BIRKHOFF SUMS OVER THE GOLDEN ROTATION

Oliver Knill   Harvard University

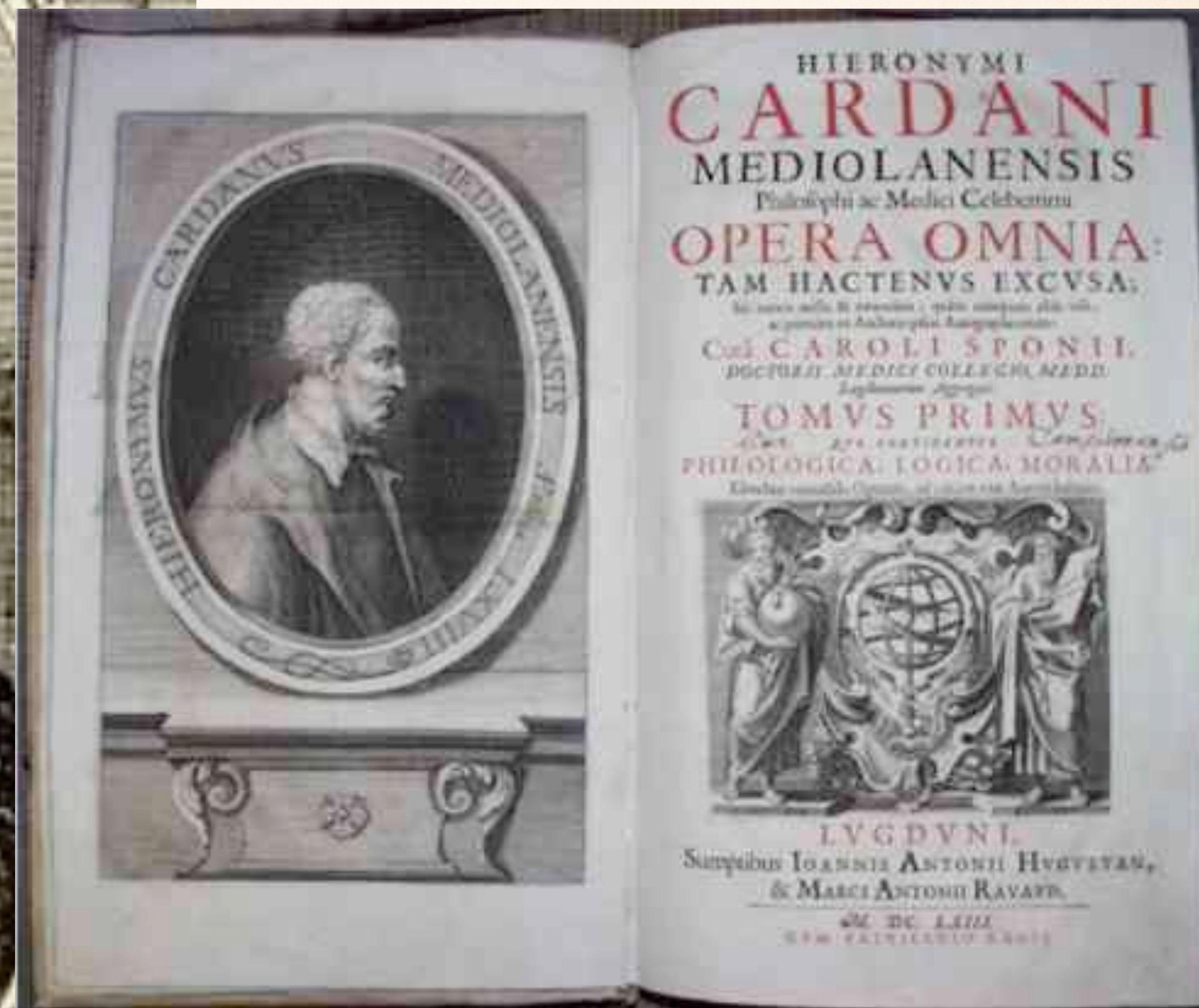
BU talk, February 23, 2015

BIRKHOFF SUM

$$S_n = \sum_{k=1}^{n} X_k$$

$$X_k = g(T^k(t))$$

GAMES



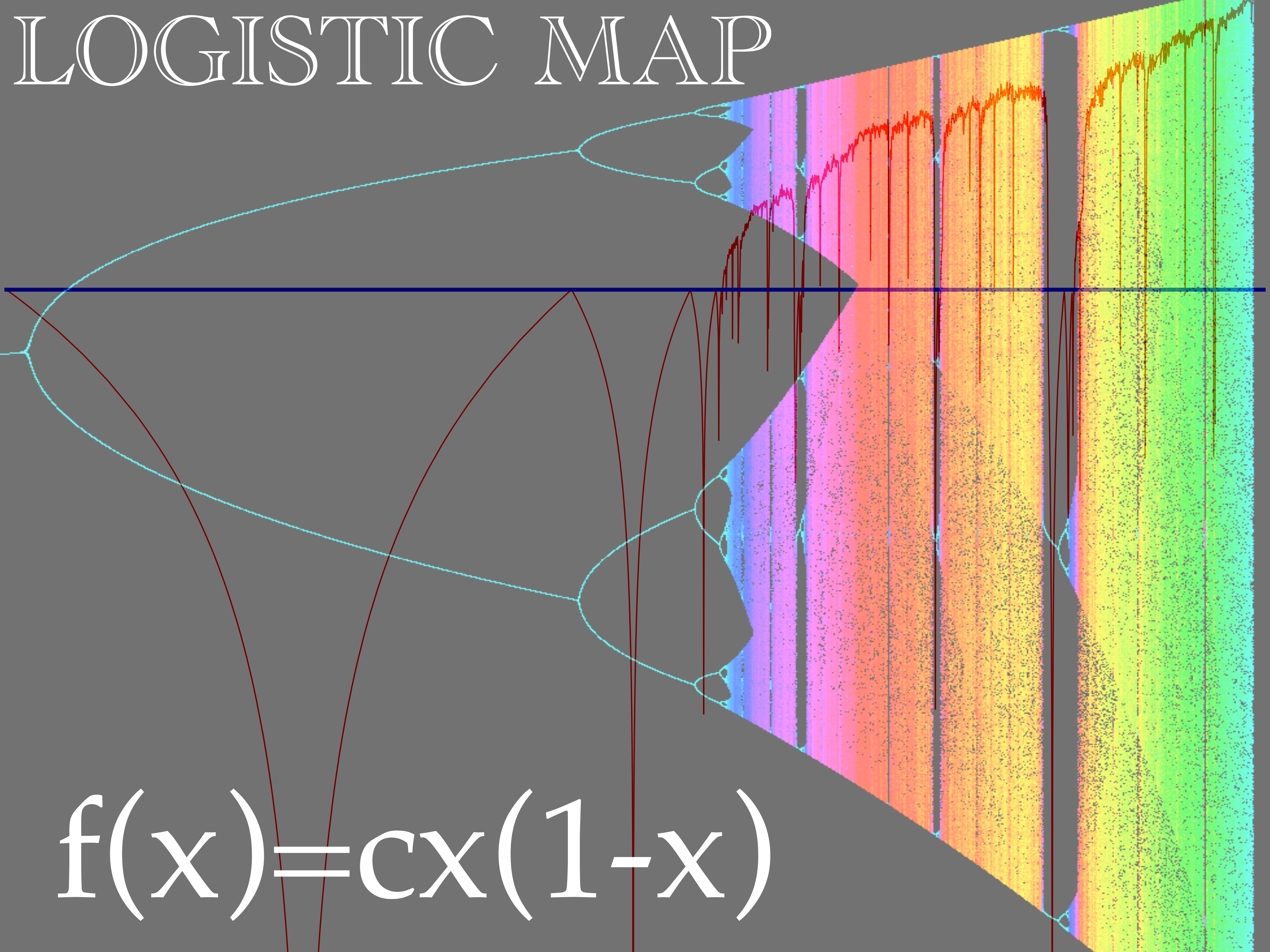
1501-1576

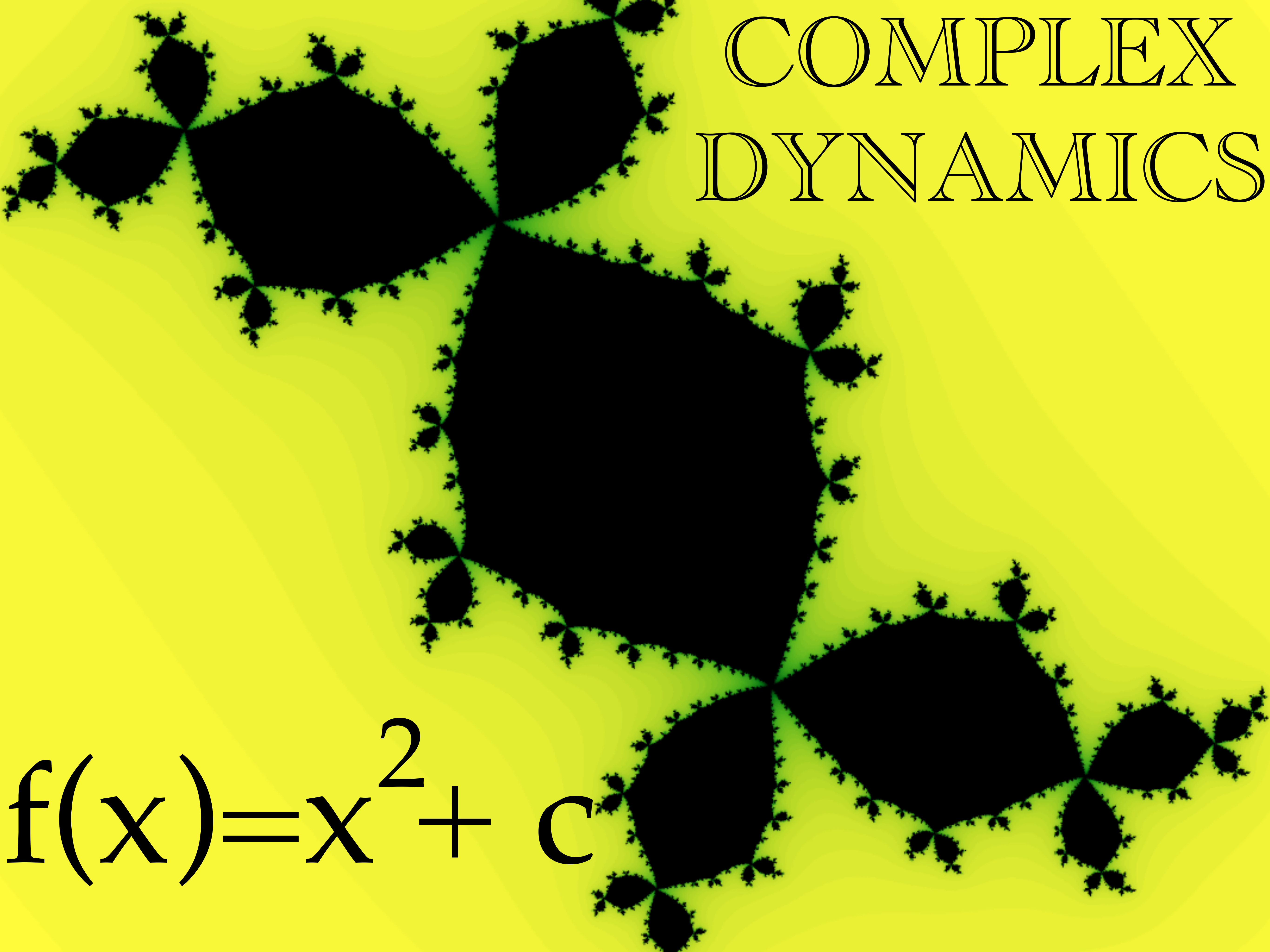
JACOBIAN

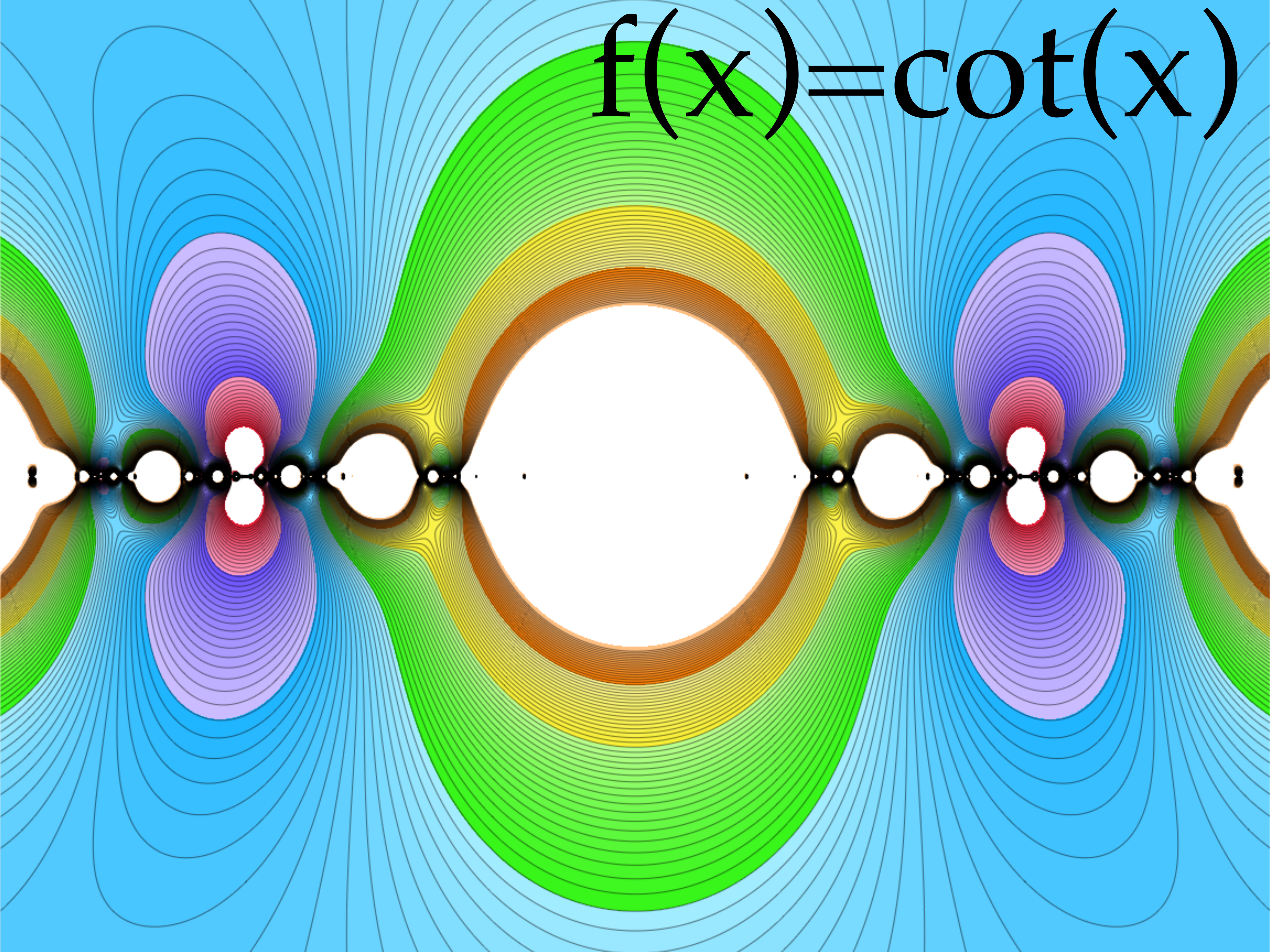
$$f: C \rightarrow C$$

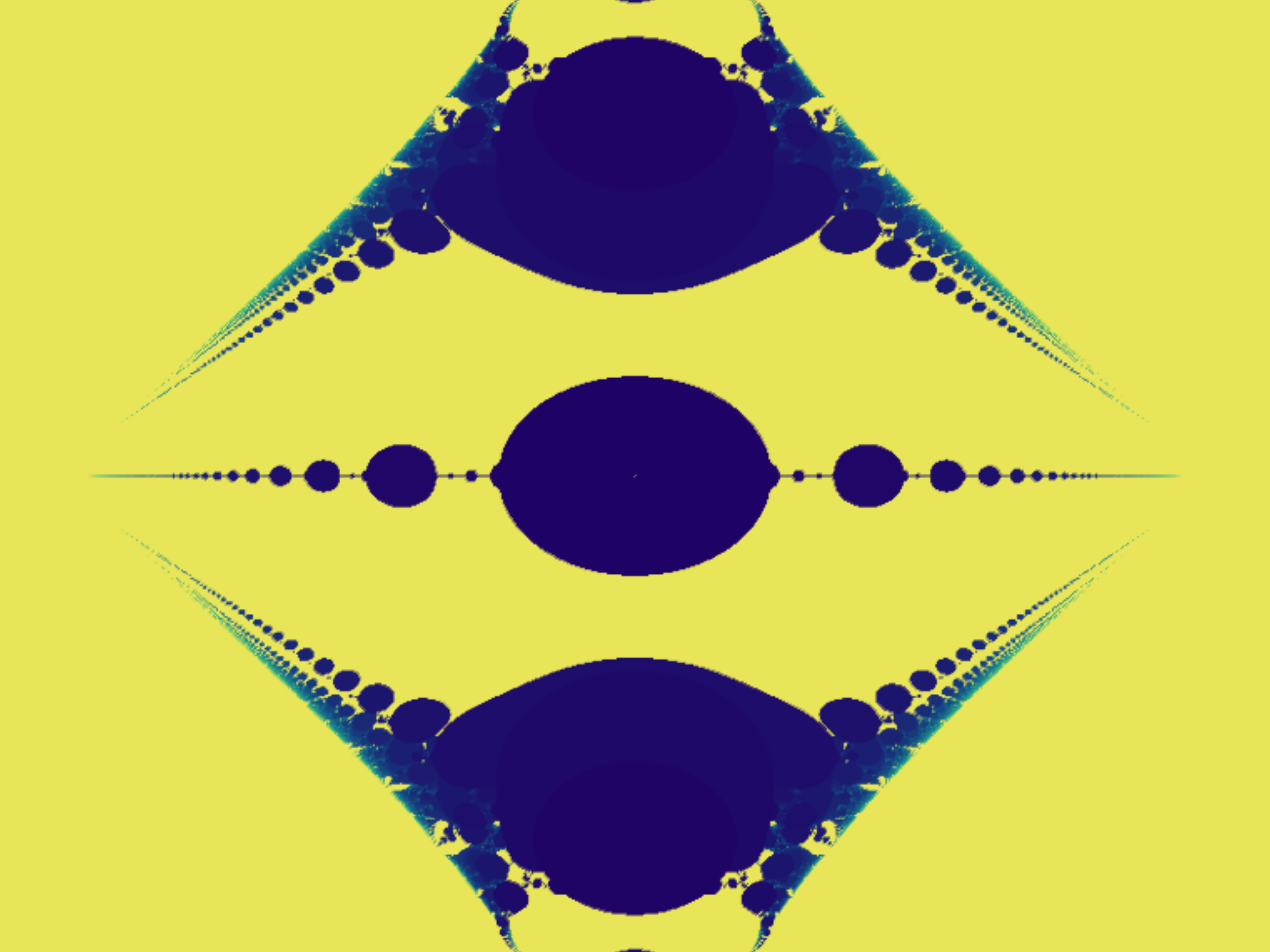
$$g(x) = \log|f'(x)|$$

$$S_n = \log|f^{(n)}(x)|$$


LOGISTIC MAP
f(x)=cx(1-x)


