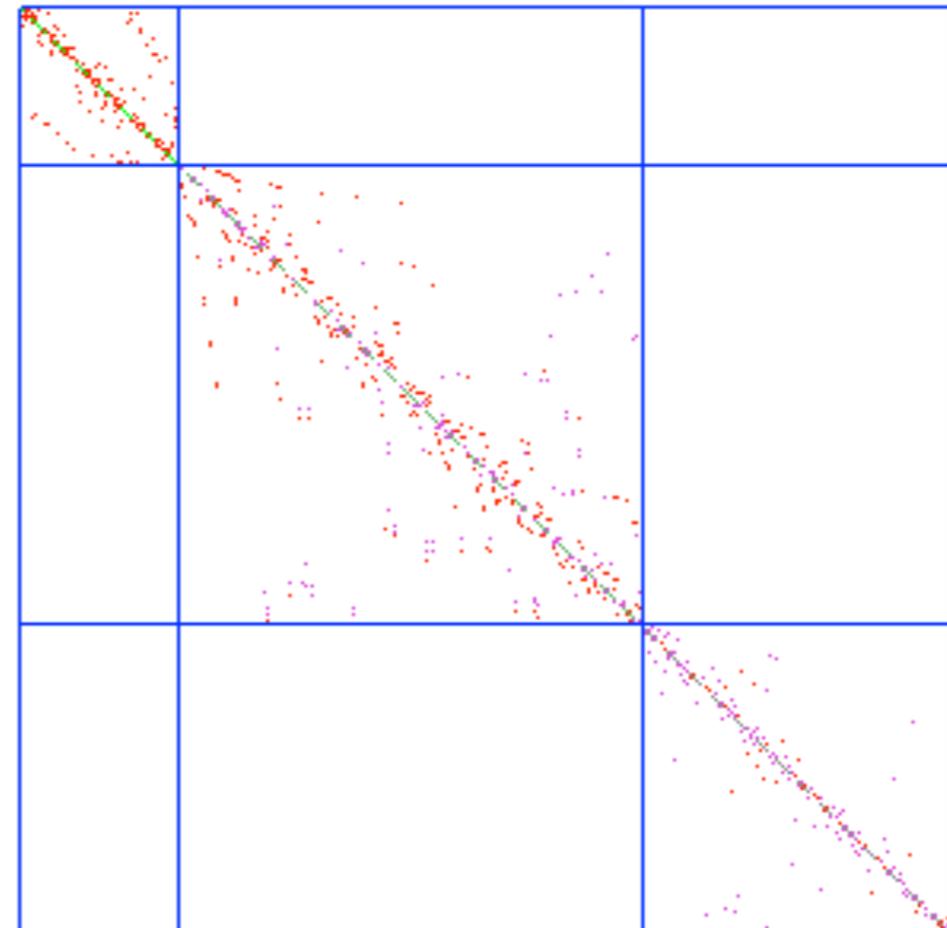
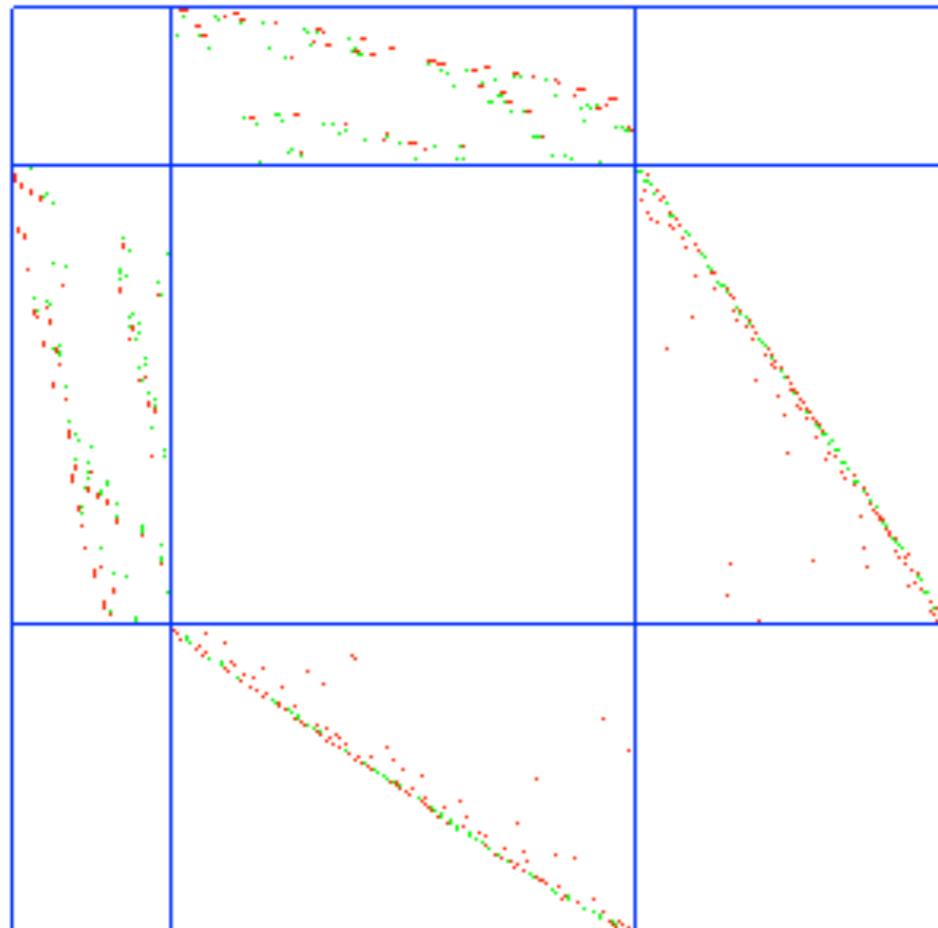
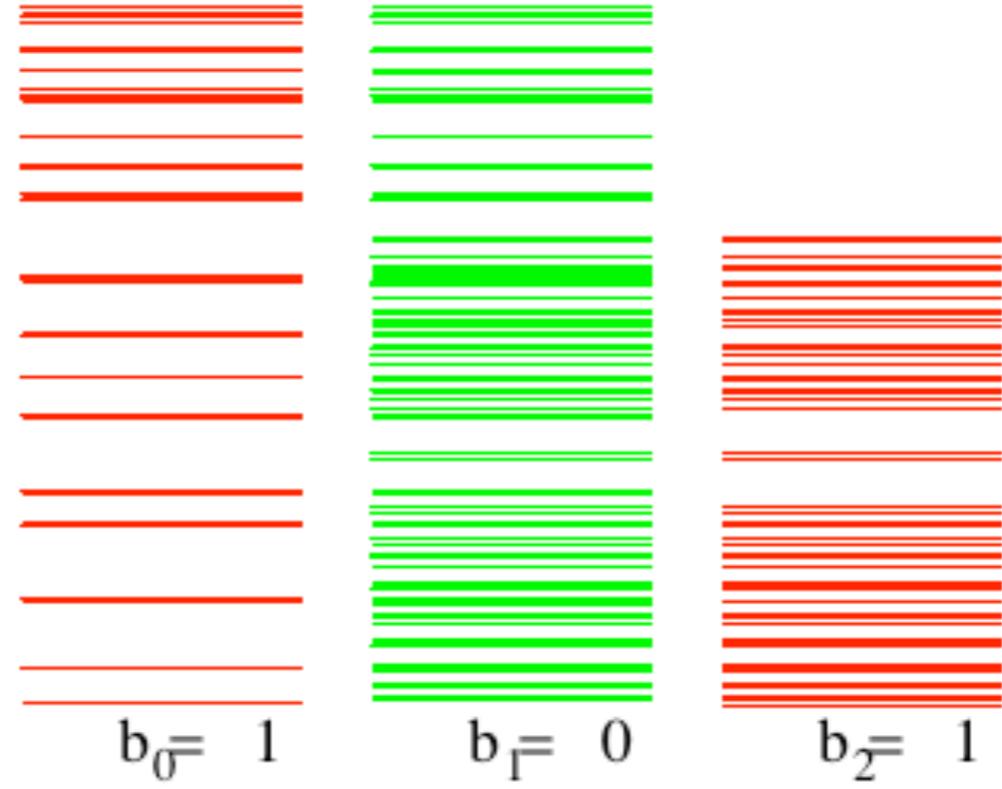
$v_{\bar{0}} = 7$ $v_{\bar{1}} = 7$ $v_{\bar{2}} = 1$
 $\chi = 1$ $\text{dim} = 1.38$

