

FROM THE SPHERE TO THE MANDELBULB

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ABSTRACT. In this 30 minutes talk, we build a visual bridge from Greek mathematics to the present, where 3D printing and 3D scanning have become affordable. We live in an exciting time, where several revolutions happen simultaneously. This short document summarizes the main points of the talk.

1. REVOLUTIONS

Science, technology and education experiences a time of rapid change. Things which would have been considered impossible 20 years ago are now possible or available on the level of consumer technology. Jeremy Rifkin once coined the term **"third industrial revolution"** and called it a time, where manufacturing becomes digital, personal and sustainable. We are in the third of the three industrial revolutions which one could coin **"steam, steal and star"** revolutions. Media revolutions like **Gutenberg's press**, the **telegraph** are now topped with the emergence of the **internet** and social networks. We had **print, talk and connect** revolutions. Information storage media sizes have exploded. We have come a long way since Clay tablets, papyrus, paper, vinyl. These physical media were in rapid succession been replaced with CDs and DVDs, tapes, harddrives and now memory sticks, SSD's. The three **optical, magnetic and electric** revolutions came within a few decades only. In education, we have seen **"New math"** in the sixties, the **"Math wars"** in the eighties and **Massive Open Online Courses** (MOOC's) at the beginning of the 21st century. These revolutions happened for **generation X, generation Y and generation Z**. For media, we had seen the emergence of **photograph, film** and now witness **3D scanning and printing**: these are **"picture, movie and object"** revolutions. We have first explored the **macro scale** (cosmos) by measuring out planets and stars, then the **microscale** (atomic level) with physics, chemistry and biology and conquer now the **mesoscale**, the level of our daily lives, which also includes biology. What does this have to do with math? Describing complex geometric objects by mathematical language is now possible thanks to **3D scanners**. We can copy the objects using **3D printers**. Understanding the motion of stars led to **calculus** and **topology** gaining insight into quantum mechanics required **analysis** and **algebra**, the mesoscale feeds from knowledge about **geometry** and **computer science**.

2. ARCHIMEDES

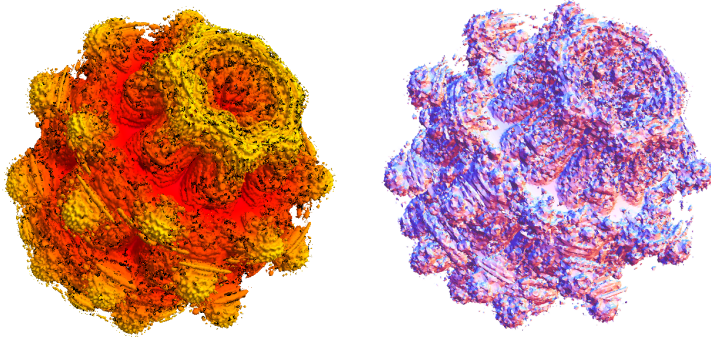
This section uses work done in [1] and [4]. Even so we live in a revolutionary time, we should not forget about the past. Early achievements in mathematics and technology made the present progress possible. Archimedes, who would now be 2300 years old, was an extraordinary mathematician and engineer. We do not have a picture of him, but artists have tried to capture the legend. A common scene with Archimedes is the moment of his death, when Syracuse was sacked. Archimedes tomb stone was decorated with a symbol of his most important achievement: the computation of the volume of the sphere. Archimedes had realized that if one cuts a sphere at height z then the area of the cross section $\pi(1 - z^2)$ is the same than the difference of the area of a cross section of a unit cylinder π and the cross section with a cone πz^2 . The volume of the half sphere is therefore the difference between the volume of the cylinder π and the volume of the cone $\pi/3$ which gives $2\pi/3$. While we know now how to compute the volume of the sphere, this had not been known before Archimedes. To translate this discovery into modern times, lets look at a problem which we presently do not know how to solve. It might be easy with an Archimedean type idea, it might also be impossible, like the quadrature of the circle: the object we are going to look at is called the **Mandelbulb set** M_8 . Unlike the **Mandelbrot set**, which was discovered 33 years ago, the set M_8 is an object in space has been discovered only recently Already **Rudy Rucker** had experimented with a variant M_2 in 1988 [?] in Mathematica. **Jules Ruis** wrote me to have written a Basic program in 1997. The first mentioning of the formulas used today is by **Daniel White** mentioned in a 2009 fractal forum. Jules Ruis also 3D printed the first models in 2010. A problem one could ask is: is there a closed expression for the volume of M_k ? What is a closed expression? The problem to find an expression of $\zeta(2) = 1 + 1/4 + 1/9 + 1/16 + \dots$ is

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called the **Basel problem**. It was solved by Euler, who found that the limit is $\pi^2/6$. While one knows similar formulas for sums $\zeta(k) = 1 + 1/2^k + 1/3^k + \dots$ for all even k , one does not have a closed formula for $k = 3$ for example. Mathematicians call $\zeta(2)$ a **period**, a number which can be expressed as an integral of an algebraic function over an algebraic domain. The You-Tube star Mandelbulb M_8 is defined as follows: for every three vector c define a map T_c in space which maps a point x to a new point $T_c(x) = x^8 + c$, where x^8 is the point with spherical coordinates $(r^8, 8\phi, 8\theta)$, if x had the spherical coordinates (r, ϕ, θ) . The set consists now of all parameter vectors c for which iterating T_c again and again starting at 0 leads to a bounded orbit. For example, if we take $c = (0, 0, 0)$ and start with $(0, 0, 0)$, then we remain at $(0, 0, 0)$. Therefore, $c = (0, 0, 0)$ is in M_8 . Now take $c = (0, 0, 2)$. Lets look at the orbit $(0, 0, 0)$, which is $(0, 0, 0), (0, 0, 2), (0, 0, 2^8), \dots$ so that $(0, 0, 2)$ is outside of M_8 . The Mandelbulb set is mathematically not well explored. It is most likely connected and simply connected like the Mandelbrot set but there could be surprises. Not much mathematical seems to be proven. Math reviews and ArXiv show no references yet. Here is an exercise for you: prove that the boundary of the Mandelbulb set is contained in the shell defined by the sphere of radius 2 and $1/2$. Now try to improve this bound. Low resolution picture of the Mandelbulb set can be computed with Mathematica's RegionPlot and spit out a STL file:



White's Mandelbulb set M_8 seen also as a printable STL file. White asked in 2008: Is it connected? Simply connected? Locally connected? Is the complement simply connected? We asked here whether the volume of M_8 is a period (a branch of mathematics called measure theory assures that the volume of the set exists). Printable Mandel and Julia bulbs have been produced in [3].

3. PICTURES

Some pictures, animations and movies shown in this part of the lecture were borrowed from my talk given at a ICTP workshop in Trieste and the article [2]. Most of the pictures were done with a 3D printer in mind. **Thinking like a printer makes pictures nicer.** One can not print surfaces of zero thickness for example. Graphics needs to be redesigned. This can pay off. The animations of the Calabi-Yau surfaces for example show the object in new beauty. An other side effect of 3D printing is that we can now plot solid graphs of functions which can be placed in Google Earth to fly around. We illustrate "impossible figures" like the Escher stairs, the Penrose triangle, as well as the Mandelbrot set as an island in the Boston Harbor in this virtual reality setup.

4. SCANNING

We first look at the problem of reconstructing a three dimensional object from projections. This is called the **structure from motion problem**. We use the affordable Microsoft Kinect device as a scanner and illustrate this. The program Artec Studio 9.1, reconstructs the object almost in real time, during the talk.

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