COMPLEX
DYNAMICS
f(x)=x²+ c


f(x)=cot(x)



SYMPLECTIC MAPS

$$T(x,y)=(2x-y+c\sin(x),x)$$

Chirikov

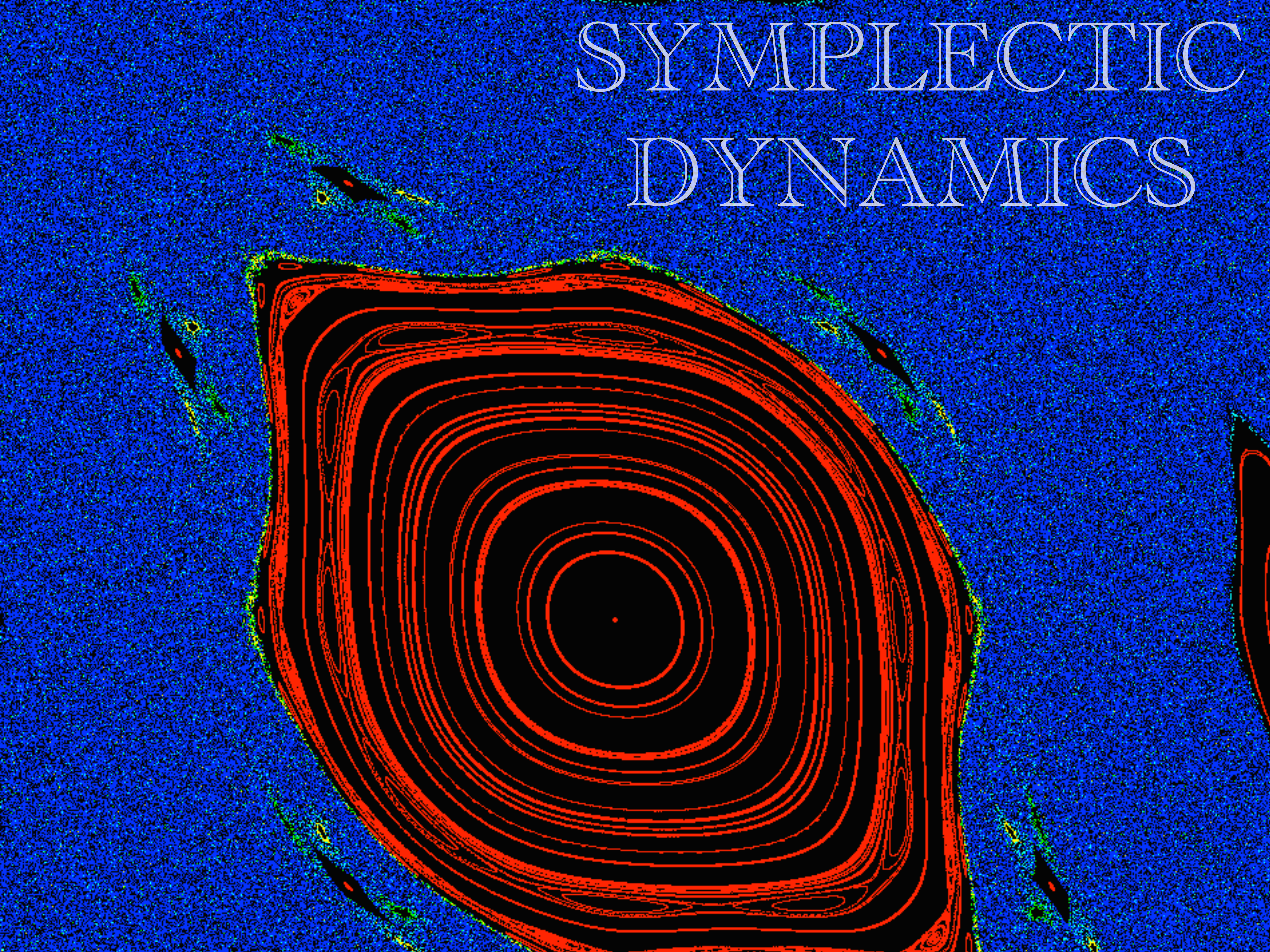
$$x_{n+1}-2x_n+x_{n-1}=c\sin(x_n)$$

Frenkel-Kontorova

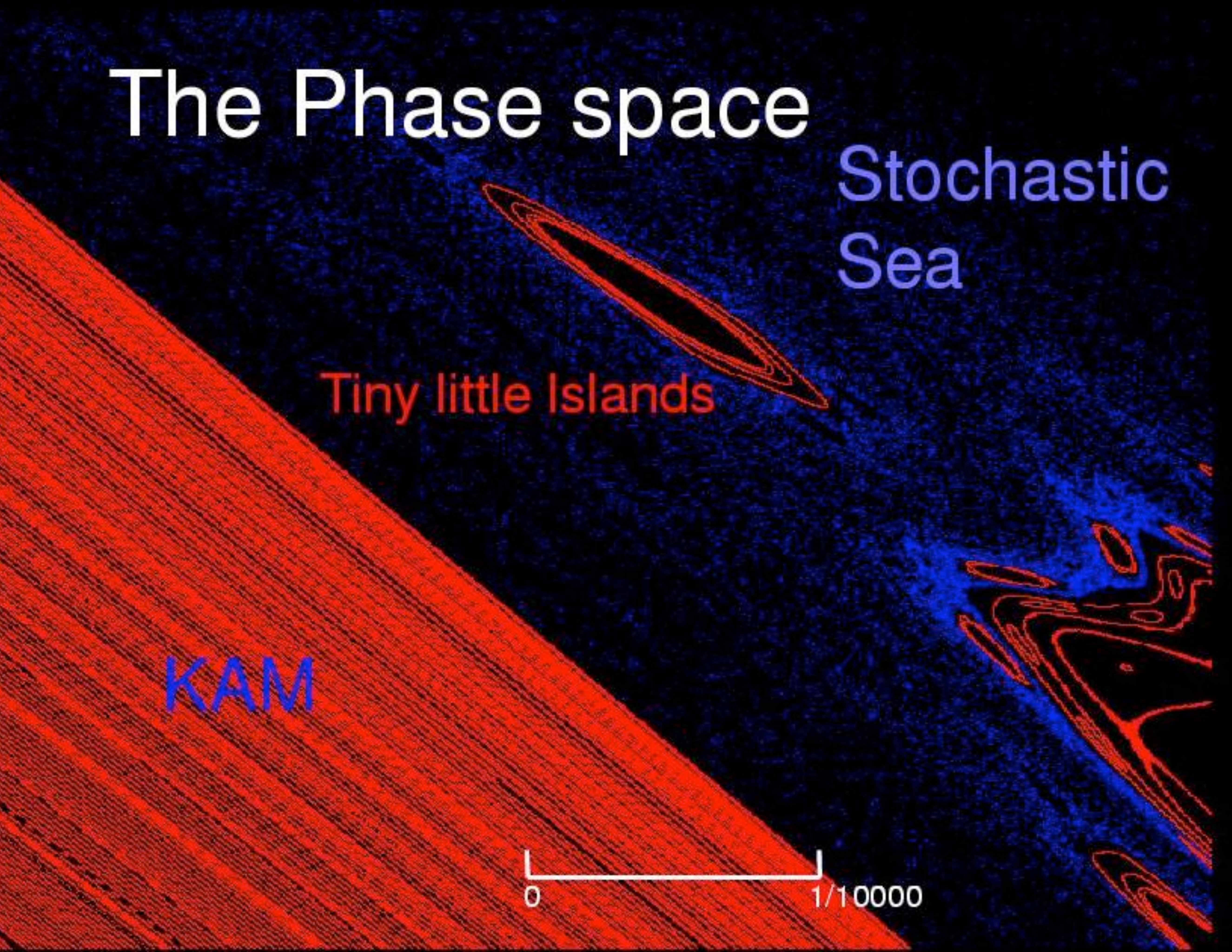
$$q(x+\alpha)-2q(x)+q(x-\alpha)=c\sin(q(x))$$

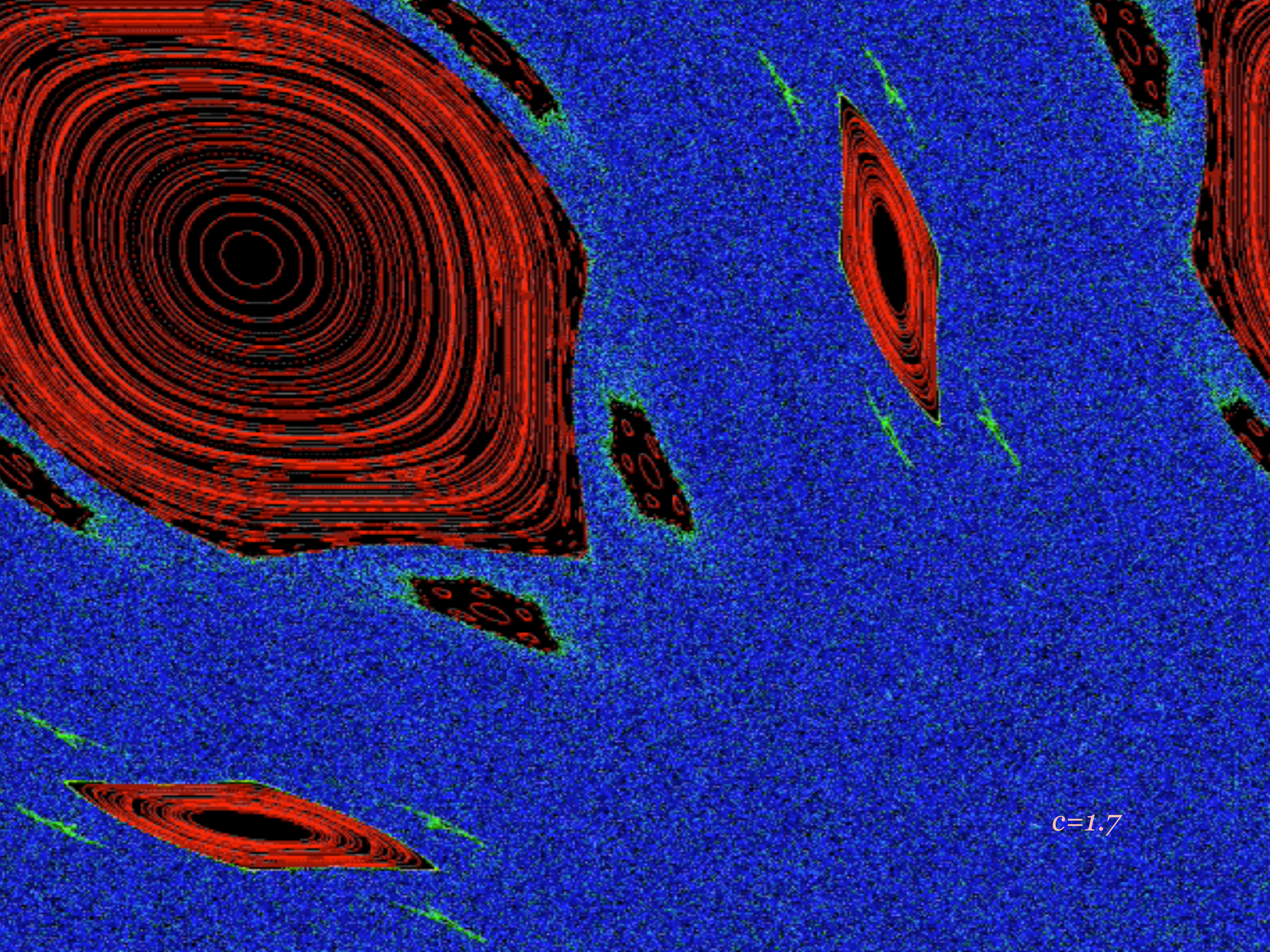
KAM circle

SYMPLECTIC DYNAMICS

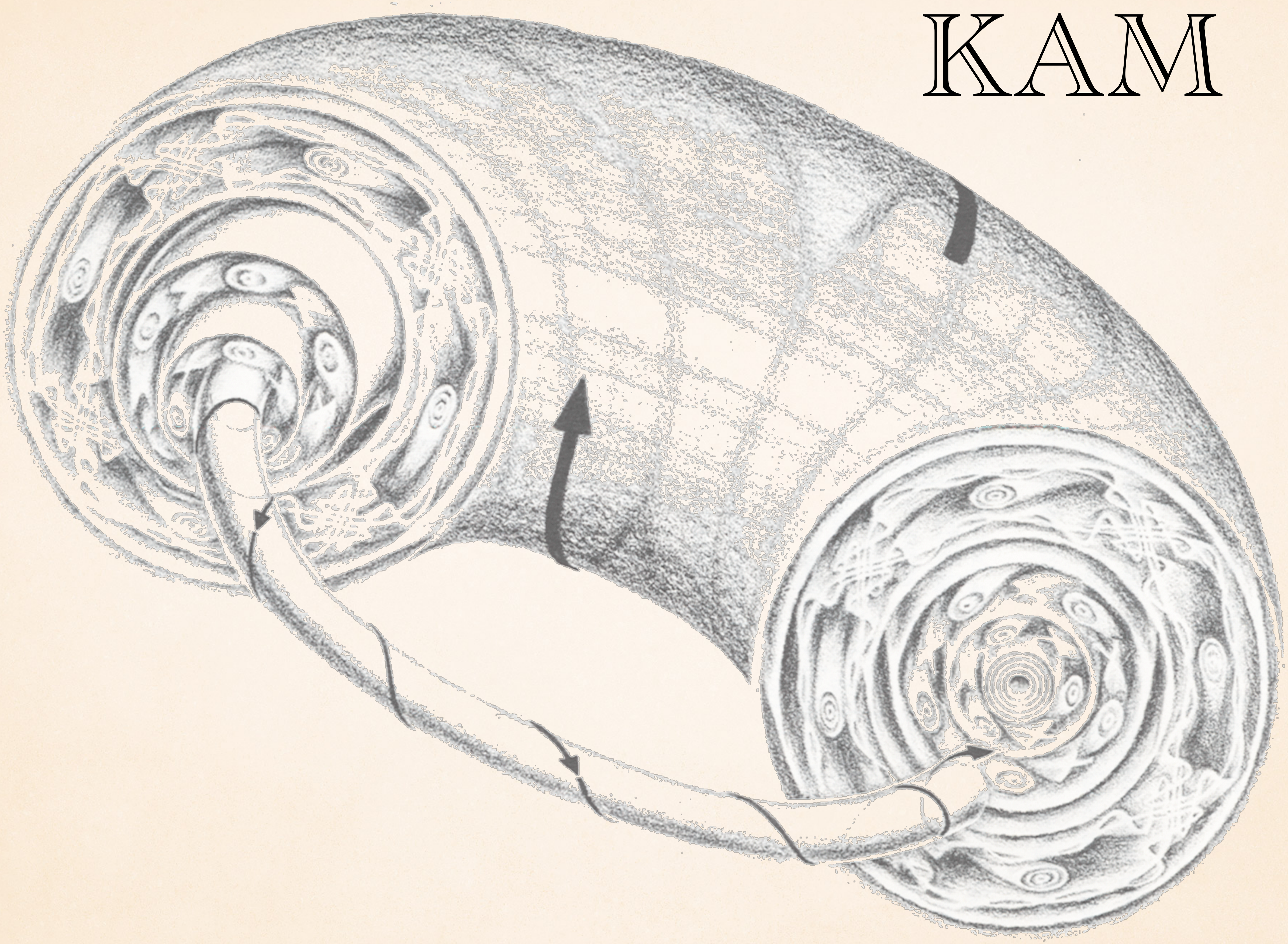


The Phase space




c=1.7

KAM



(Image source: R. Abraham and J. Marsden, 1978)

IMPLICIT FUNCTION

$$q(x+\alpha)-2q(x)+q(x-\alpha)=c\sin(q(x))$$

$$c=0 \;\Rightarrow\; q(x)=x$$

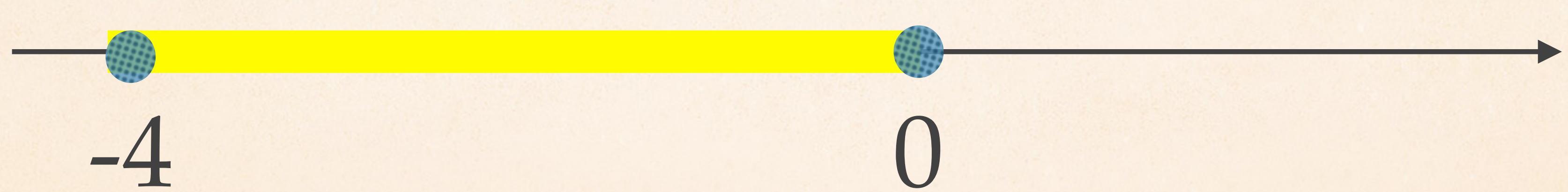
Euler eq.