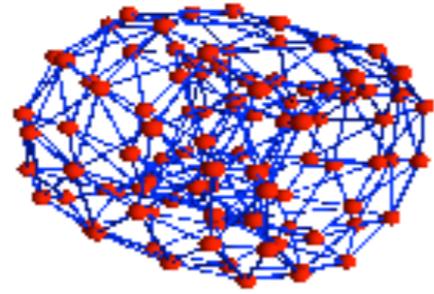




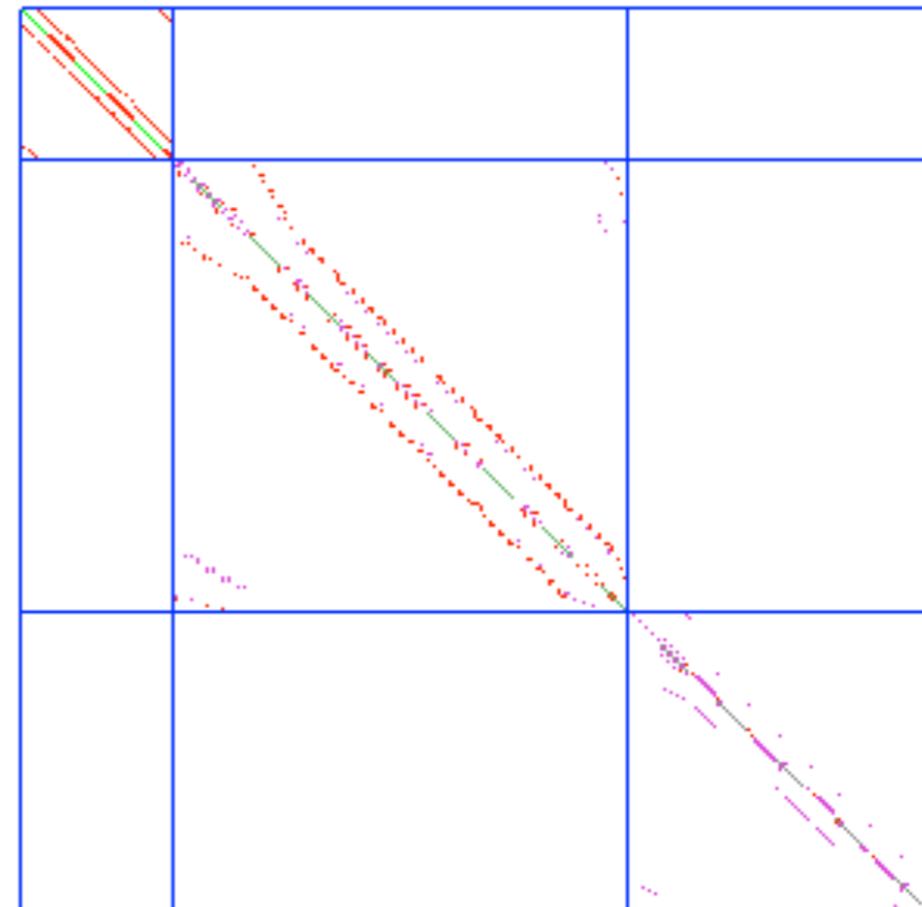
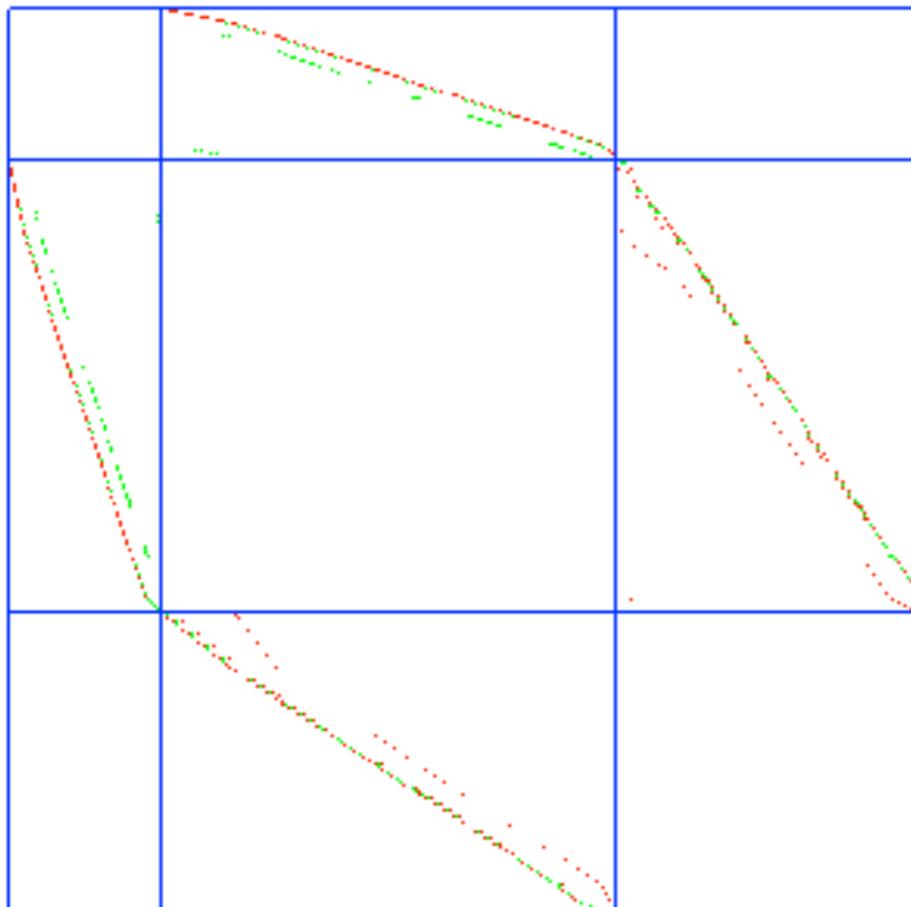
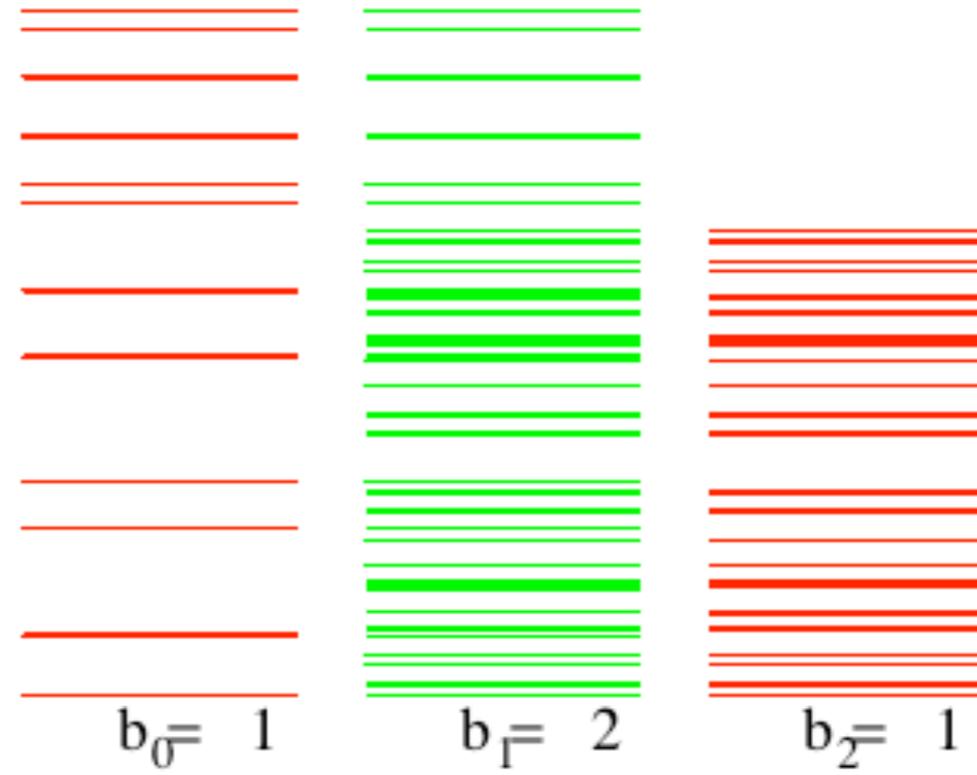
$$v_{\bar{0}} = 92 \quad v_{\bar{1}} = 270 \quad v_{\bar{2}} = 180$$