$$\hat{L}\,u = u(x+\alpha)-2u(x)+u(x-\alpha)$$

Hessian

$$\hat{L}\,\hat{u}_n = (2\cos(n\alpha)-2)u_n$$

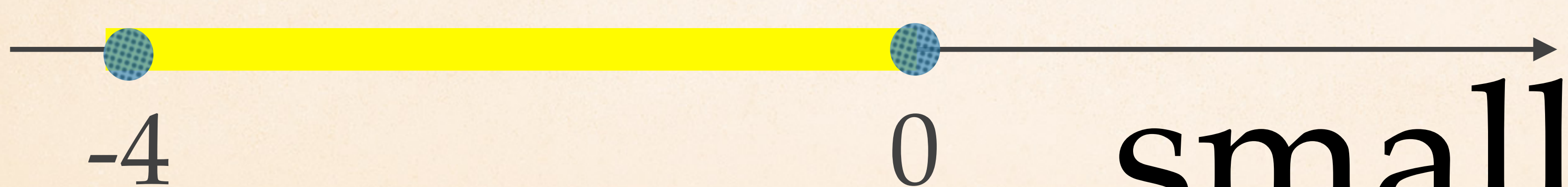
diagonal



IMPLICIT FUNCTION

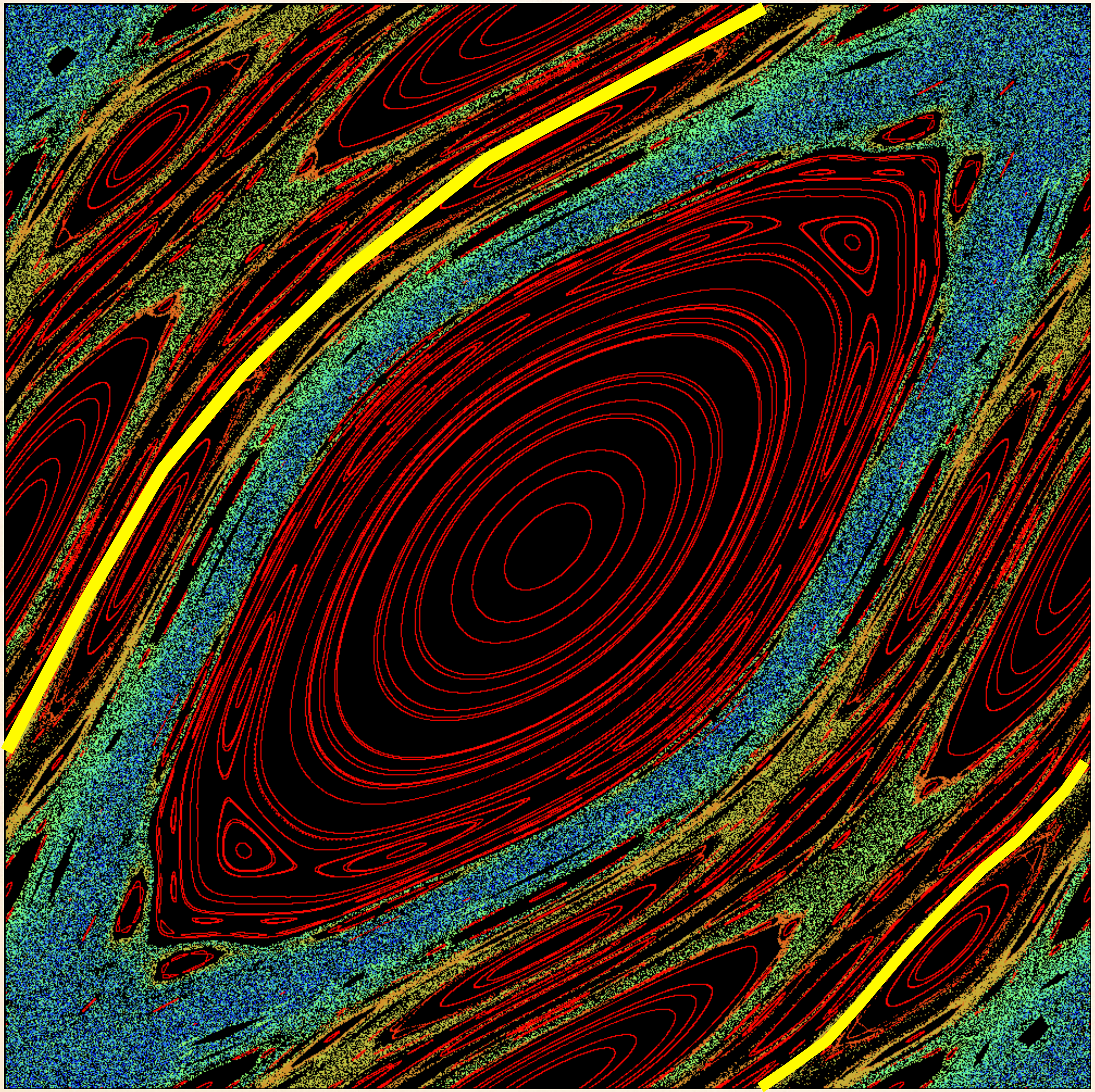
$$\hat{L} = \begin{pmatrix} 2\cos(\alpha)-2 & & & \\ & 2\cos(2\alpha)-2 & & \\ & & 2\cos(3\alpha)-2 & \\ & & & 2\cos(4\alpha)-2 \end{pmatrix}$$

$$\det(L(n)) = \prod_{k=1}^{n} p(k)$$



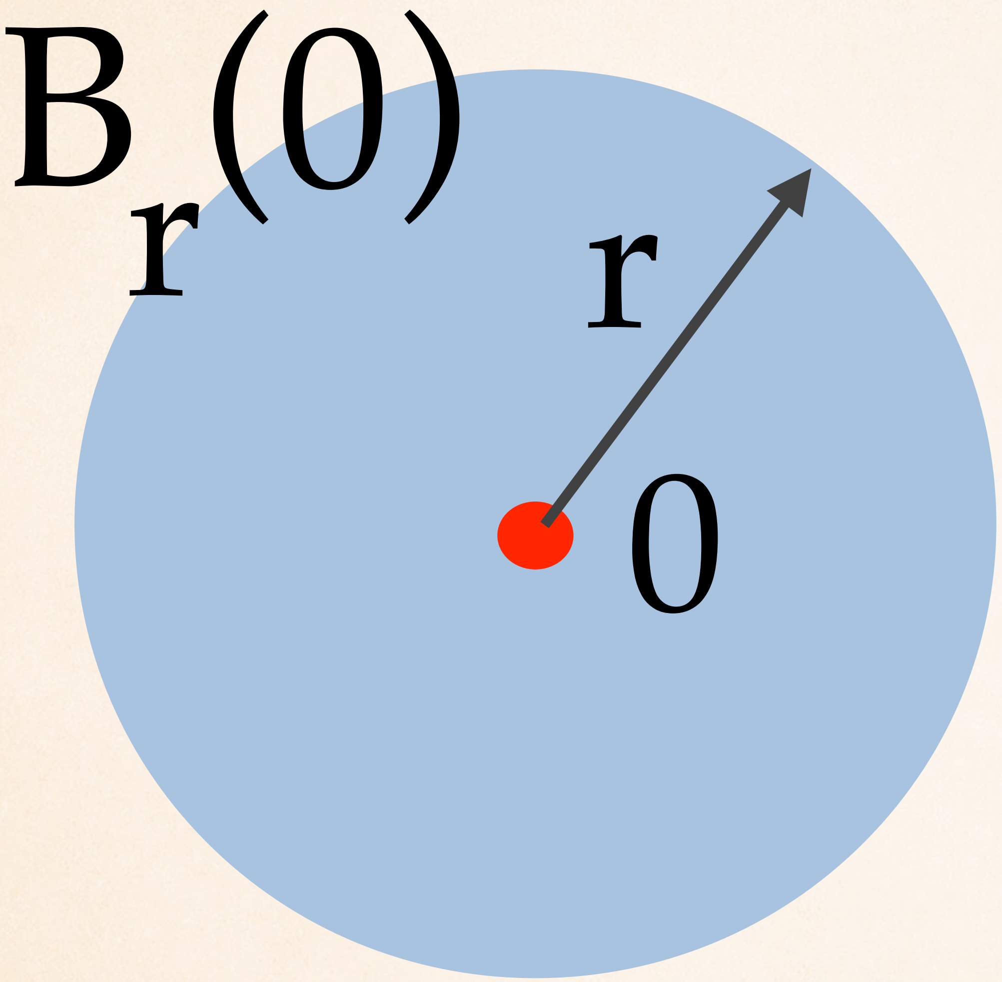
small divisors

WHAT BROUGHT US IN?



Work with
John Lesieutre, 2008

NEUBERGER THEOREM

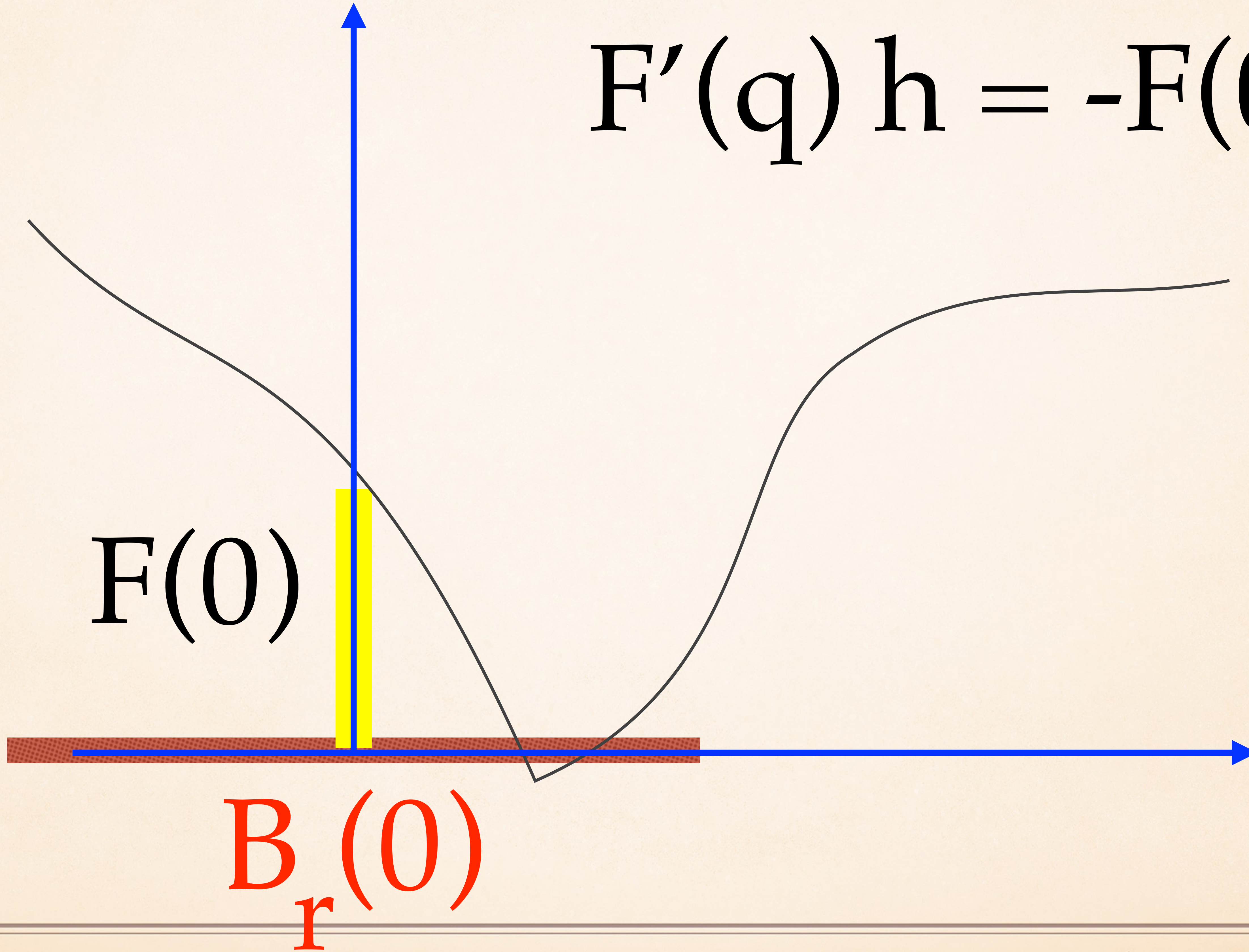


$F'(q)\,h = -F(0)$
can be solved
on dense set
with h in $B_r(0)$

Then there is q
with $F(q)=0$

EXAMPLE

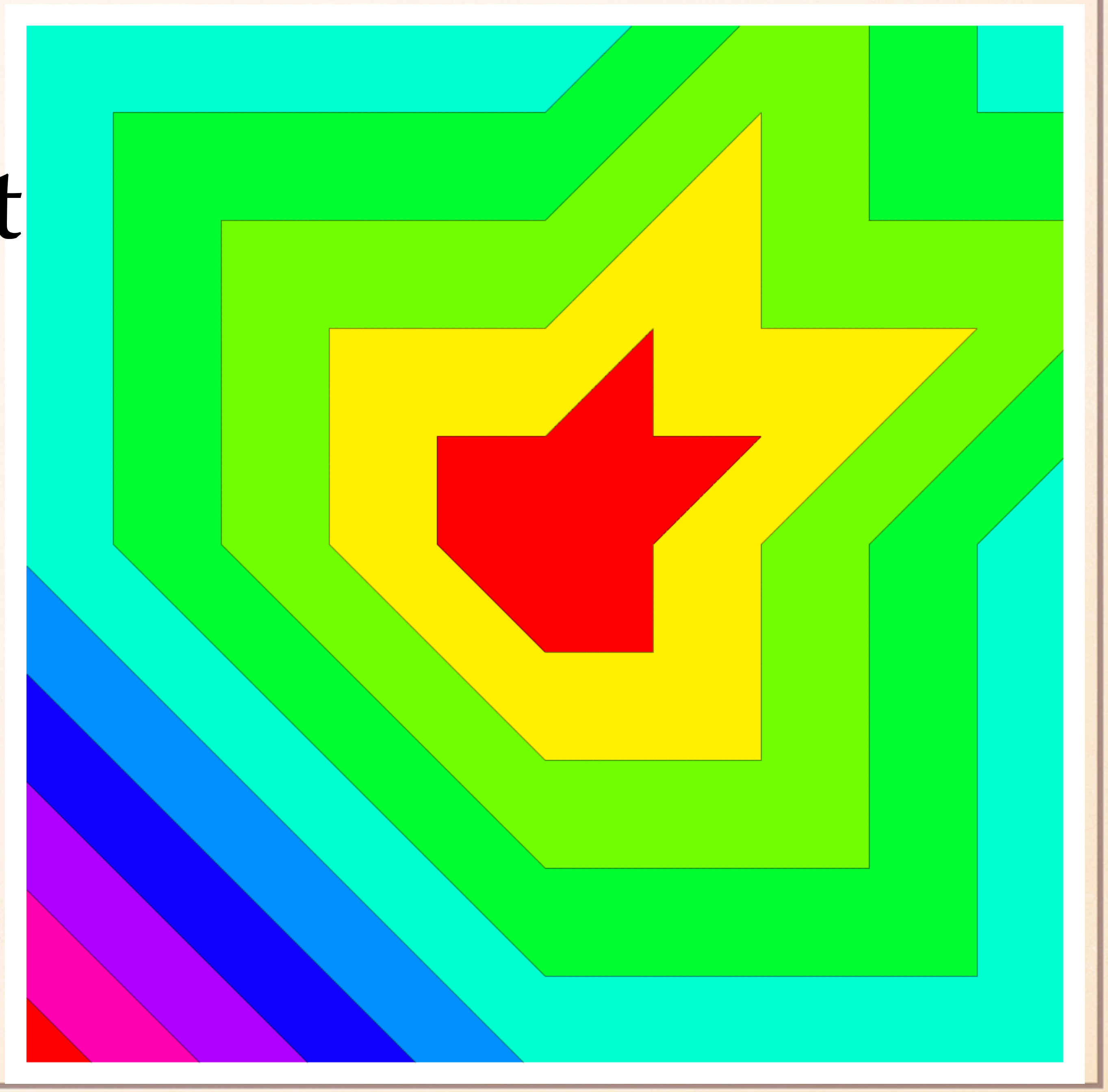
$$F'(q)\,h = -F(0)$$



INTEGRALS

$$\frac{S_n}{n} \rightarrow I$$

T invariant



T(x,y)=

(|x|-y,x)

Knuth Map

ANALYTIC MAP

$$f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

Folkert Tangerman 2010

$$f(z,w)=(cz, w-wz)$$

$$c = \exp(2\pi i \alpha)$$

$\{|z|=1\} \times \mathbb{C}$ invariant

PARTITIONS

$$\prod_{k=1}^{n} (1-x^k)^{-1}$$

```
CoefficientList[ Series[
Product[1/(1 - x^k), {k, 1, 10}] ,
{x, 0, 10}],x]
```

```
Table[PartitionsP[k], {k, 10}]
```

{1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42}

1

2

3

5

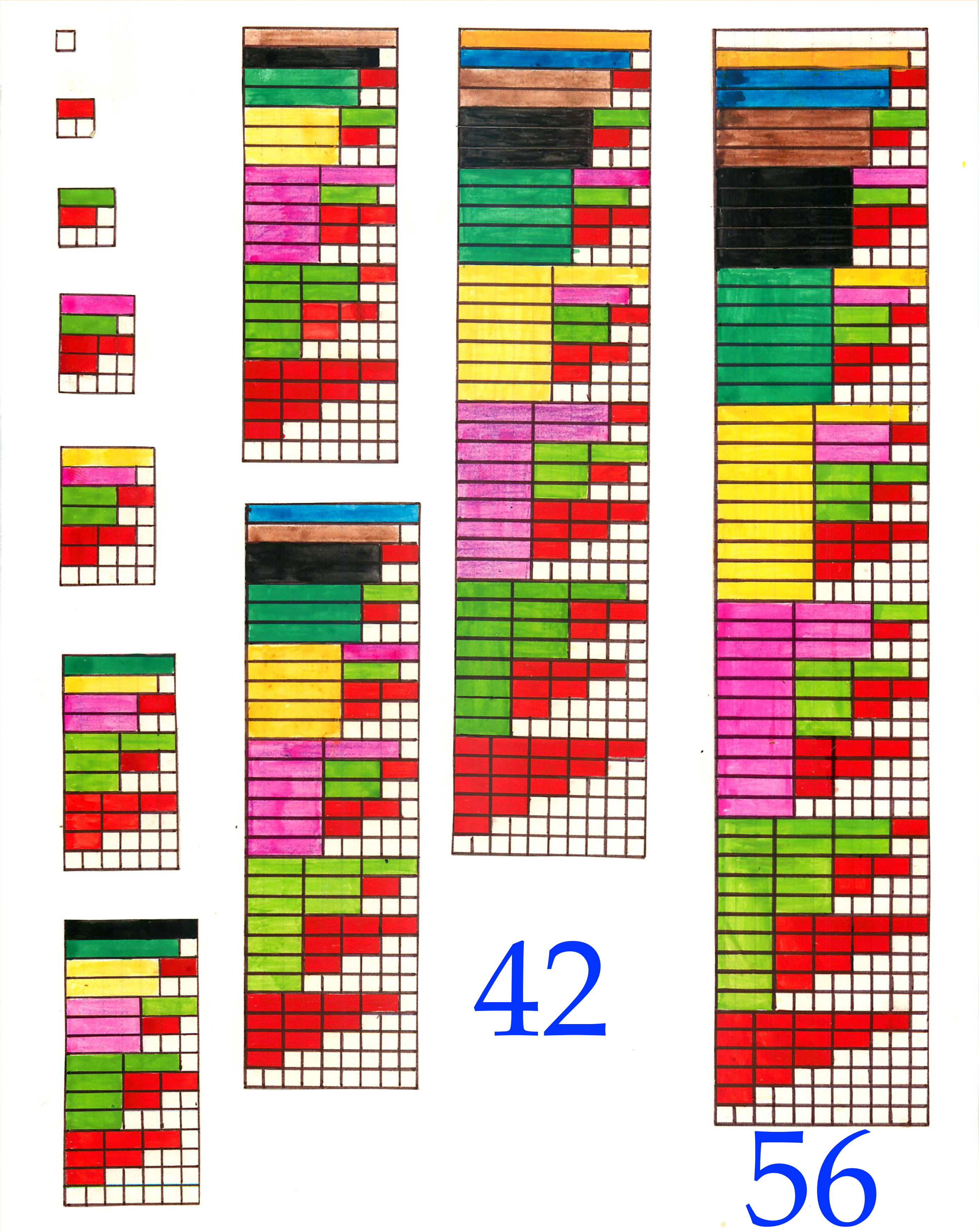
7

11

15

22

30



RESTRICTED PARTITION FUNCTION

$$\prod_{k=1}^{n} (1-x^k)^{-1}$$

Theodore Motzkin (1955)

Culbreth Sudler (1964)

Freiman Halberstam 1988

THEORY

BIRKHOFF

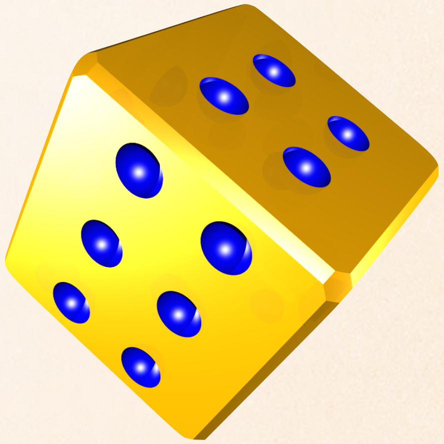
$$\frac{S_n}{n} \longrightarrow I$$

I is a T invariant function, constant if T is ergodic, an integral of motion if T is integrable

LAW OF LARGE NUMBERS

Jacob Bernoulli

$$\frac{S_n}{n} \longrightarrow E[X]$$



with probability 1

1713

OXTOBY 1952

Uniform convergence

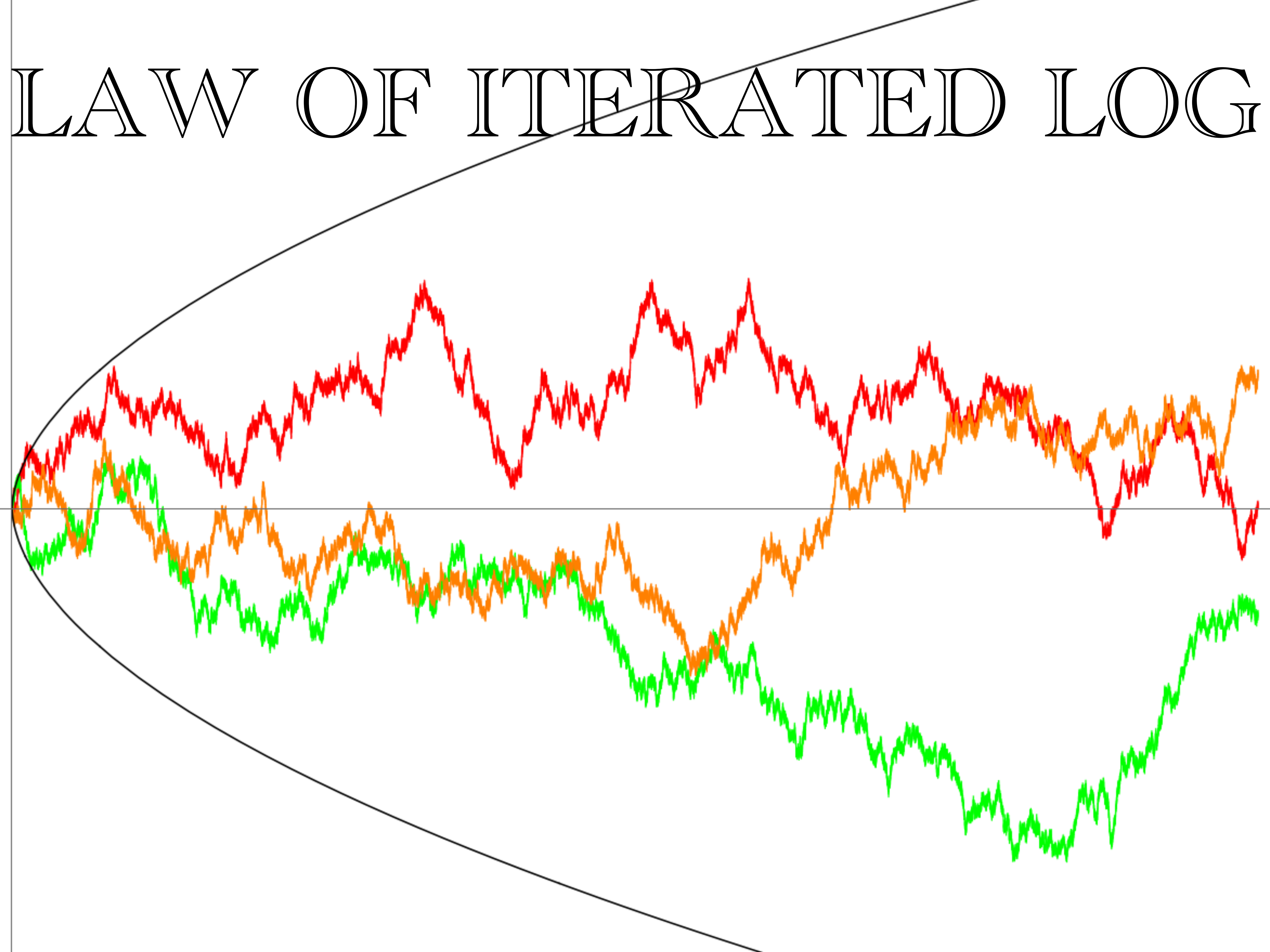
$$\frac{S_n}{n} \longrightarrow E[X]$$

if T uniquely ergodic, g continuous

ZERO EXPECTATION

Assume E[g]=0.
How fast
does the sum grow?