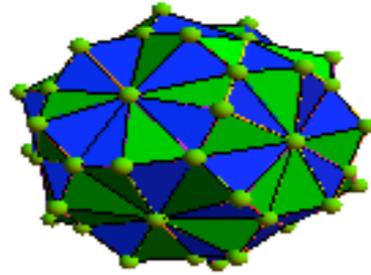
$$\chi = 2 \quad \text{dim} = 2.$$





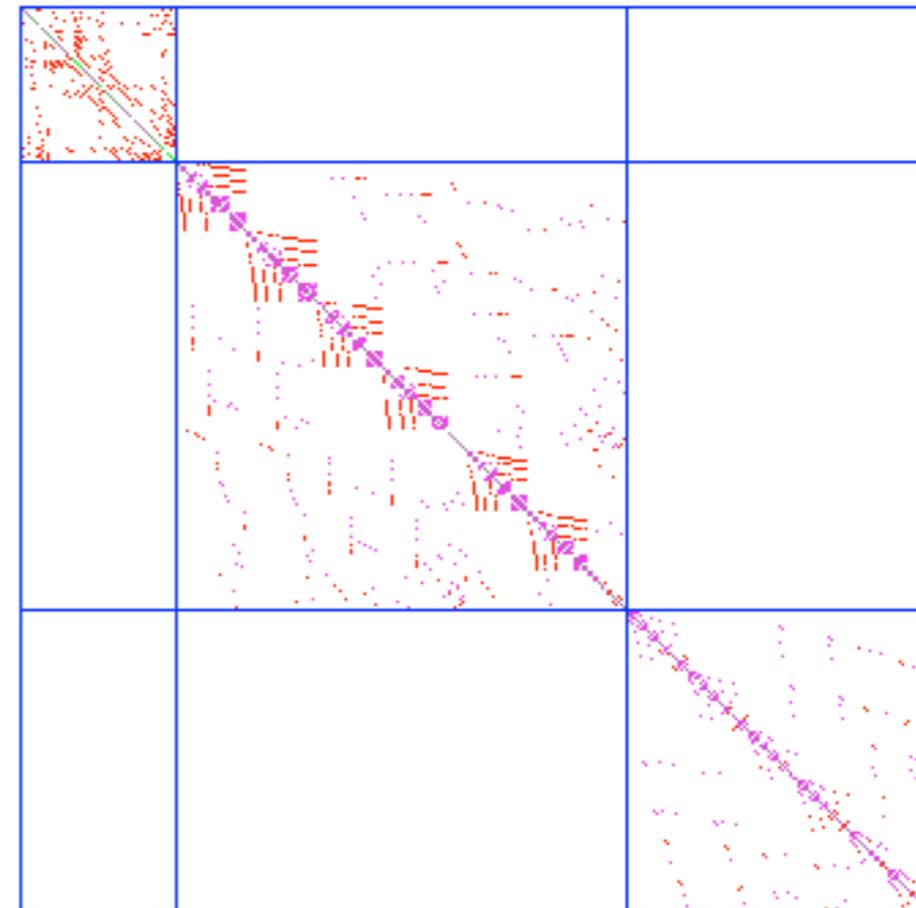
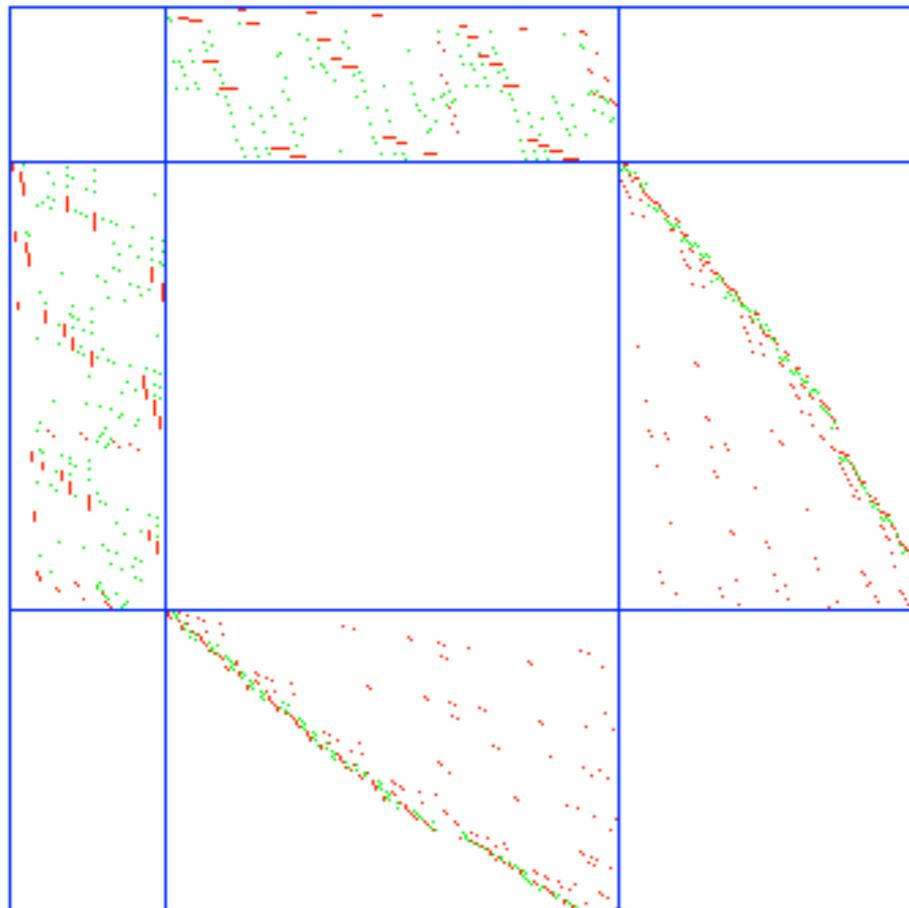
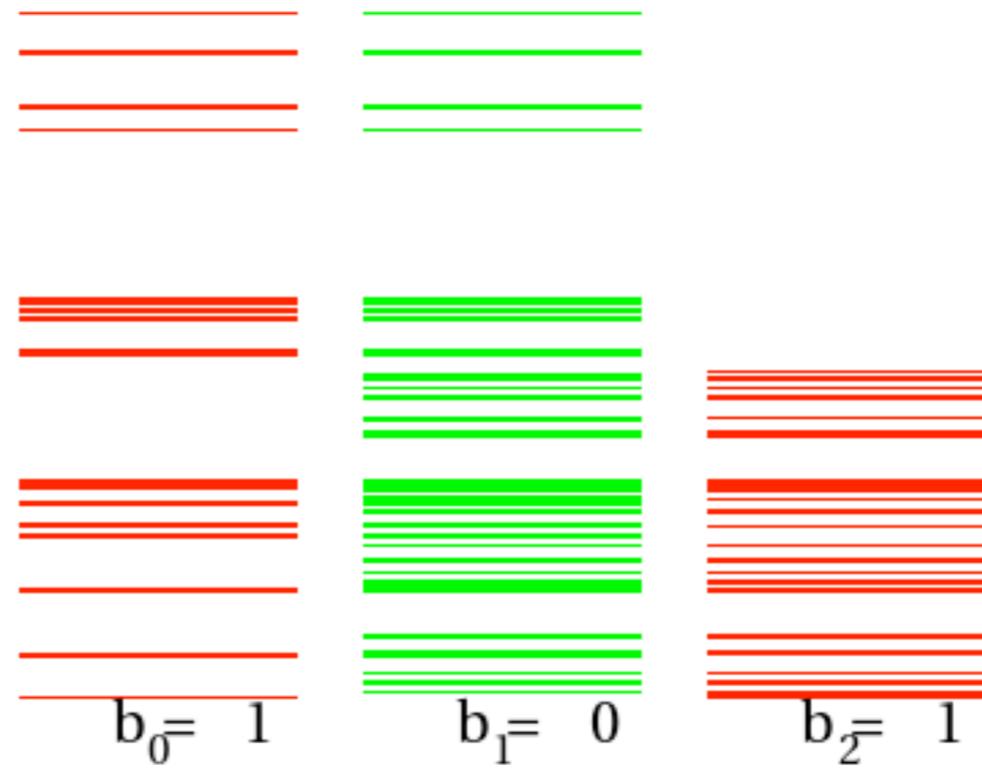
$v_0 = 100$ $v_1 = 300$ $v_2 = 200$
 $\chi = 0$ $\dim = 2.$

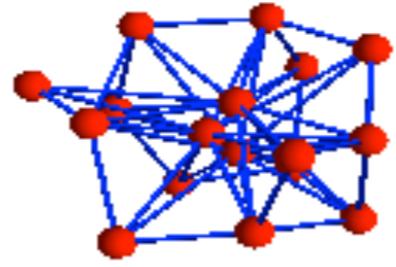




$$v_{\bar{0}} = 62 \quad v_{\bar{1}} = 180 \quad v_{\bar{2}} = 120$$

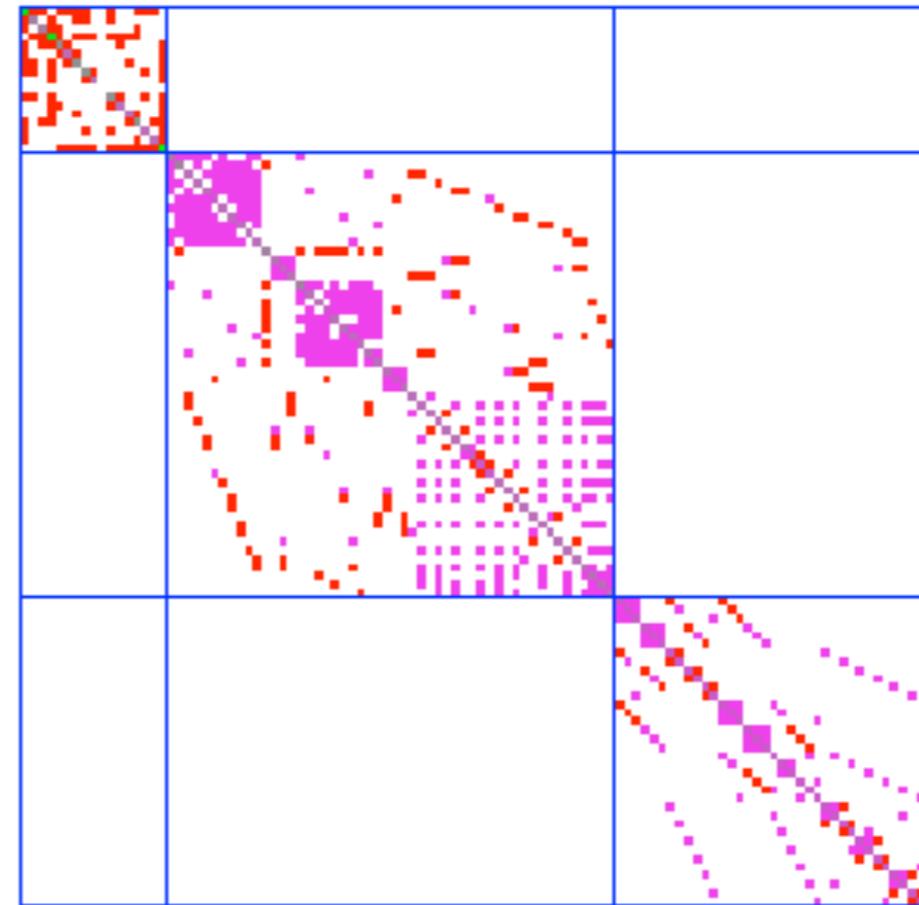
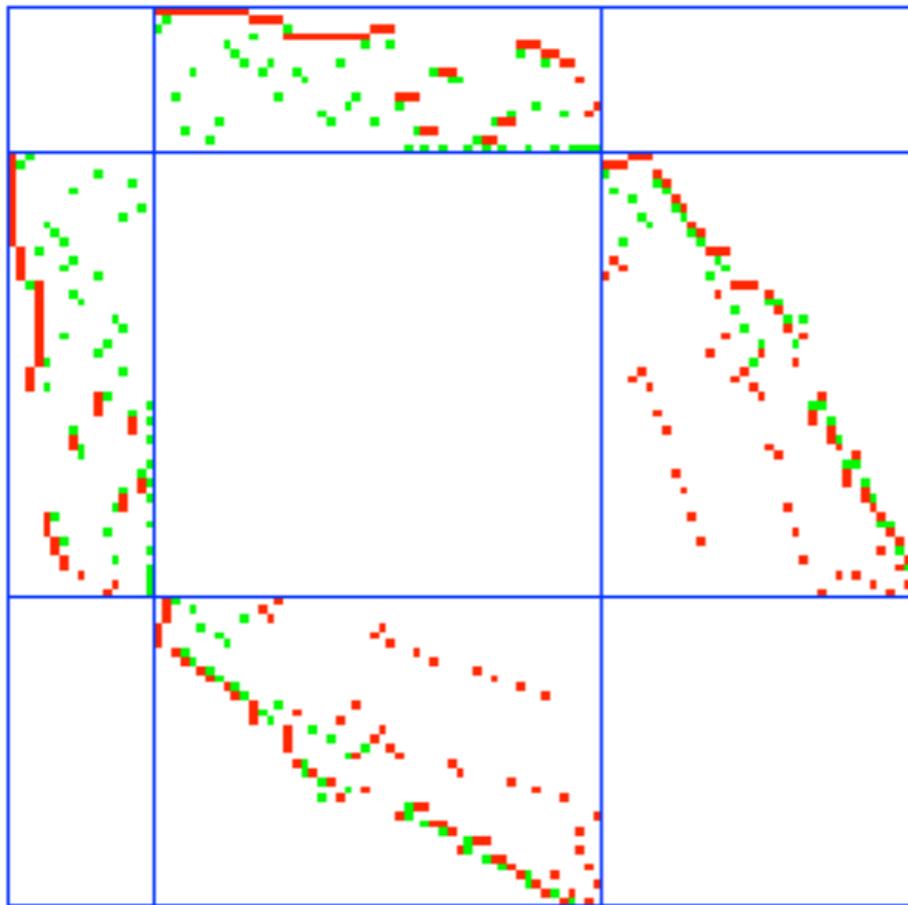
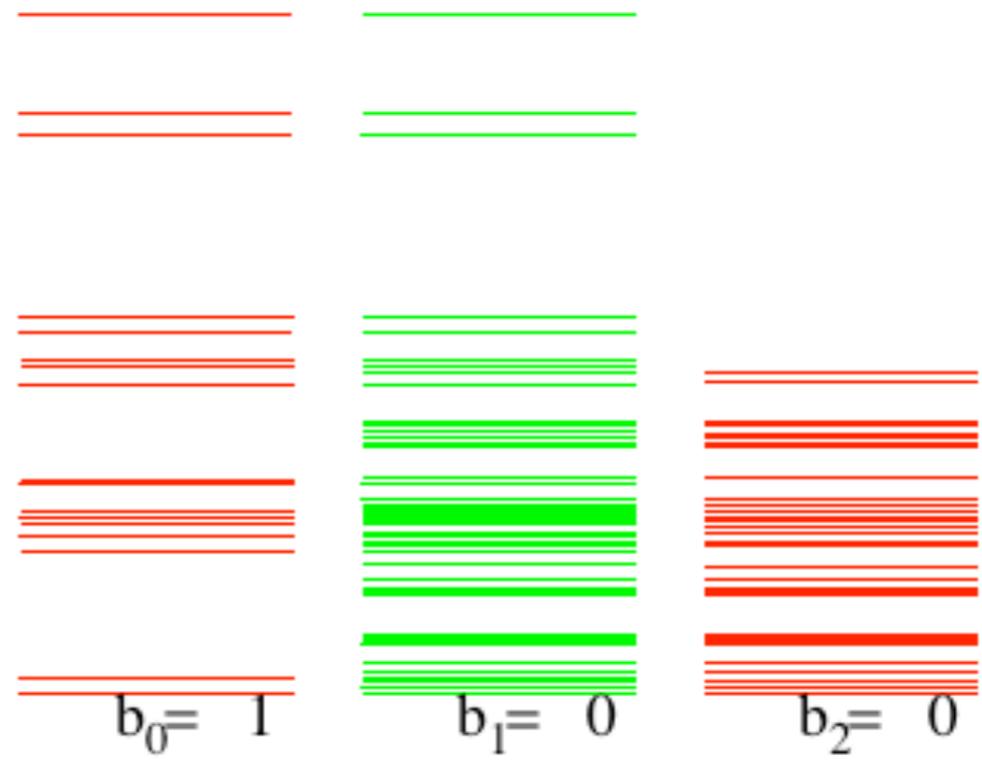
$$\chi = 2 \quad \dim = 2.$$