LAW OF ITERATED LOG



WEYL SUMS

g real analytic

α Diophantine

$g(x) = f(x+\alpha)-f(x)$

cohomology

COBOUNDARY

$$g(x) = h(T(x)) - h(x)$$

$$S_n(x) = h(T^n(x)) - h(x)$$

$$\hat{g}_k = \hat{h}_k (e^{i\alpha} - 1)$$

small divisors

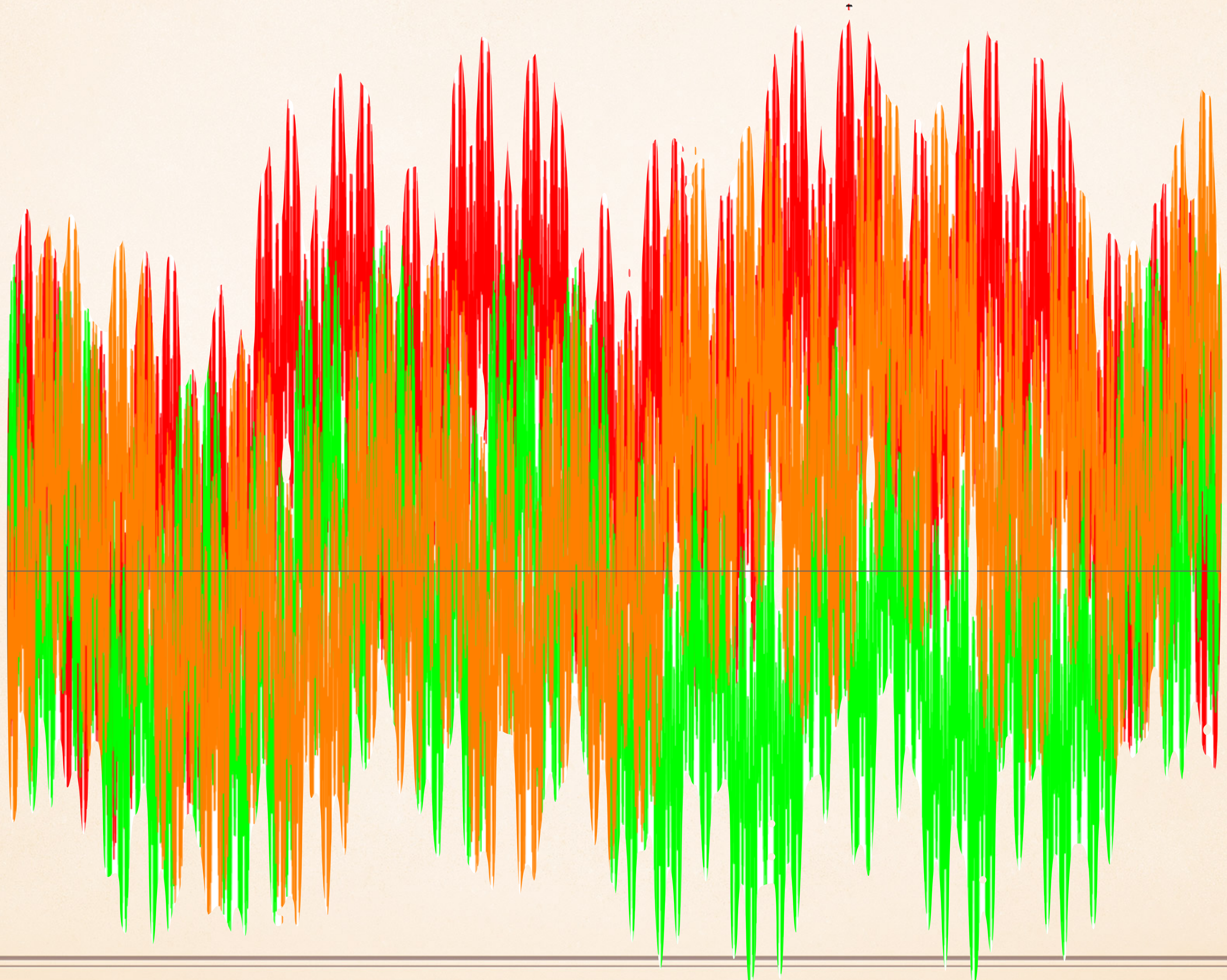
$$g = Uf - f$$

$$f = (U-1)^{-1} g$$

$$(1 + U + U^2 + \ldots)\, g$$

small divisors

DENJOY-KOKSMA



DENJOY-KOKSMA

$$|\alpha p - q| < 1/q$$

$$S_n \leq \mathrm{Var}[g] \log(n)$$

Arnauld Denjoy

Jurjen Koksma

JITOMIRSKAYA

$$|\alpha p - q| < 1/q^r \qquad r > 1$$

$$S_n \leq \mathrm{Var}[g] \log(n)\, n^{1-r}$$

GOTTSCHALK-HEDLUND

bounded S = g is coboundary

Walter Gottschalk

Gustav Hedlund

1955

THE COT CASE

THE MOST NATURAL CASE

$$\alpha = [1,1,1,1,....]$$ golden number

$$\hat{g} = [1,1,1,1,....]$$ golden function

$$G(x) = \log(2\sin(\pi x))$$

$$= \frac{1}{2}\log(2\text{-}2\cos(2\pi x))$$

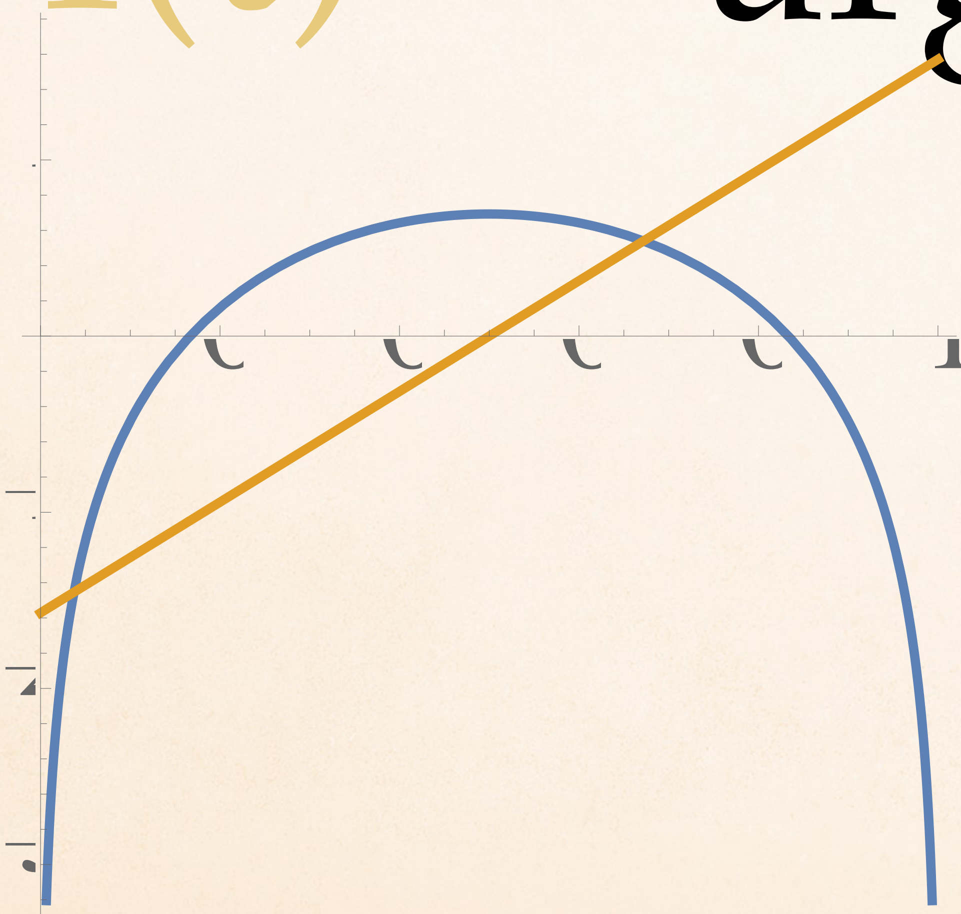
$$= \log|1\text{-}e^{2\pi ix}|$$

$$G'(x) = \pi\cot(\pi x)$$

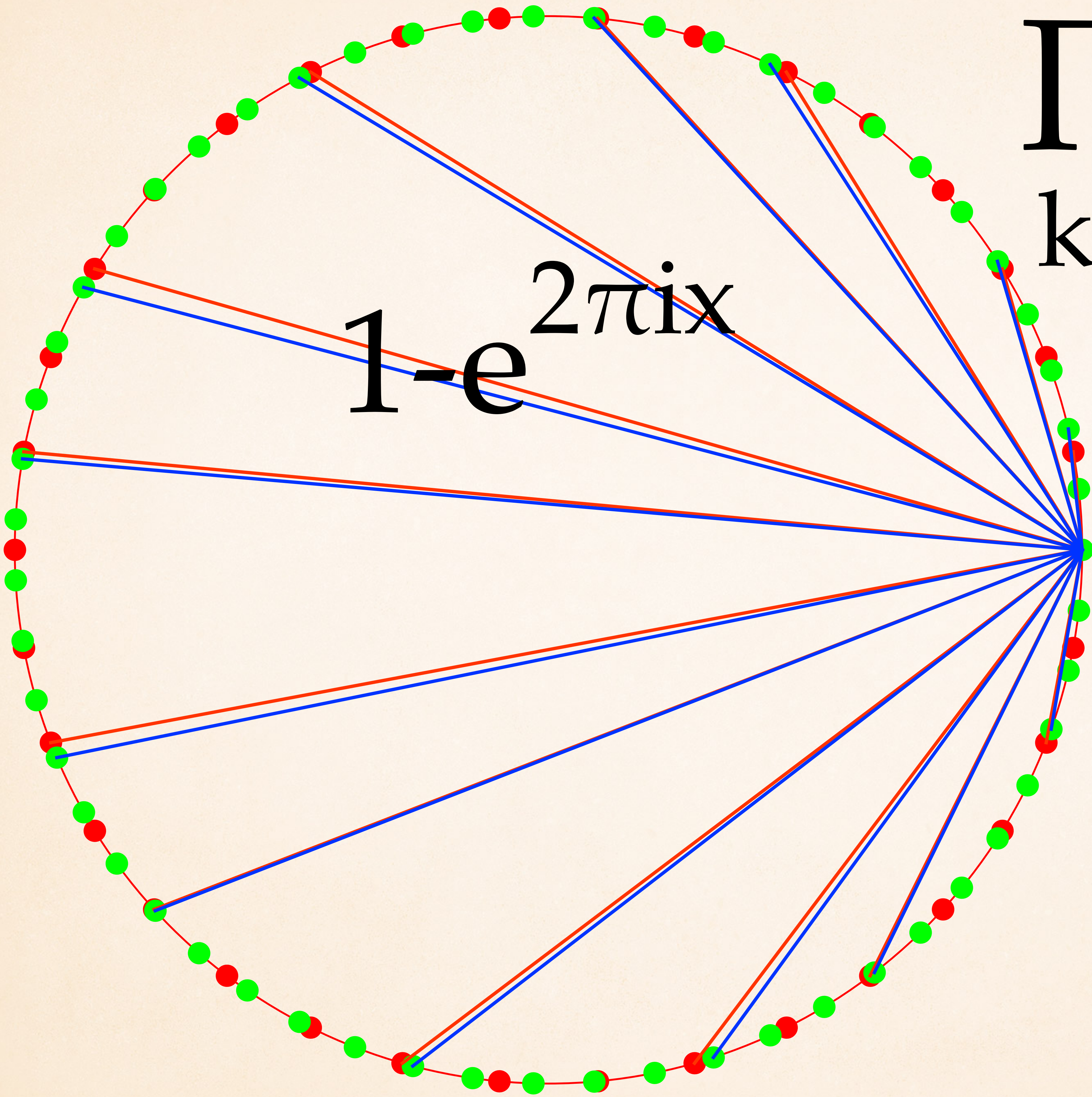
HILBERT TRANSFORM

$$G(t) = \log|1\text{-}e^{2\pi ix}|$$

$$H(t) = \arg|1\text{-}e^{2\pi ix}|$$

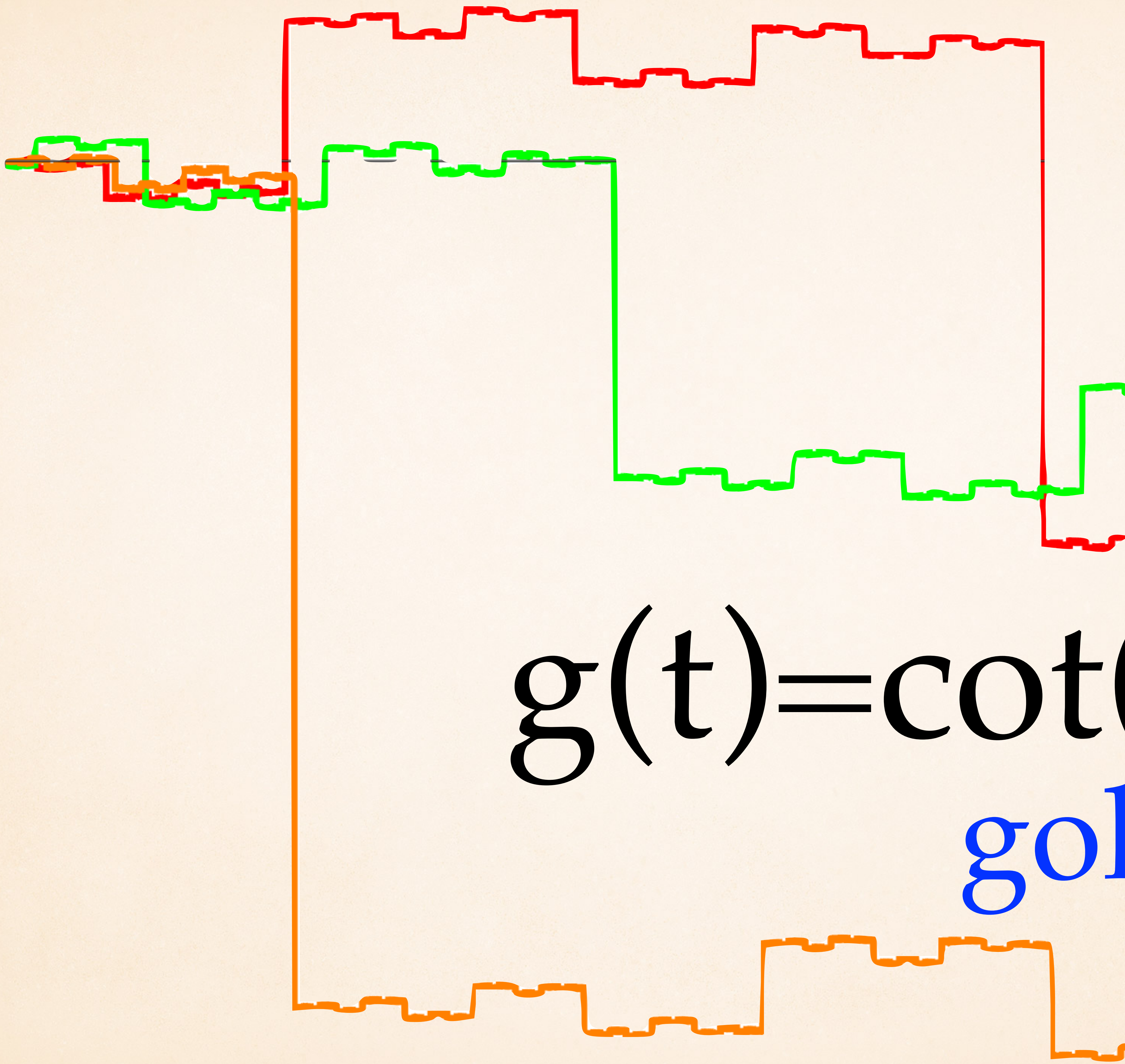


Erich Hecke



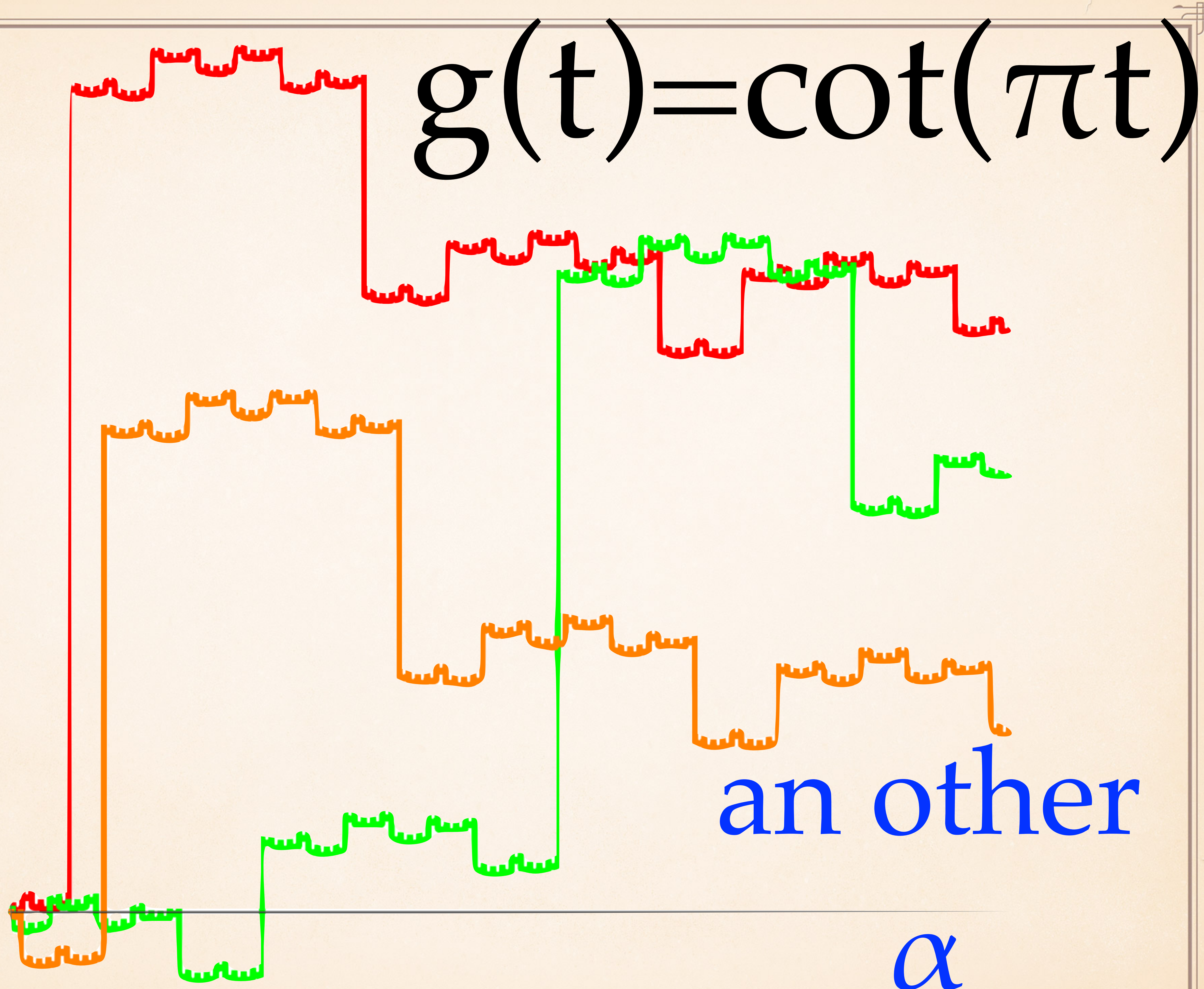
$$\prod_{k=1}^{n-1} 1 - e^{\frac{2\pi i k}{n}} = n$$

EXPERIMENTS



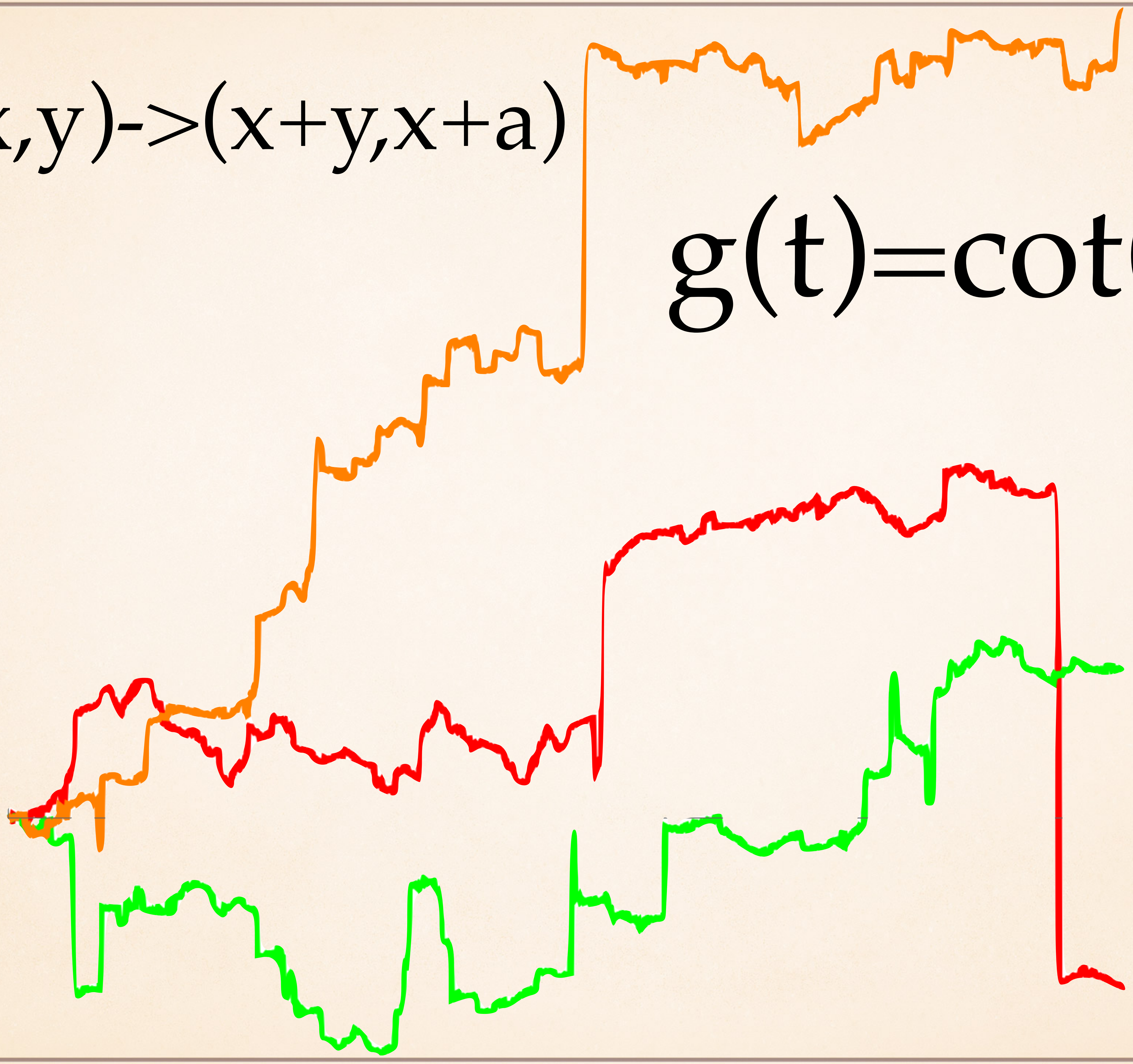
$$g(t)=\cot(\pi t)$$

golden


g(t)=cot(πt)
an other
α

(x,y)->(x+y,x+a)

$g(t)=\cot(\pi t)$



Dow Jones Ind. (1.5 Yr) Daily
1 day bars Histo
D: 12/10/29 O: 259.17 H: 263.98 L: 255.52 C: 262.20 V: 3,650,000
U.S. Markets Close
Modulated Avr (1 day) Volume
Crash
2 months
Great
Depres
2 years a
2 month
down-tr
© MarketVolume™
© MarketVol
Black Tuesday
After-Crash
Recovery
5 months
Patent Pending
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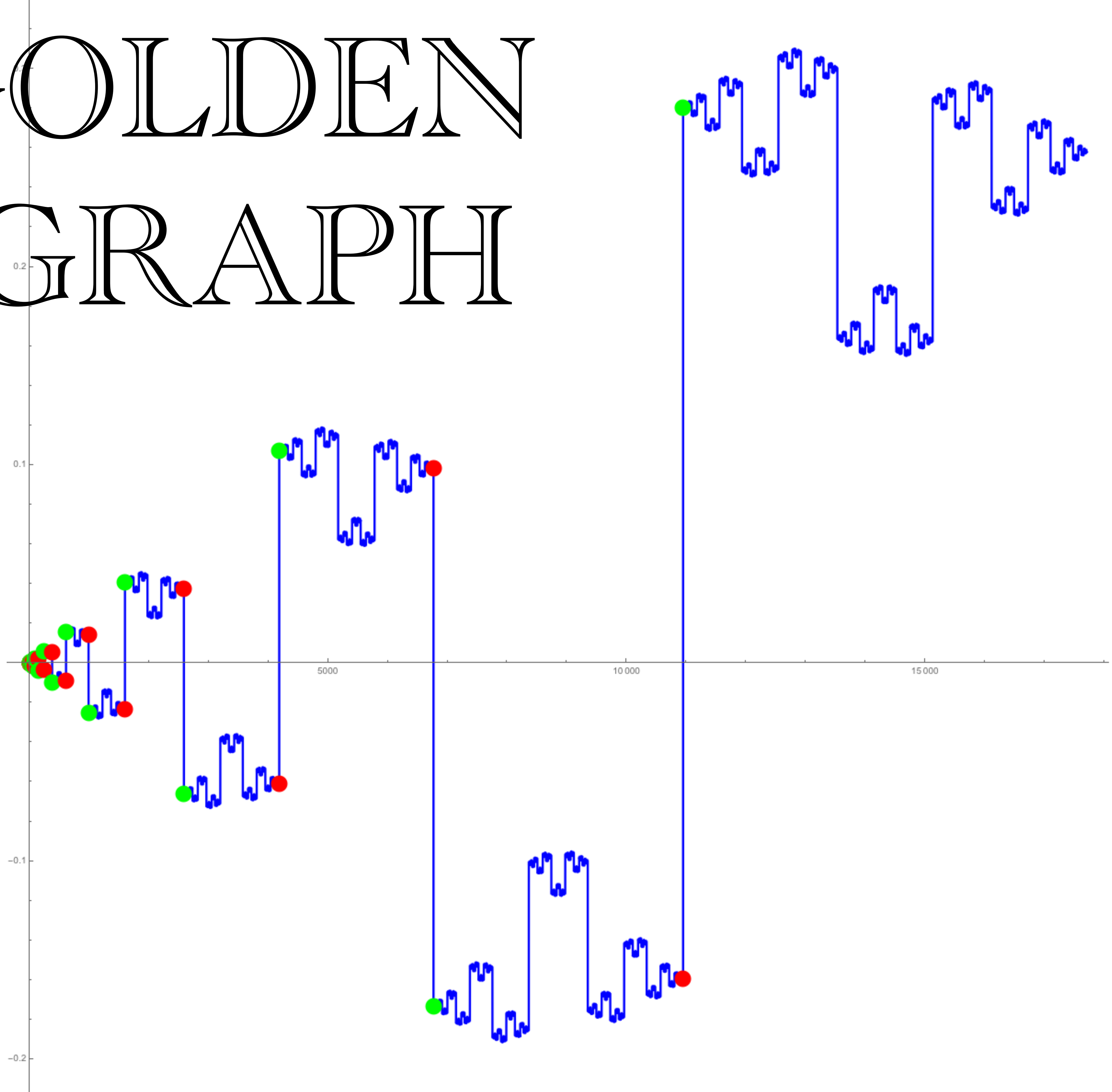
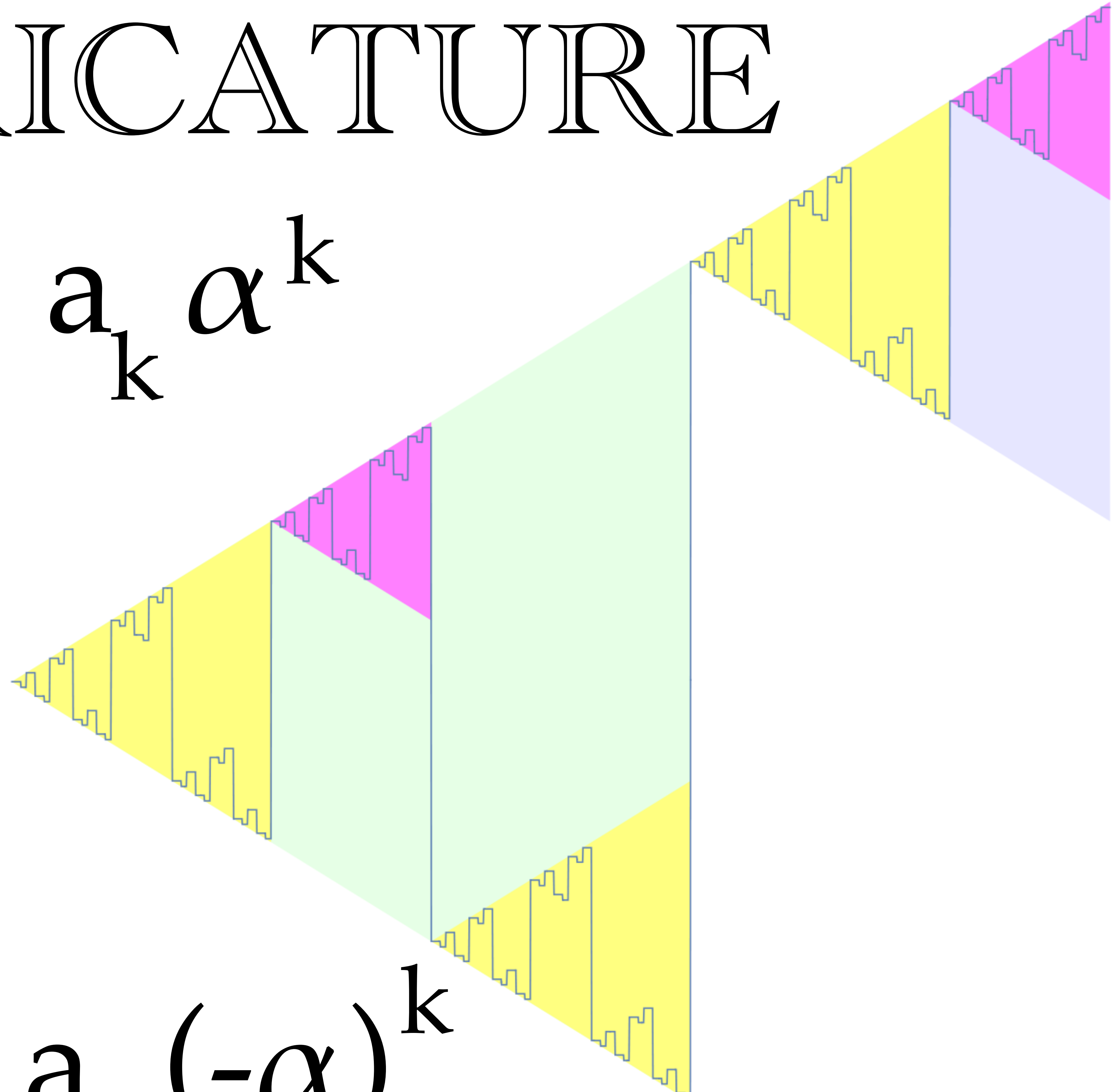
Jul 1929
Sep 1929
Nov 1929
Jan 1930
Mar 1930
May 1930
Hide Draw.
Hide Studies

GOLDEN GRAPH



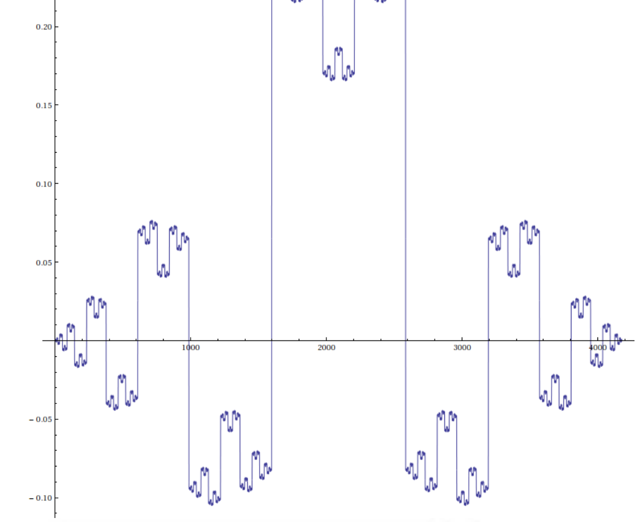
CARICATURE

$$x=\sum a_k \alpha^k$$



$$f(x)=\sum a_k(-\alpha)^k$$

$\tau(y)=(\pi/3)\,y - (\pi/45)\,y^3 \quad \ldots$



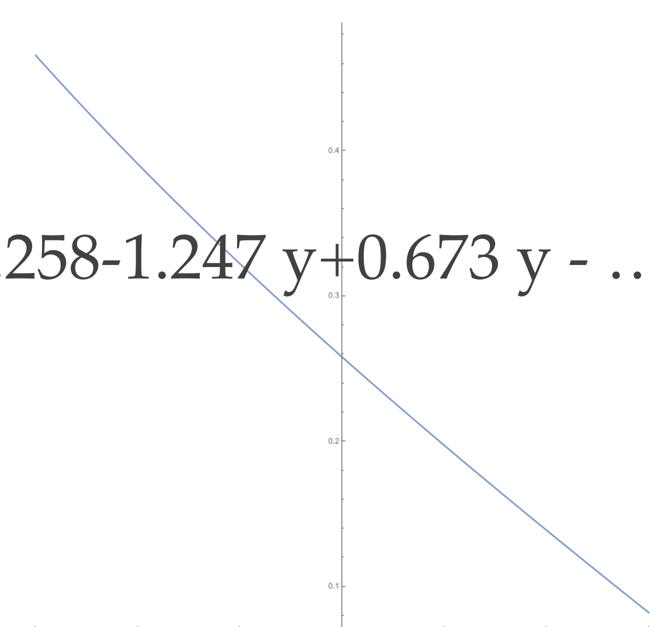
$\tau(y) = t(1^-, y)$ $\qquad$ $t(x, y)$ $\longrightarrow$ $t(x) = t(x, 0)$

(Fig 2)

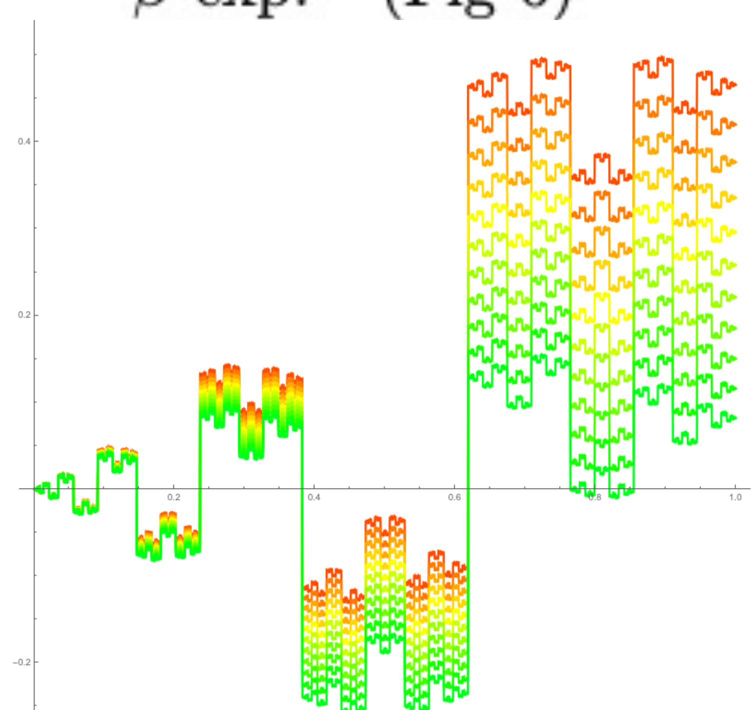
Steps $\downarrow$ $\qquad$ $\uparrow$ Taylor

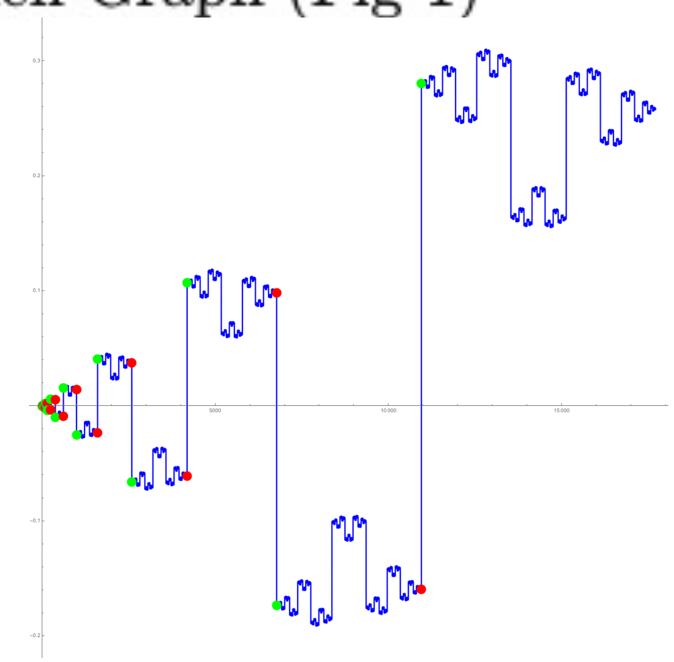
$\sigma(y) = s(1^-, y)$ $\longrightarrow$ $s(x, y)$ $\longrightarrow$ $s(x) = s(x, 0)$

(Fig 7) $\qquad$ β-exp. $\quad$ (Fig 6) $\qquad$ Golden Graph (Fig 1)



$\sigma(y)=0.258-1.247\,y+0.673\,y - \ldots$



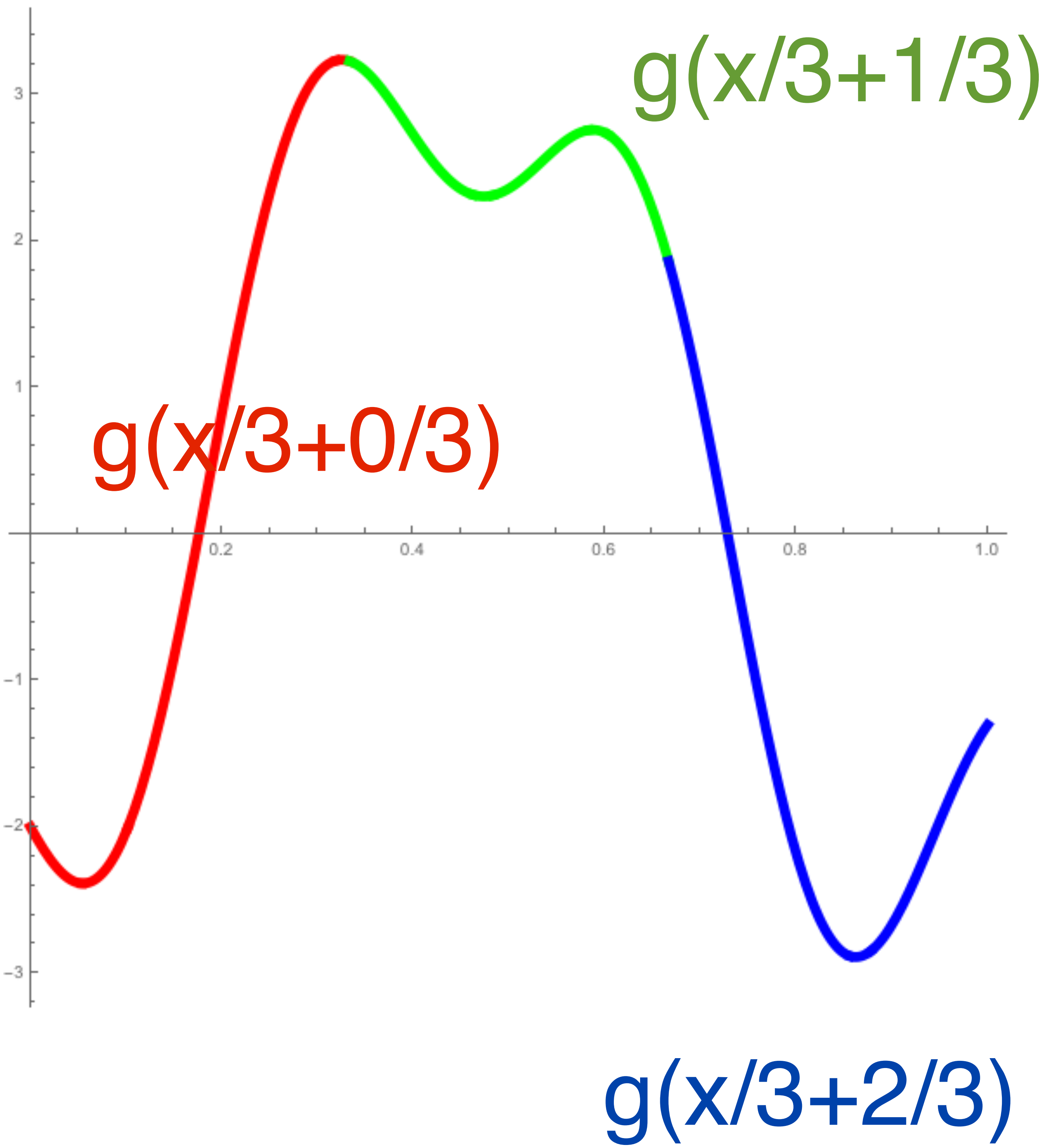


KUBERT-LANG

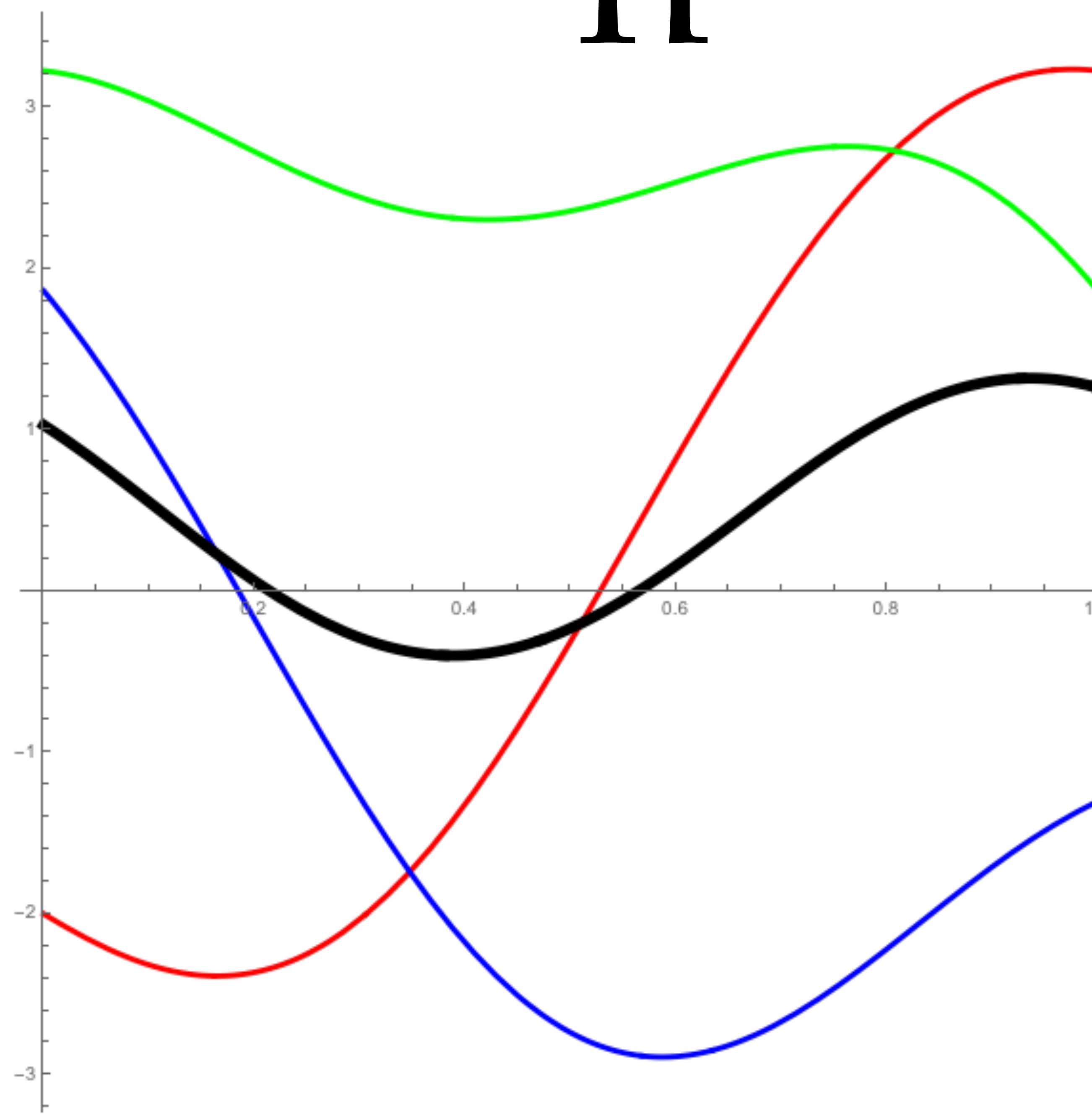
$$f(t) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{t+k}{n}\right)$$