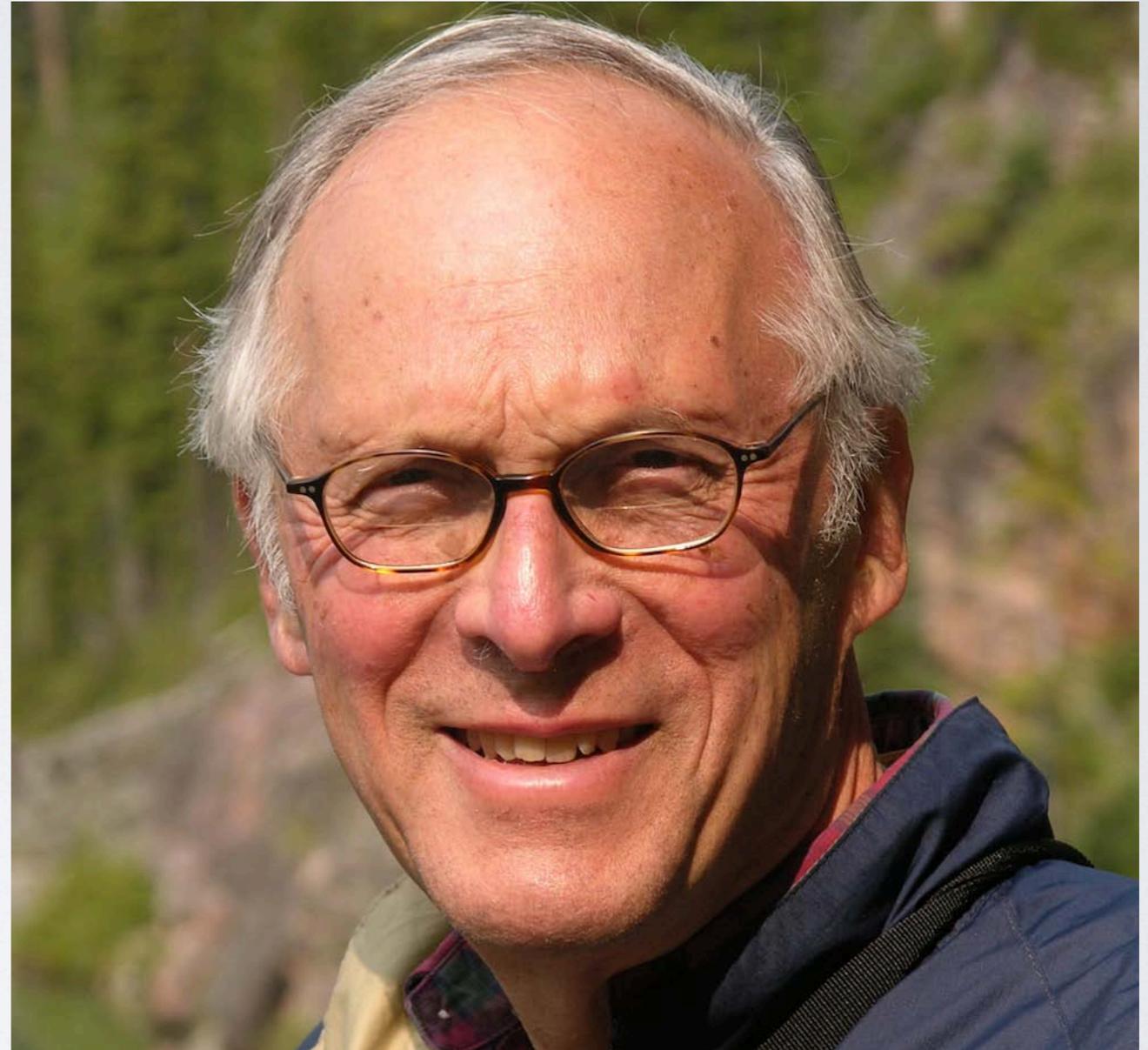
$$v_0 = 17 \quad v_1 = 52 \quad v_2 = 36$$

$$\chi = 1 \quad \text{dim} = 2.$$



MARK KAC

1914-1984



HEAR THE SHAPE?

- Marc Kac: Can one hear the shape of a drum?
- Norman Biggs, Fan Chung and others: To which degree can one hear the shape of a graph?



MATH FOR EVERYONE

TALKS AND DISCUSSIONS ESPECIALLY FOR UNDERGRADS



Carolyn Gordon
Dartmouth College

In spectroscopy, one attempts to recover the shape or chemical composition of an object from the characteristic frequencies of sound or light emitted. Mark Kac's question, "Can one hear the shape of a drum?" asks whether two membranes (drumheads) which vibrate at the same characteristic frequencies must have the same shape. We answer Kac's question negatively by constructing pairs of exotic-shaped sound-alike drums. We also listen to a computer simulation, produced by Dennis DeTurck, of the sounds of these drums.

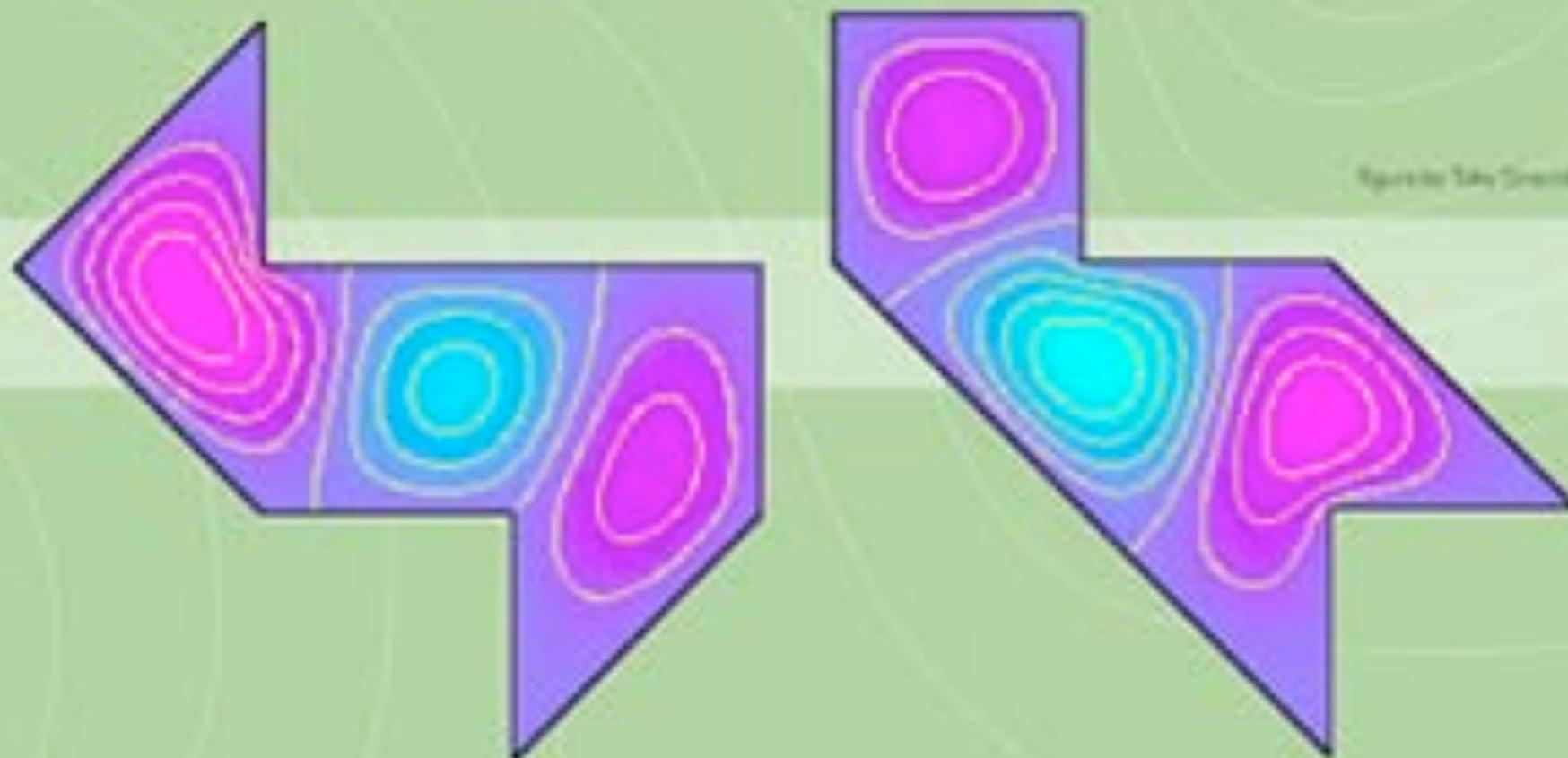


Figure by Toby Driscoll

"YOU CAN'T HEAR THE SHAPE OF A DRUM"

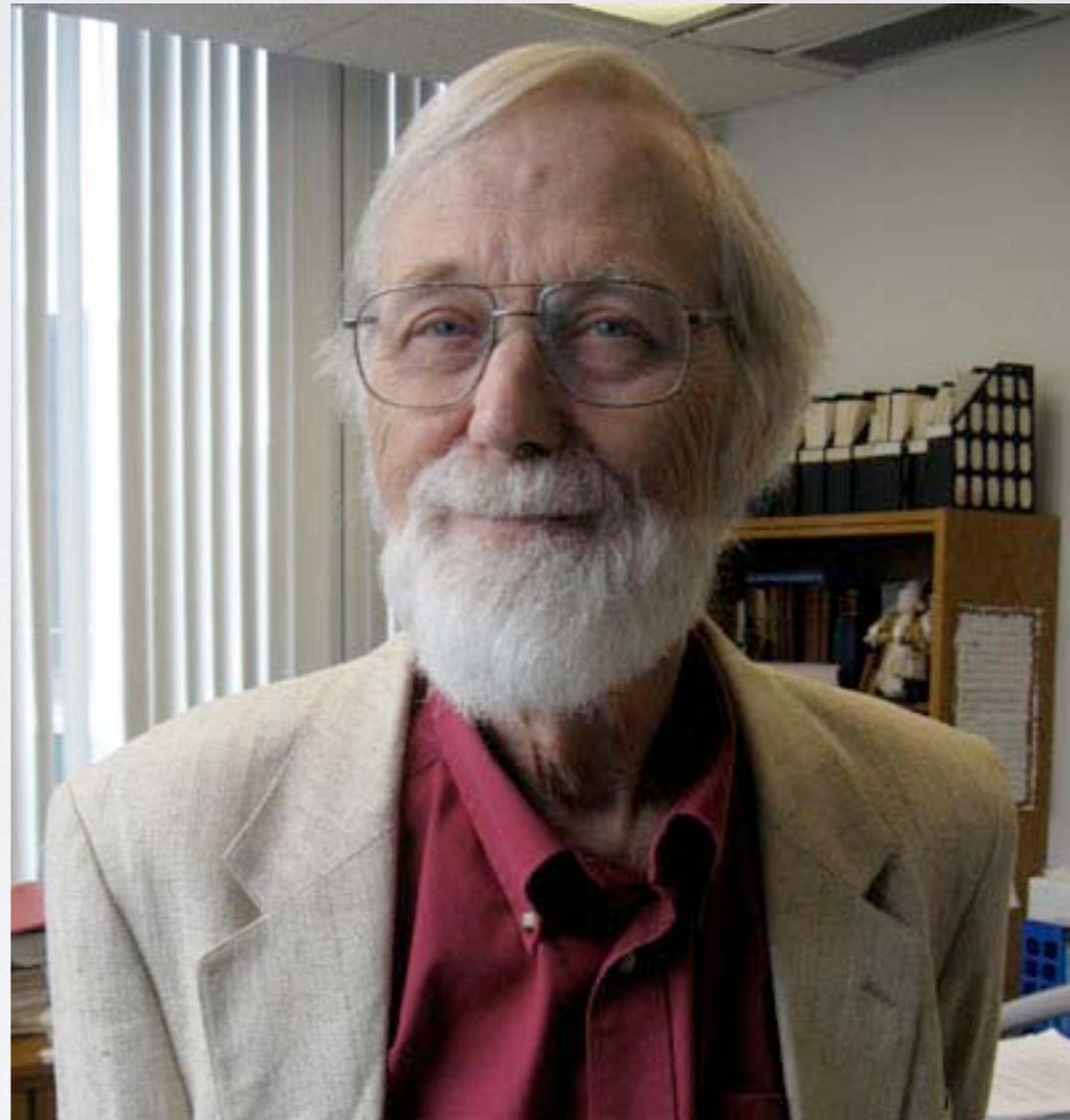
THURSDAY, APRIL 4, 2013
5:00PM
127 HAYES-HEALY CENTER