f odd

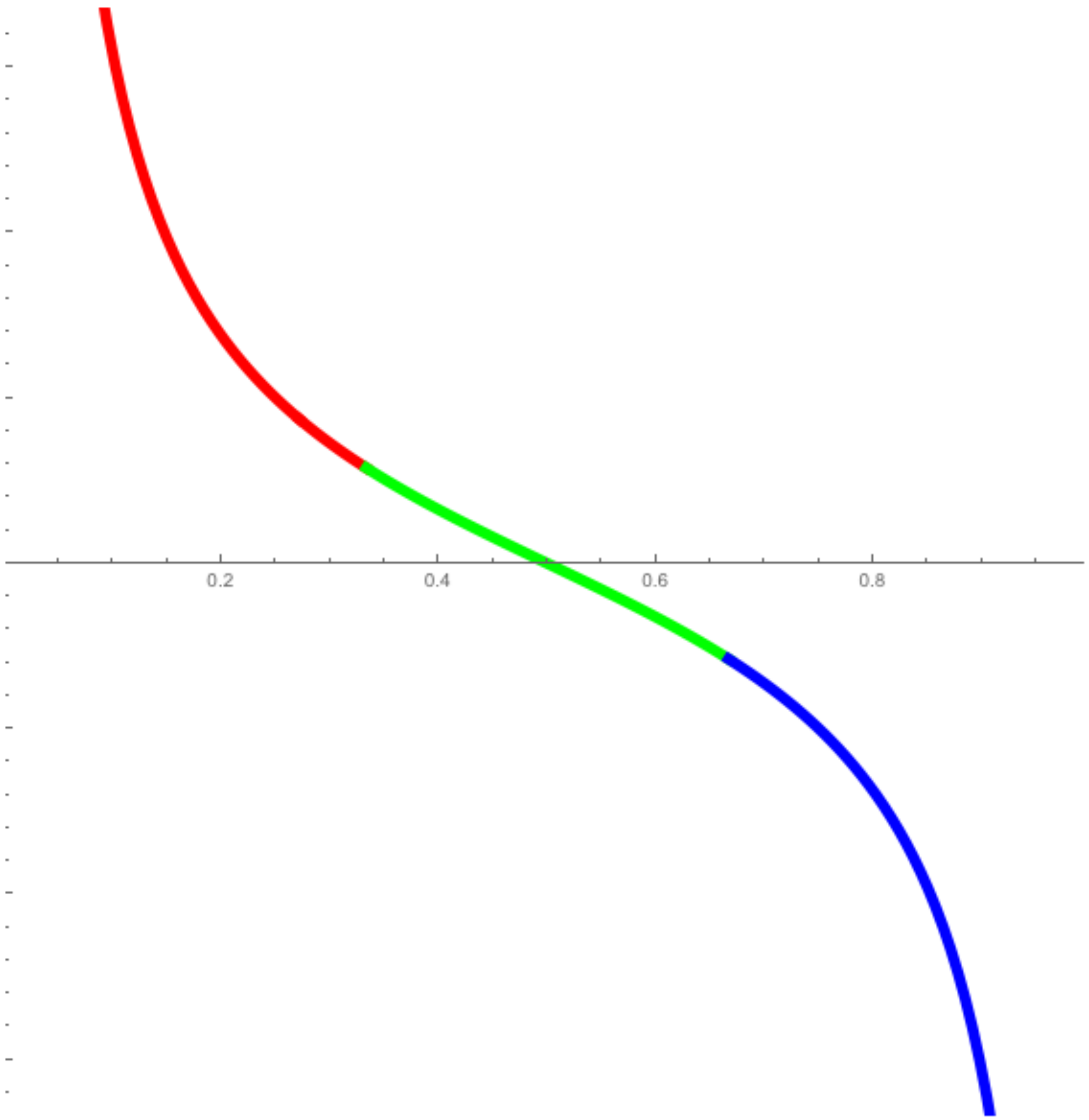
$$g(t)$$

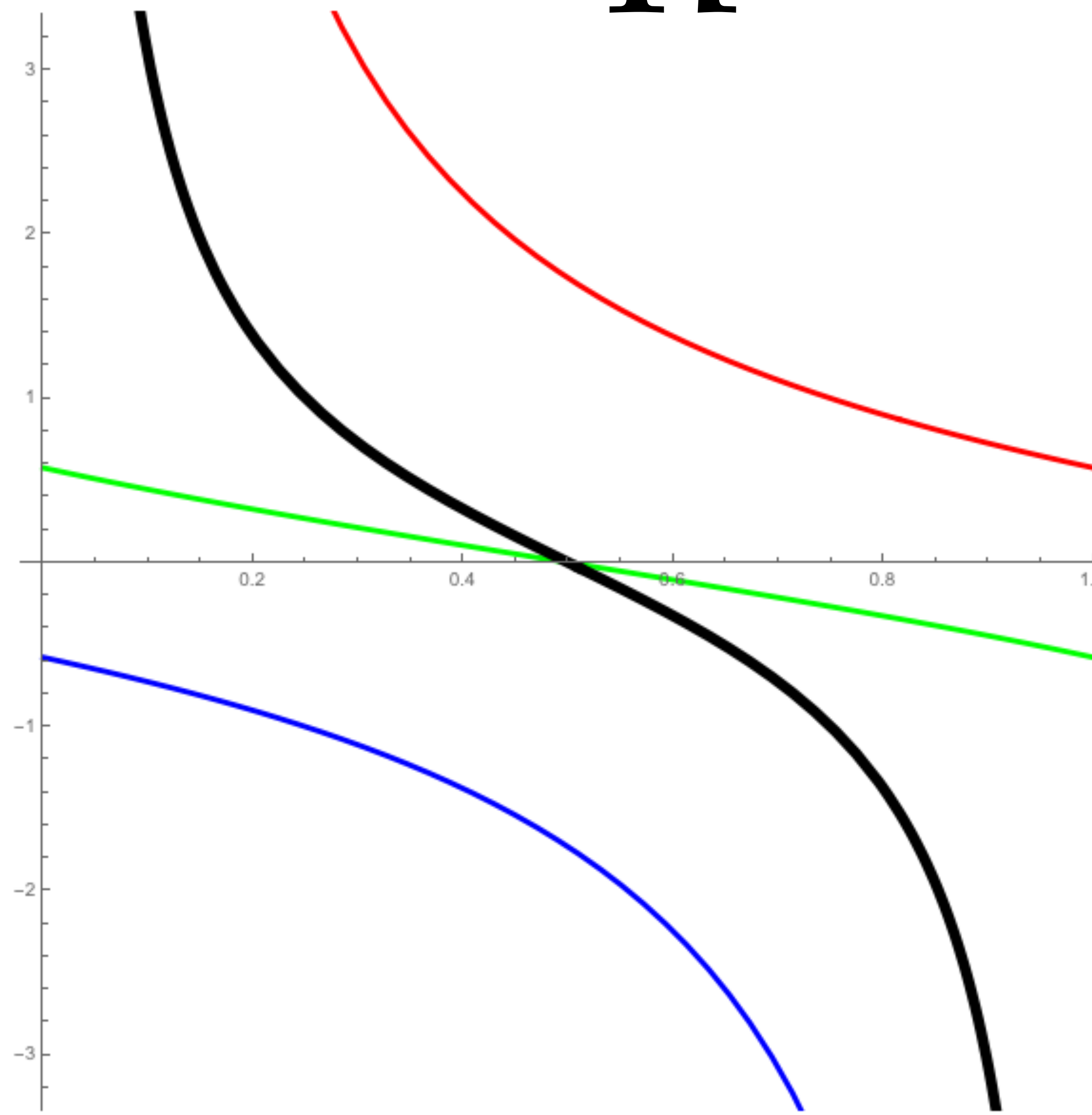


$$g\left(\frac{t+k}{n}\right)$$



$$\cot(t) \qquad \cot\left(\frac{t+k}{n}\right)$$





L'Enseignement Mathématique, t. 29 (1983), p. 281-322

ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES

by John Milnor

§1. Introduction

D. Kubert [12] has studied functions $f(x)$, where x varies over $\mathbf{Q}/\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$, which satisfy the identity

$$(*_s) \qquad f(x) = m^{s-1} \sum_{k=0}^{m-1} f((x+k)/m)$$

for every positive integer m. (See also Lang [16-18], as well as Kubert and Lang [13-15].) Here s is some fixed parameter. Note that $(x+k)/m$ varies precisely over all solutions y to the equation $my = x$ in the group $\mathbf{Q}/\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$. However, the equation is set up so that it also makes sense for x in the interval $(0, 1)$ or $(0, \infty)$. Evidently it would suffice to assume the equation $(*_s)$ for prime values of m.

Classical examples of such functions are provided by the uniformly convergent Fourier series $l_s(x) = \sum_{n=1}^{\infty} e^{2\pi inx}/n^s$ for $x \in \mathbf{R}/\mathbf{Z}$ and $Re(s) > 1$, the Hurwitz function

$$\zeta_{1-s}(x) = x^{s-1} + (x+1)^{s-1} + \ldots$$

-2	-1	0	1	2
$\zeta_3(x) + \zeta_3(1-x)$	$csc^2 \pi x$	$\beta_0(x) = 1$	$\log(2 \sin \pi x)$	$\beta_2(x) = x^2 - x + \frac{1}{6}$
$\cos \pi x / \sin^3 \pi x$	$\zeta_2(x) - \zeta_2(1-x)$	$\cot \pi x$	$\beta_1(x) = x - \frac{1}{2}$	$\Lambda(\pi x)$