GORDON-WEBB



JOHN MILNOR

1931-



ISOSPECTRAL TORI

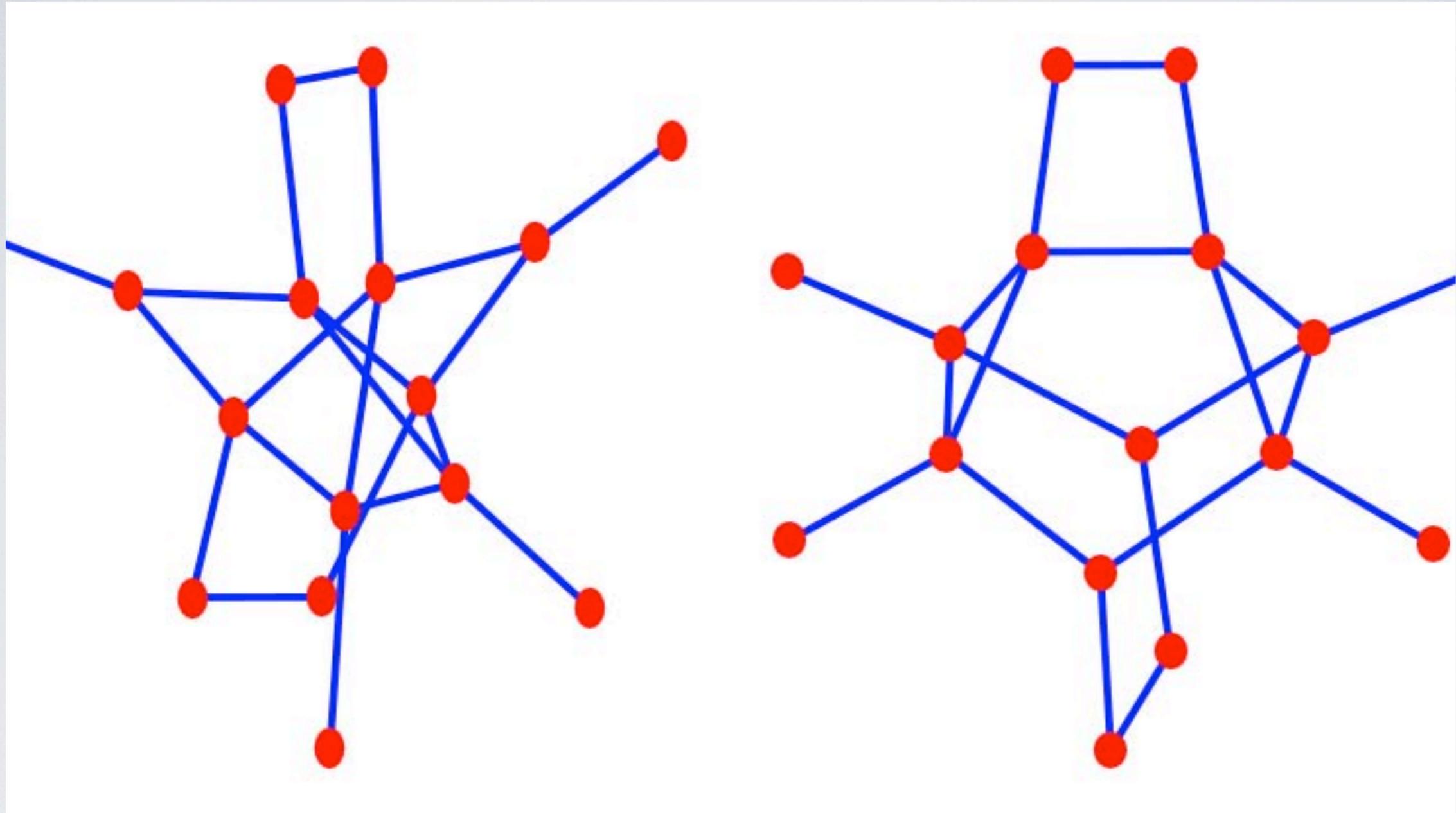
Also Dirac isospectral

Sunada: general technique

HUNGERBUHLER-HALBEISEN



ISOSPECTRAL GRAPHS

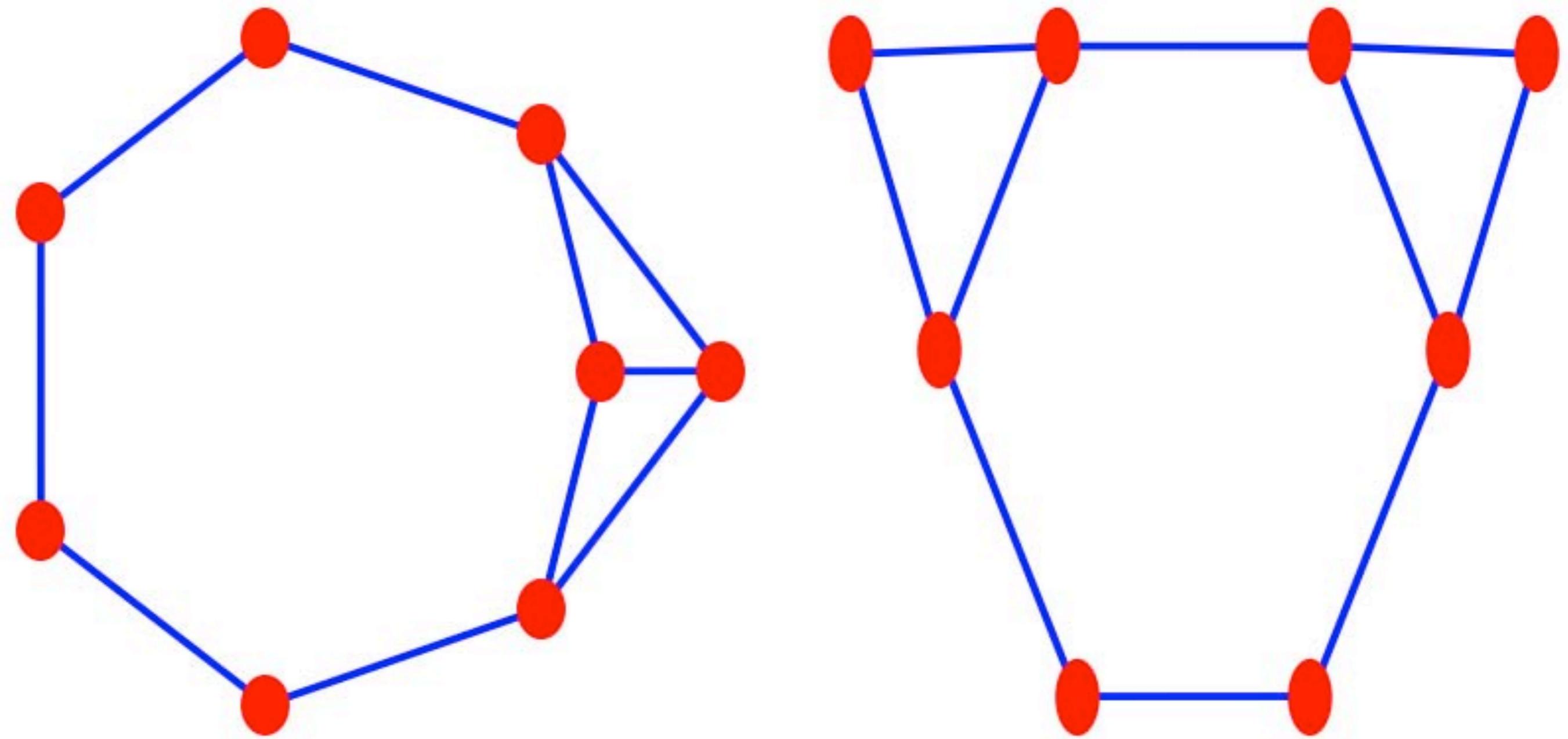


H-H Graphs are Dirac Isospectral!

proof: L_0, L_2 , McKean Singer: L_1

ISOSPECTRAL GRAPHS

Isospectral with respect to L_0 but not for L_1



Haemers-Spence

THE END

5 topics currently cooking:

- 1) Graph limits
- 2) Zeta Functions
- 3) Integrable Deformation
- 4) Cauchy-Binet
- 5) Pseudo Determinants



VICTOR LIDSKII

1924-2008



LIDSKII

A, B selfadjoint, $C=A-B$

$\lambda_1, \lambda_2, \lambda_3 \dots$ eigenvalues of A

$\mu_1, \mu_2, \mu_3 \dots$ eigenvalues of B

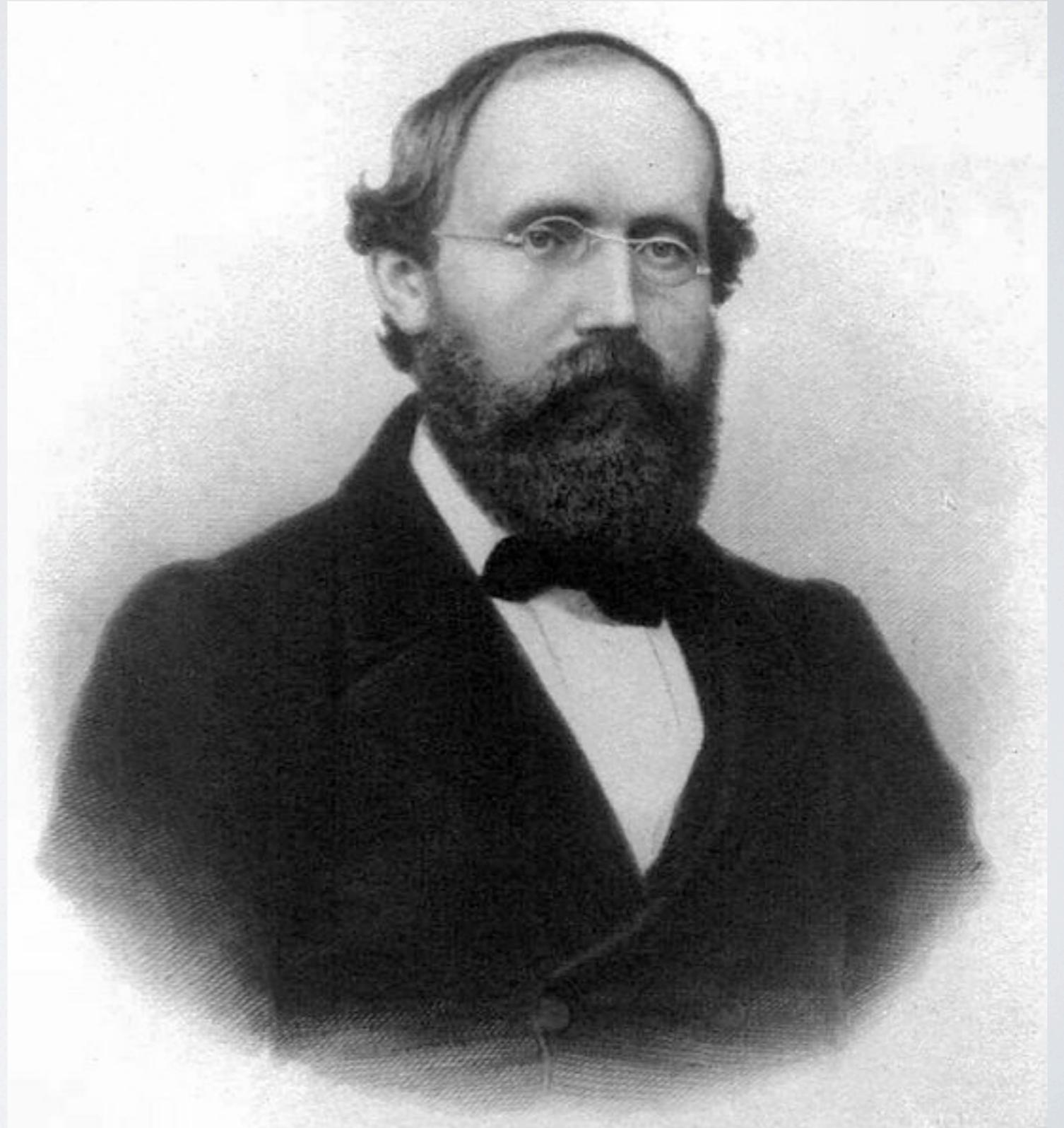
$$\sum_j |\lambda_j - \mu_j| \leq \sum_{k,l} |C_{kl}|$$

$$\begin{aligned} \sum_i |\gamma_i| &= \sum_i (-1)^{m_i} \gamma_i = \sum_{i,k,l} (-1)^{m_i} U_{ik} C_{kl} U_{il} \\ &\leq \sum_{k,l} |C_{kl}| \cdot \left| \sum_i (-1)^{m_i} U_{ik} U_{il} \right| \leq \sum_{k,l} |C_{kl}| \cdot \end{aligned}$$



BERNHARD RIEMANN

1826-1866



ZETA FUNCTION

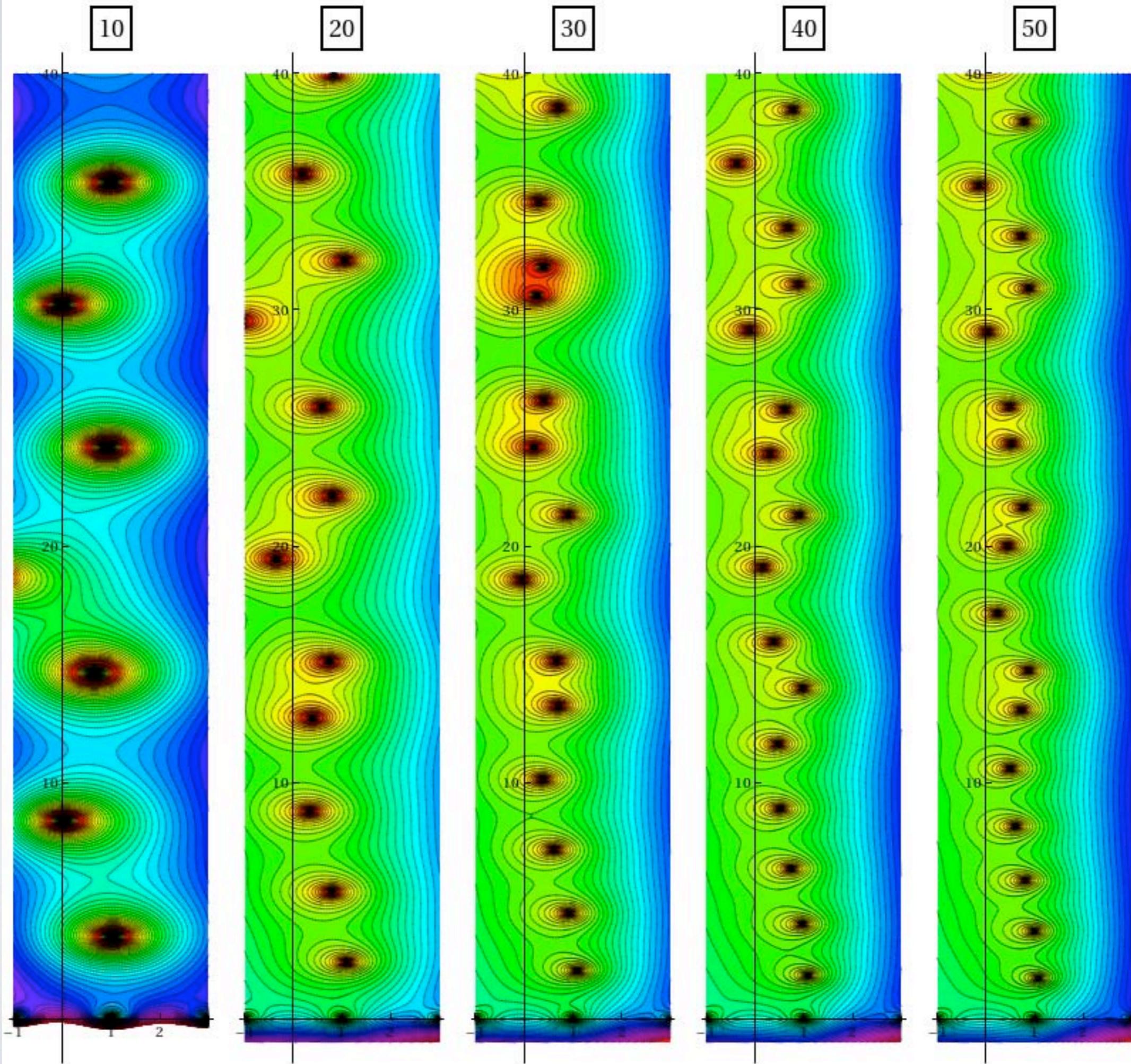
$$\zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s} \quad (\lambda \text{ eigenvalues of } D)$$

$$= (1 + \exp(i\pi s)) \sum_{\lambda > 0} \lambda^{-s}$$

(choose branch)

-|s| (1 + exp(πs)) times Riemann ζ function in Circle case (analytic!).

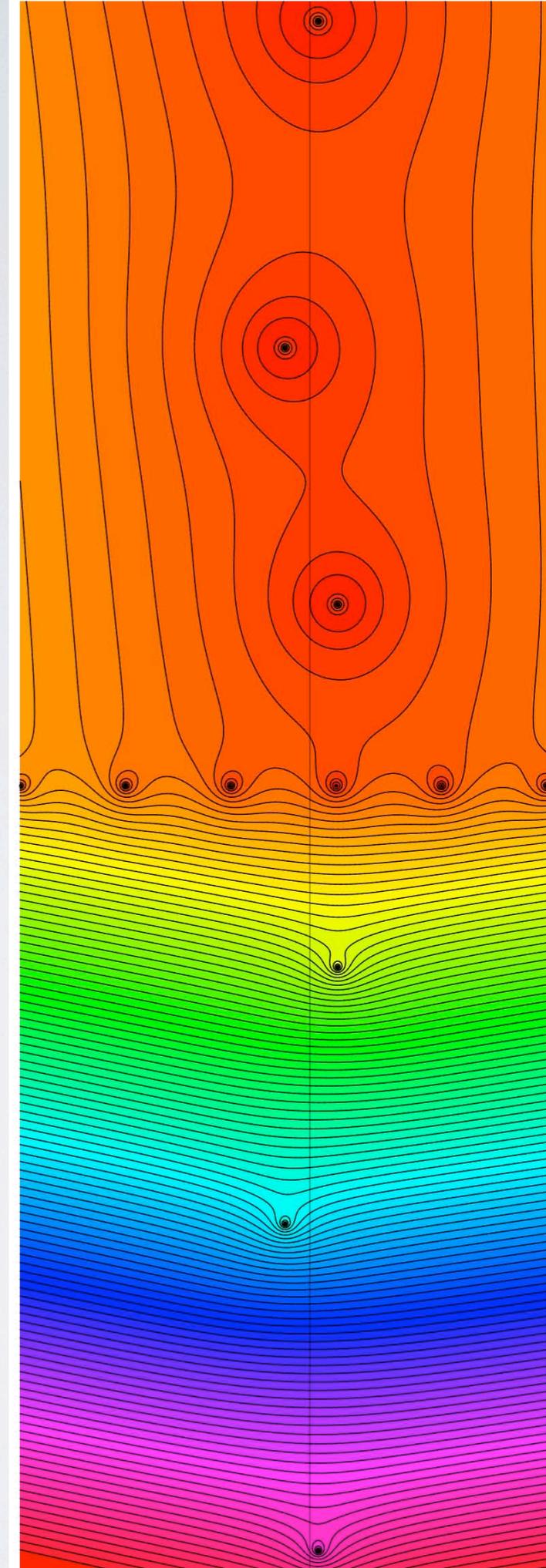
-|s| always analytic for graphs. Explicit in C_n case.



$$\zeta(C_n) = (1 + \exp(i\pi s)) \sum_{k=1}^{n-1} (2 \sin(\pi k/n))^{-s}$$

we see an animation with the roots in the case from

$n=10$ to 650



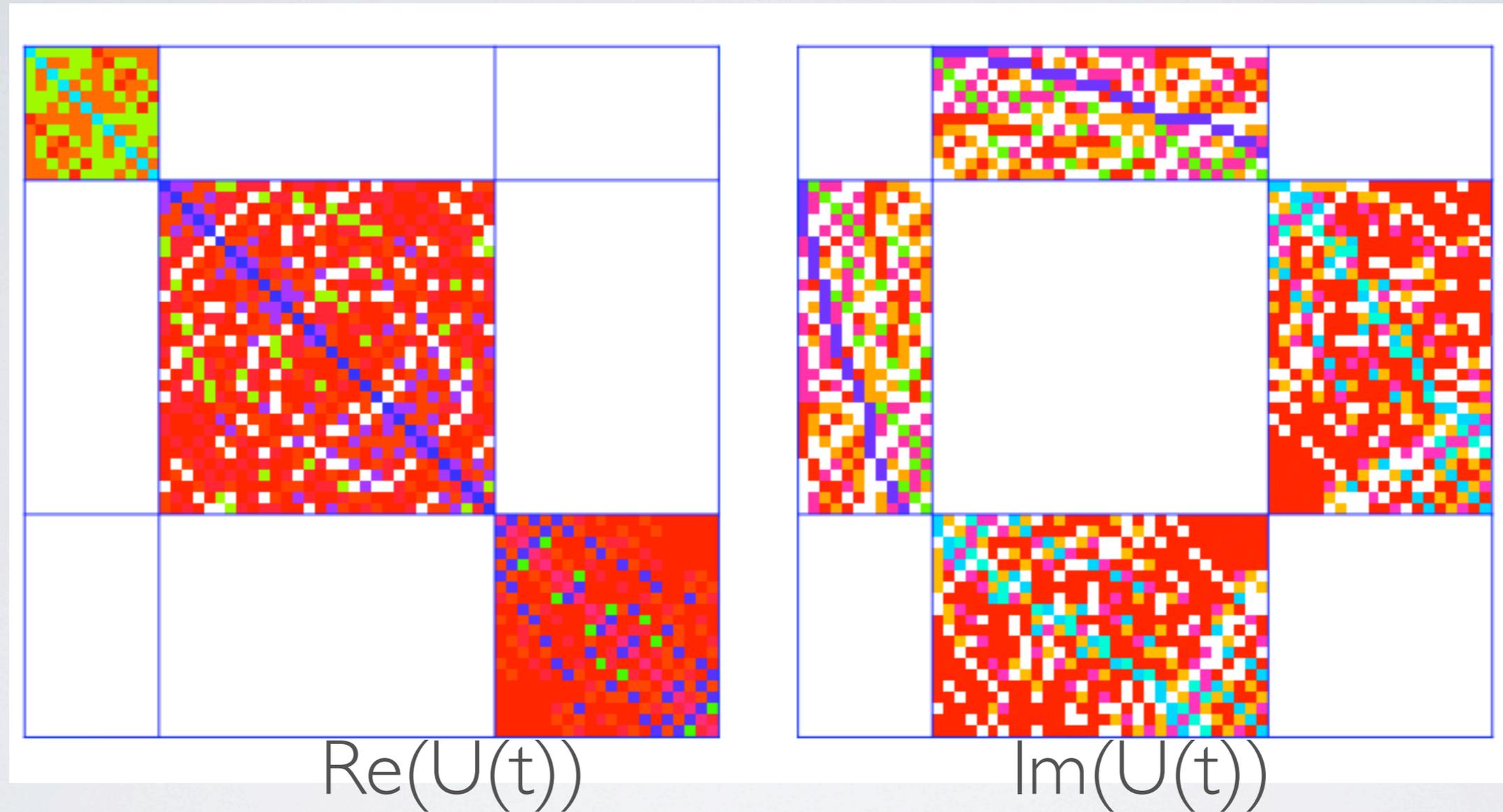


PETER LAX

1926-



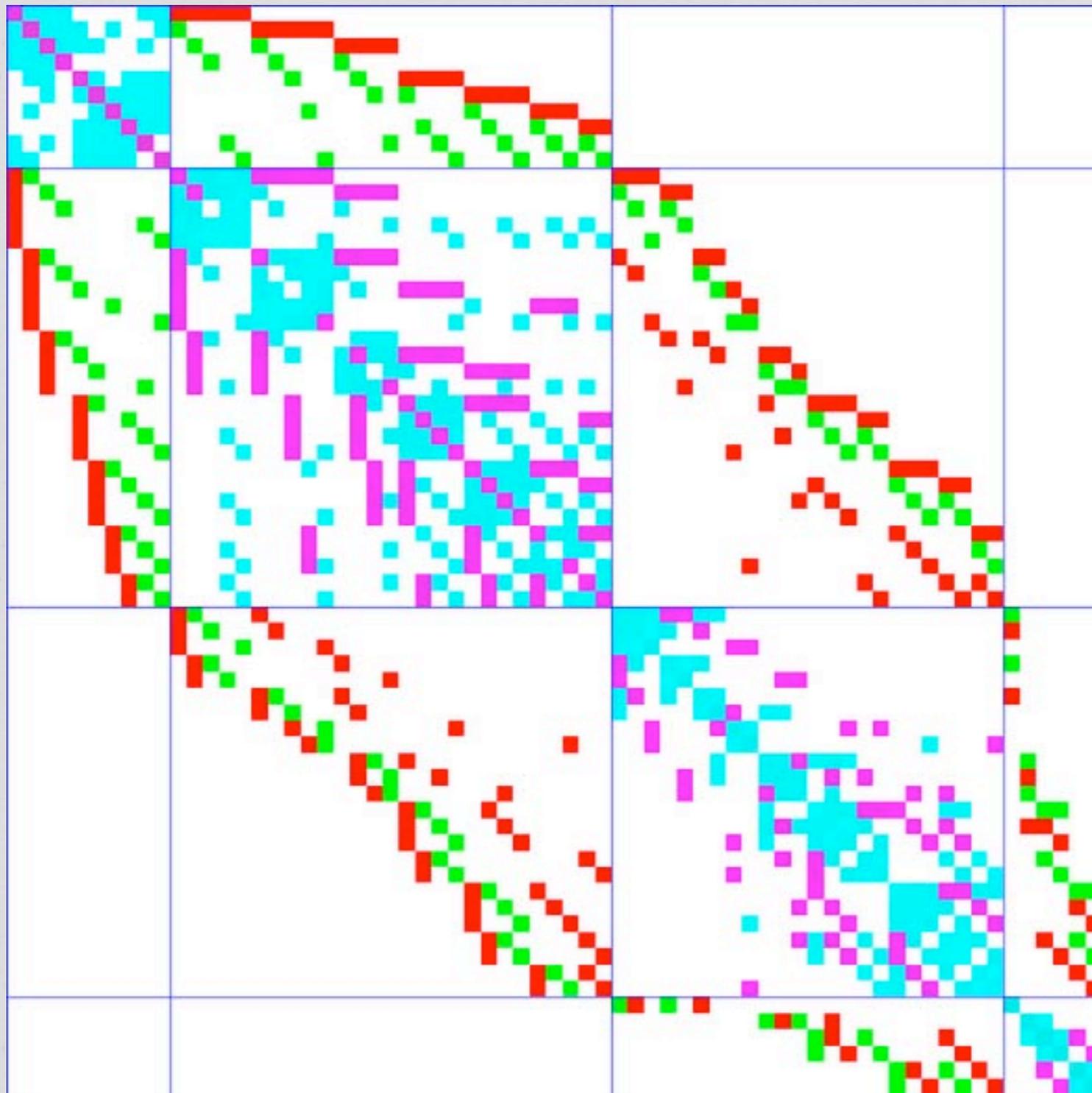
ISOSPECTRAL EVOLUTIONS



$$D' = [B, D] \quad , \quad B = d - d^*$$

isospectral deformation

$$D(t) = d^* + d + b$$



$$D(t) = U(t)^* D(0) U(t)$$



JACQUES PHILIPPE MARIE BINET AUGUSTIN-LOUIS CAUCHY

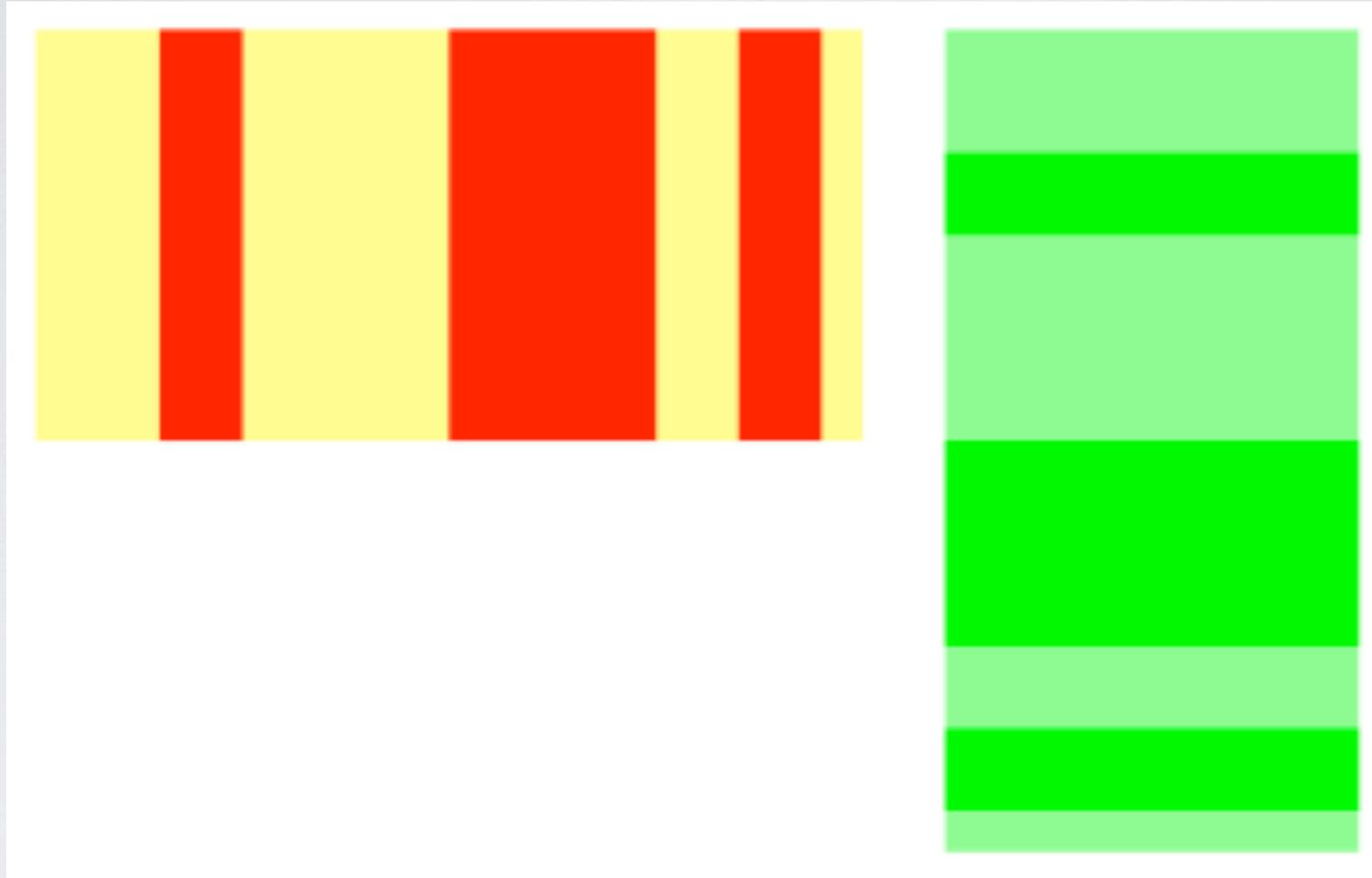


1786-1856



1789-1857

CAUCHY BINET



$$\det(F^T G) = \sum_P \det(F_P) \det(G_P)$$

classical Cauchy-Binet (1812)

CAUCHY BINET FOR PSEUDO DETERMINANTS

$\text{Det}(A) =$ product of nonzero eigenvalues
The sum is over all $k \times k$ minors with
 $k = \min(\text{Ran}(F^T G), \text{Ran}(F G^T))$

$$\text{Det}(F^T G) = \sum_P \det(F_P) \det(G_P)$$

counts signed trees in a double cover of
simplex graph (matrix tree theorem)

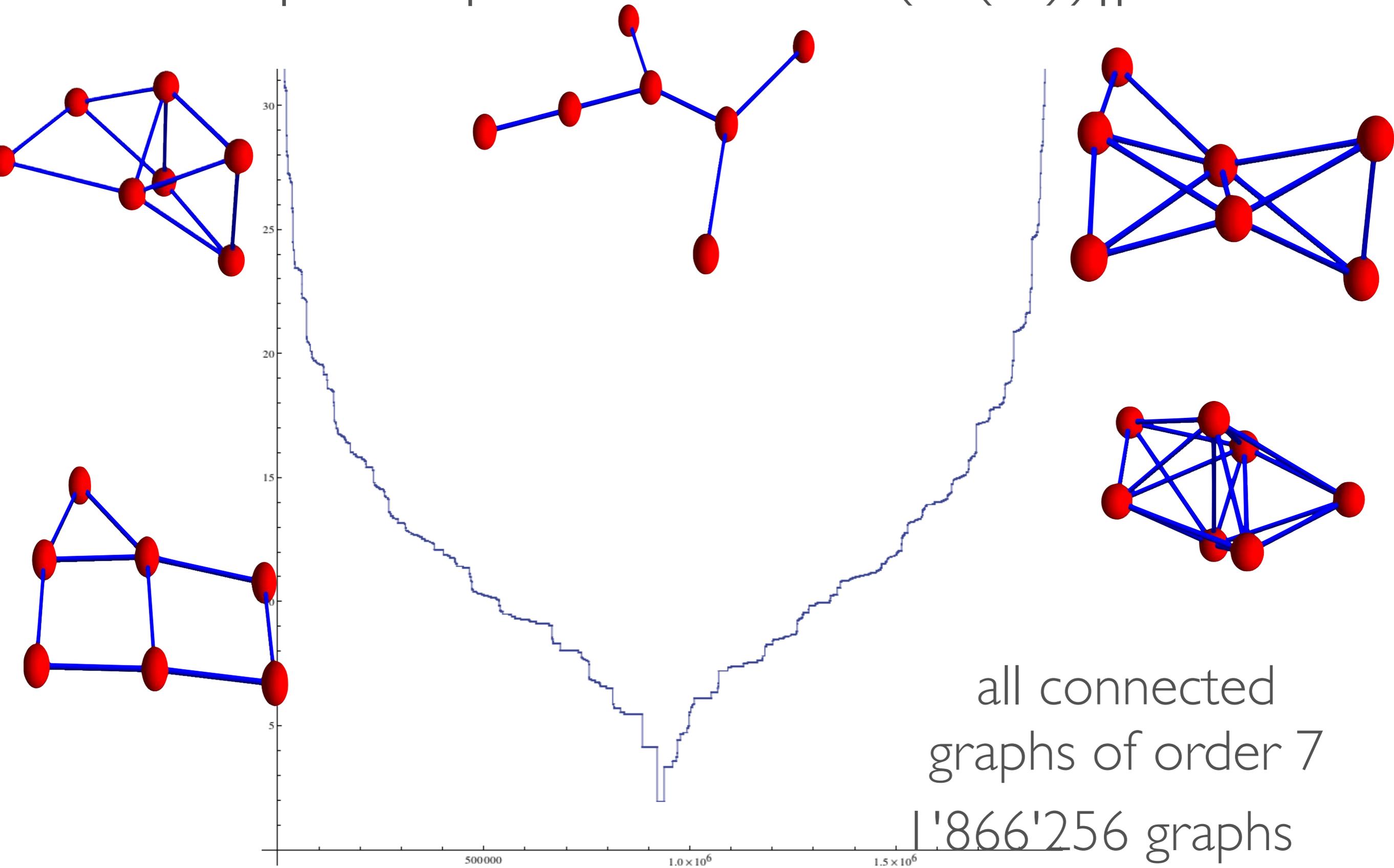


PAUL ERDOS

1913-1996



STATISTICS OF $|\text{LOG}|\text{PSEUDODET}(D(G))||$



THE END