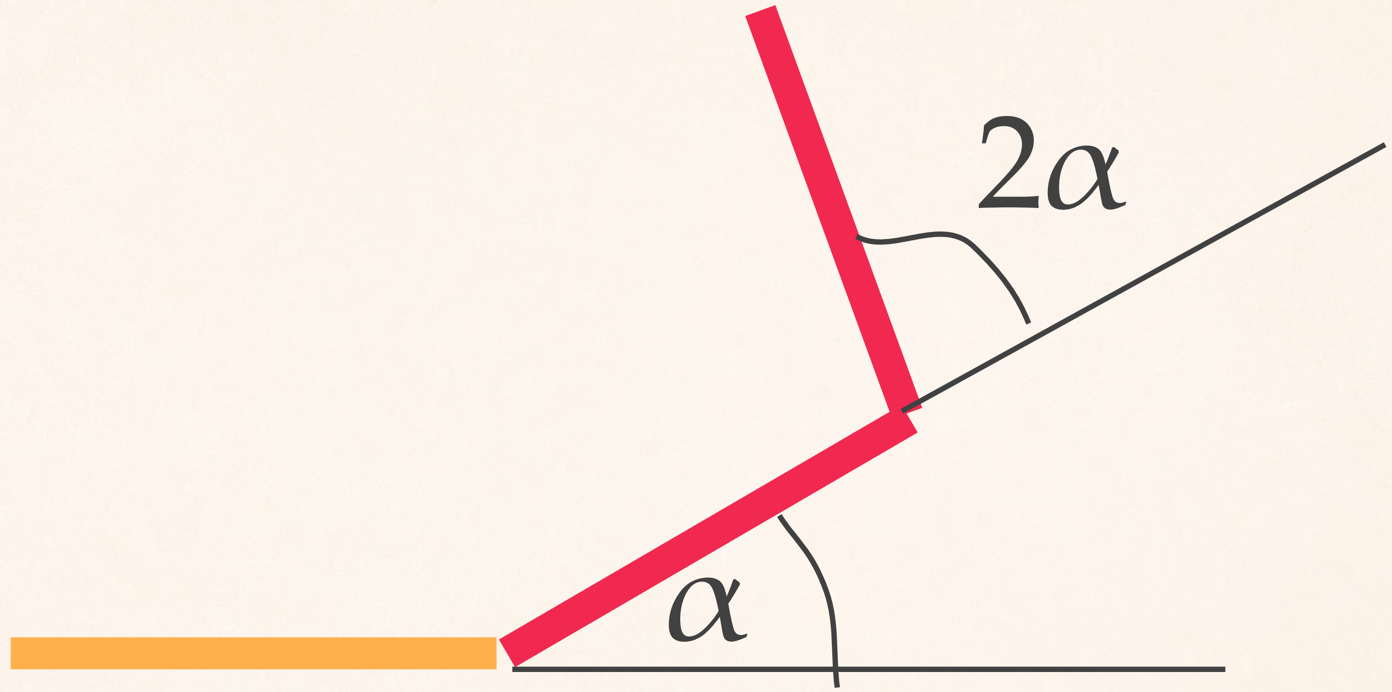
SINAI-ULCIGRAI

$$g(t)=2\,(1-\exp(i\,x))^{-1}$$

$= 1+i\cot(x/2)$


26544

CURLICUE



$$\sum_{k=1}^{n} e^{i(k^2+k)\alpha/2}$$

THETA FUNCTIONS GAUSS SUMS

$$\sum_{k=1}^{n} e^{ik^2\alpha} = \sum_{k=1}^{n} q^{k^2}$$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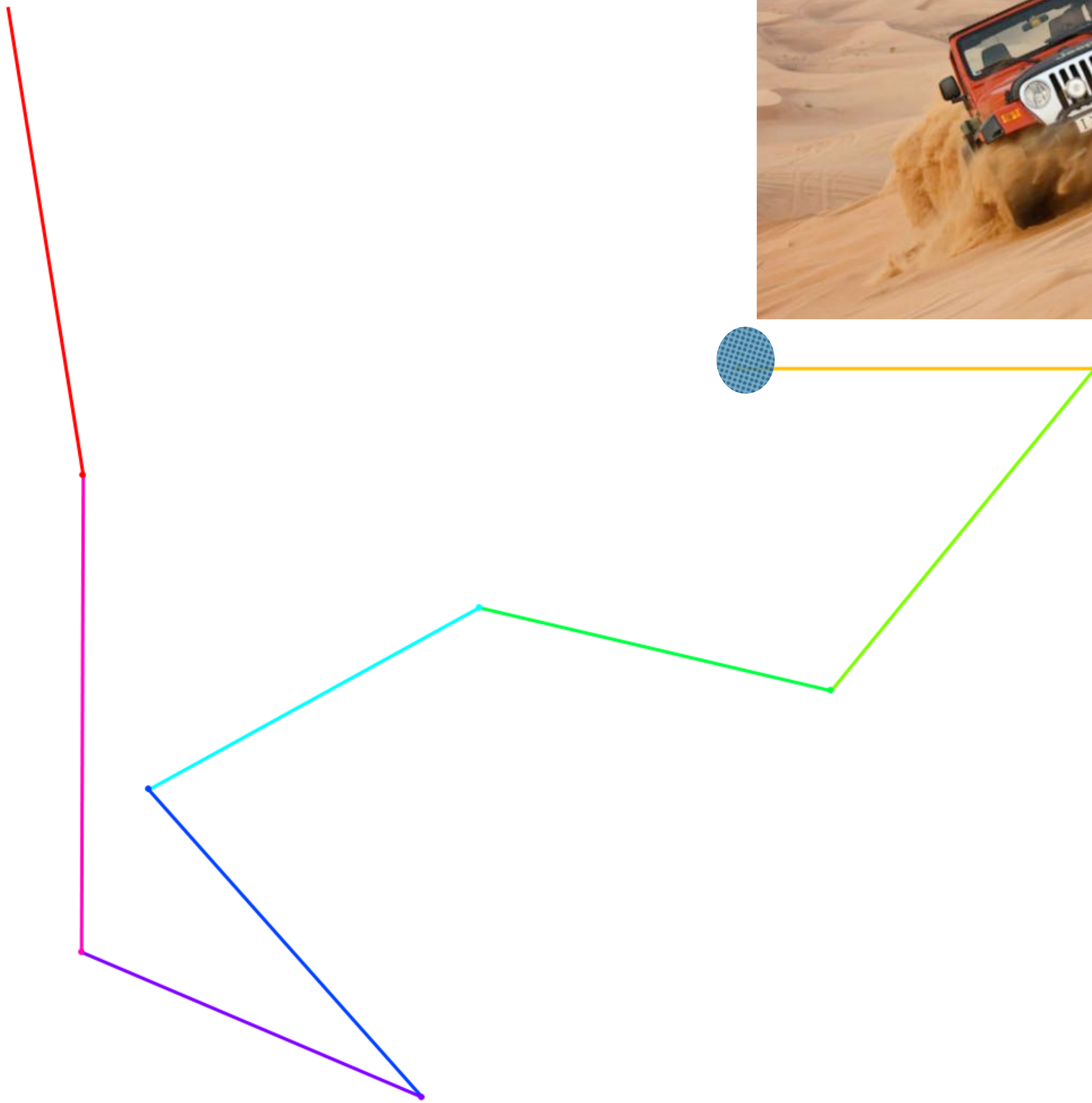
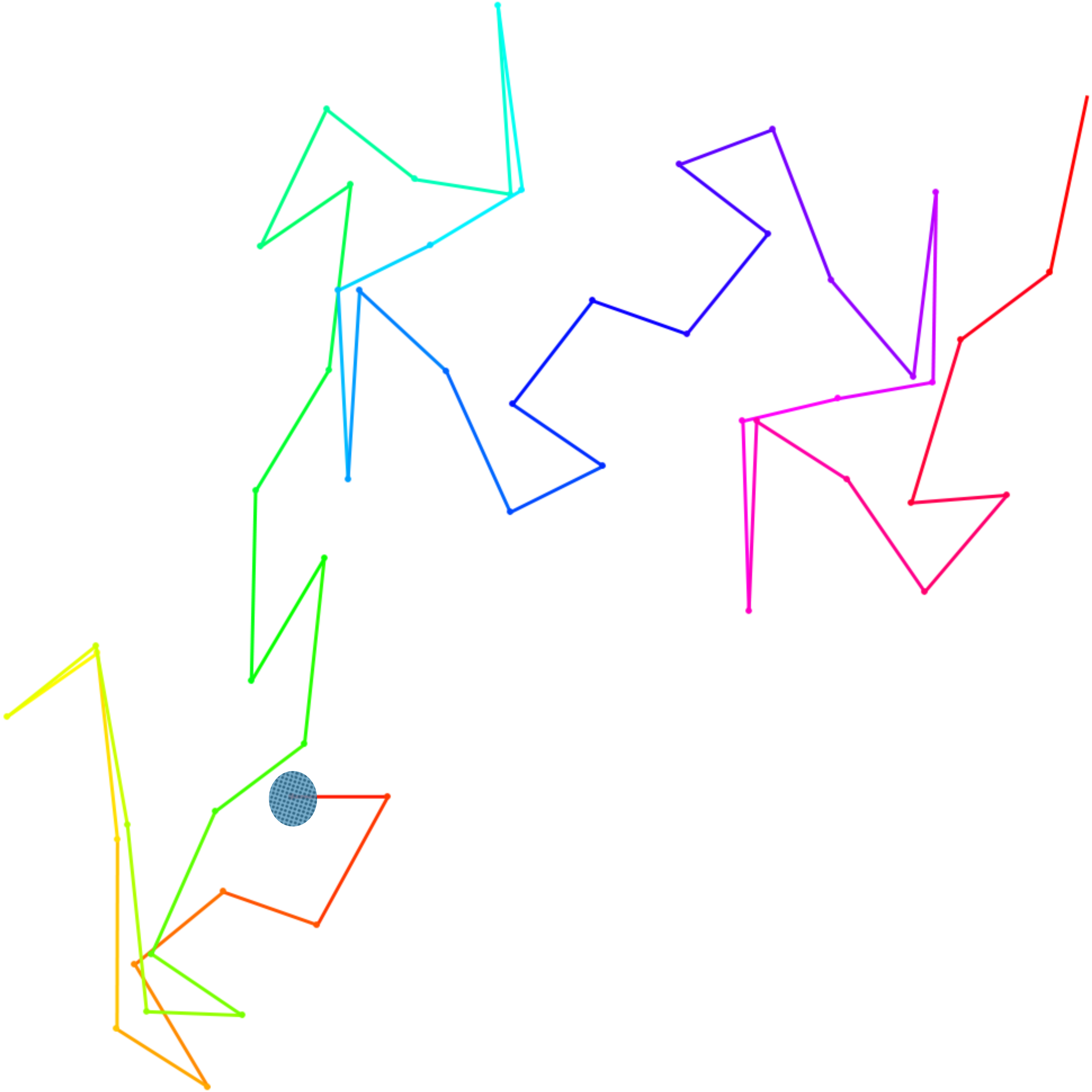
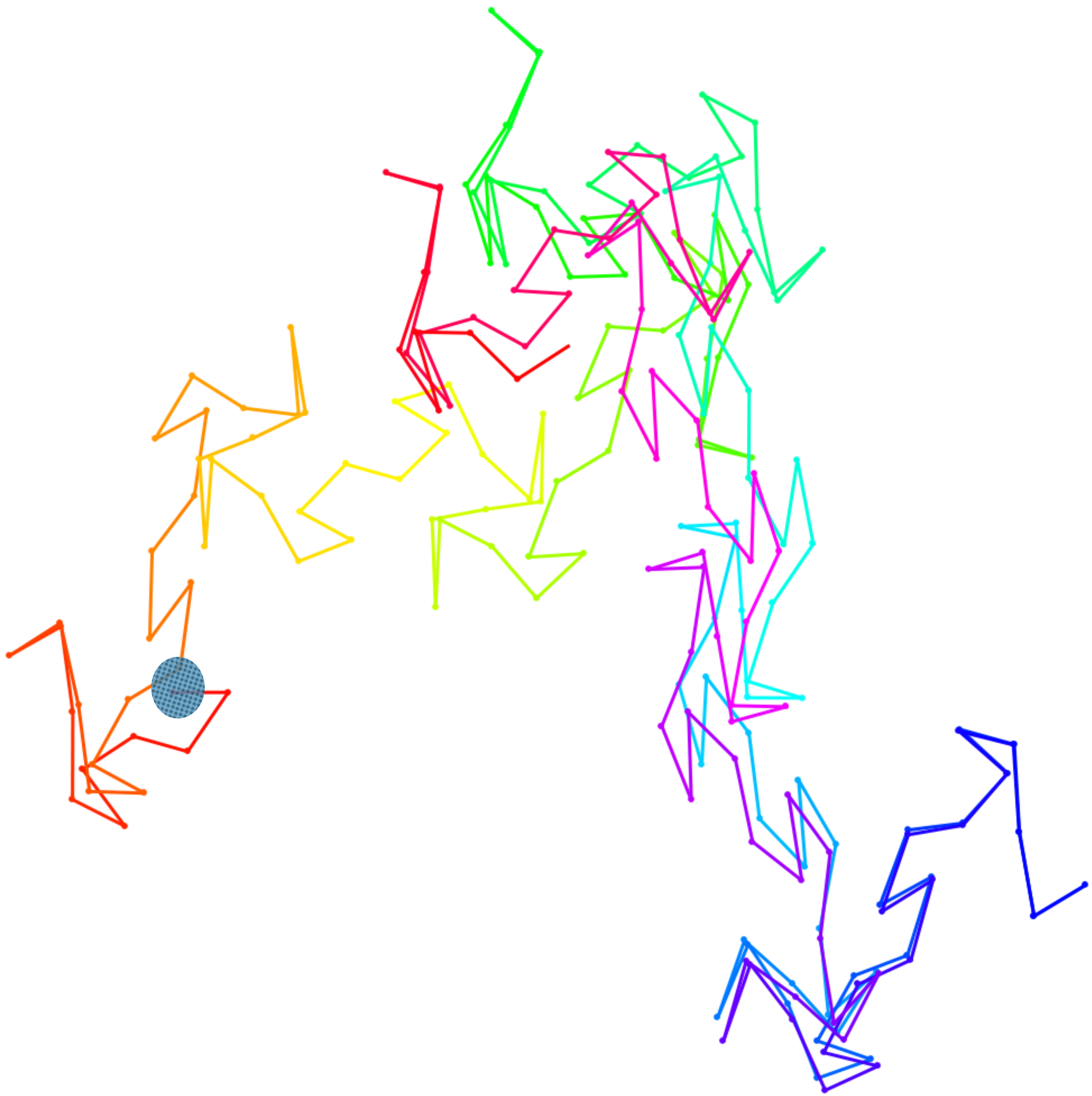
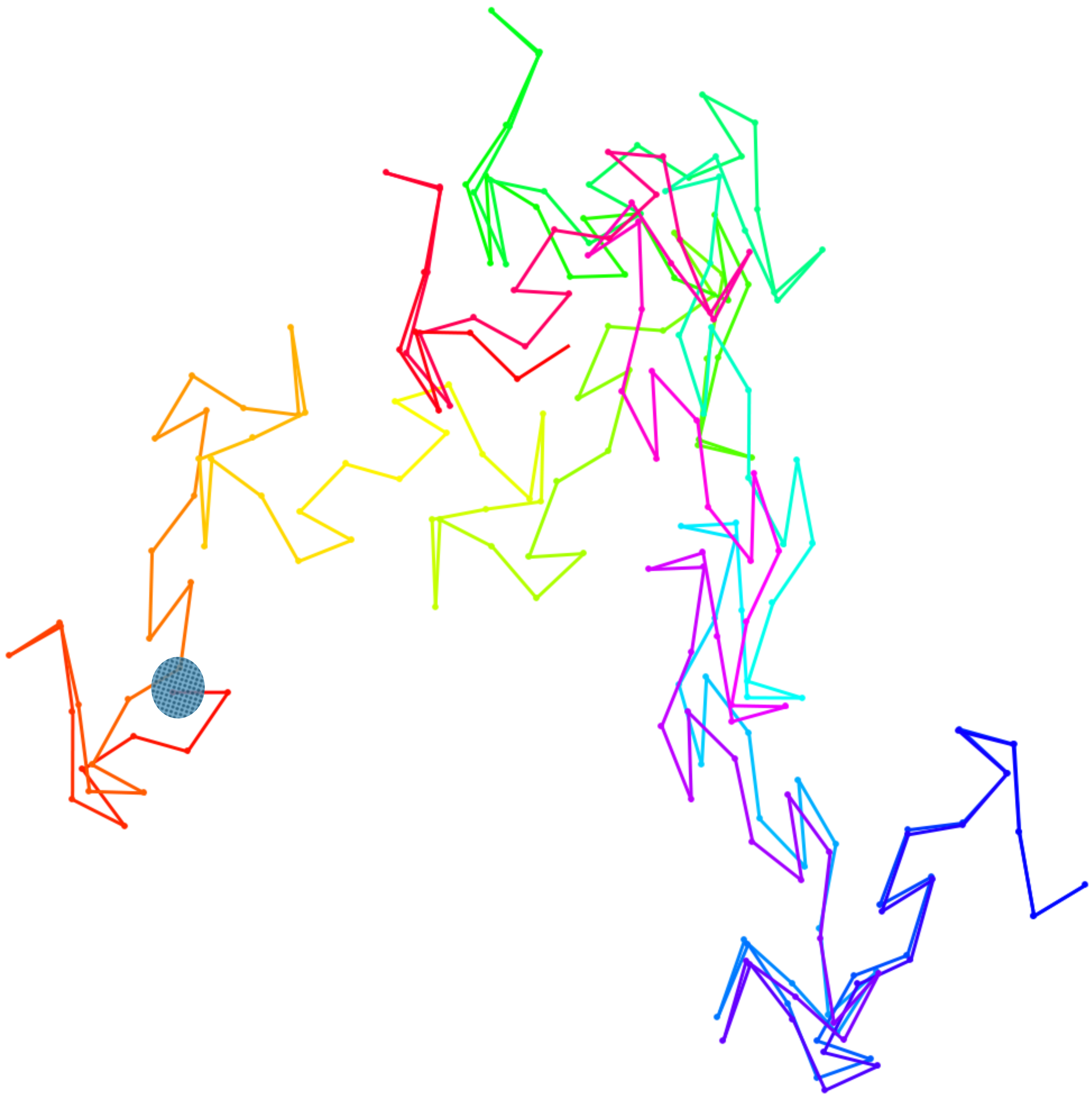
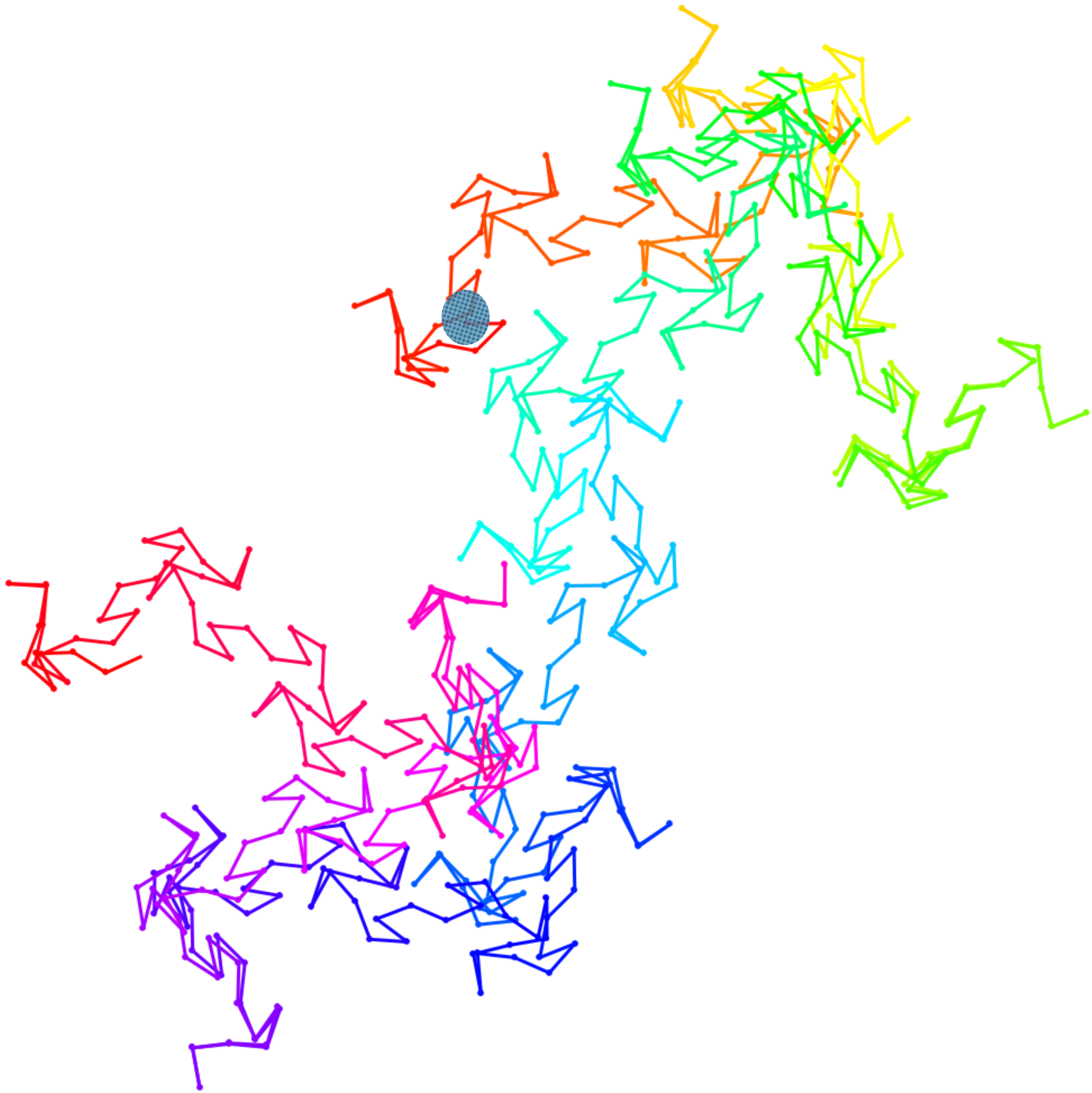
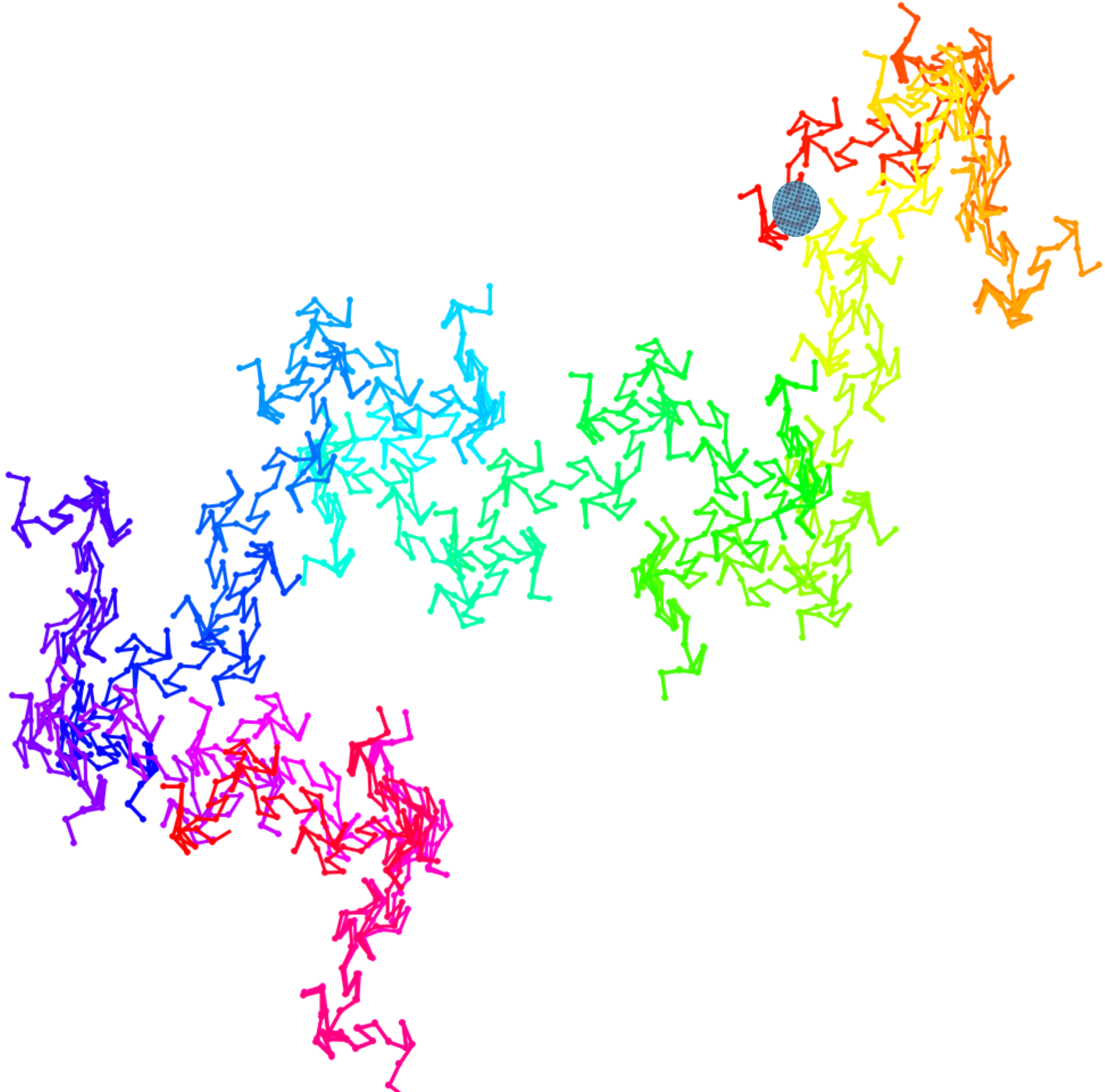


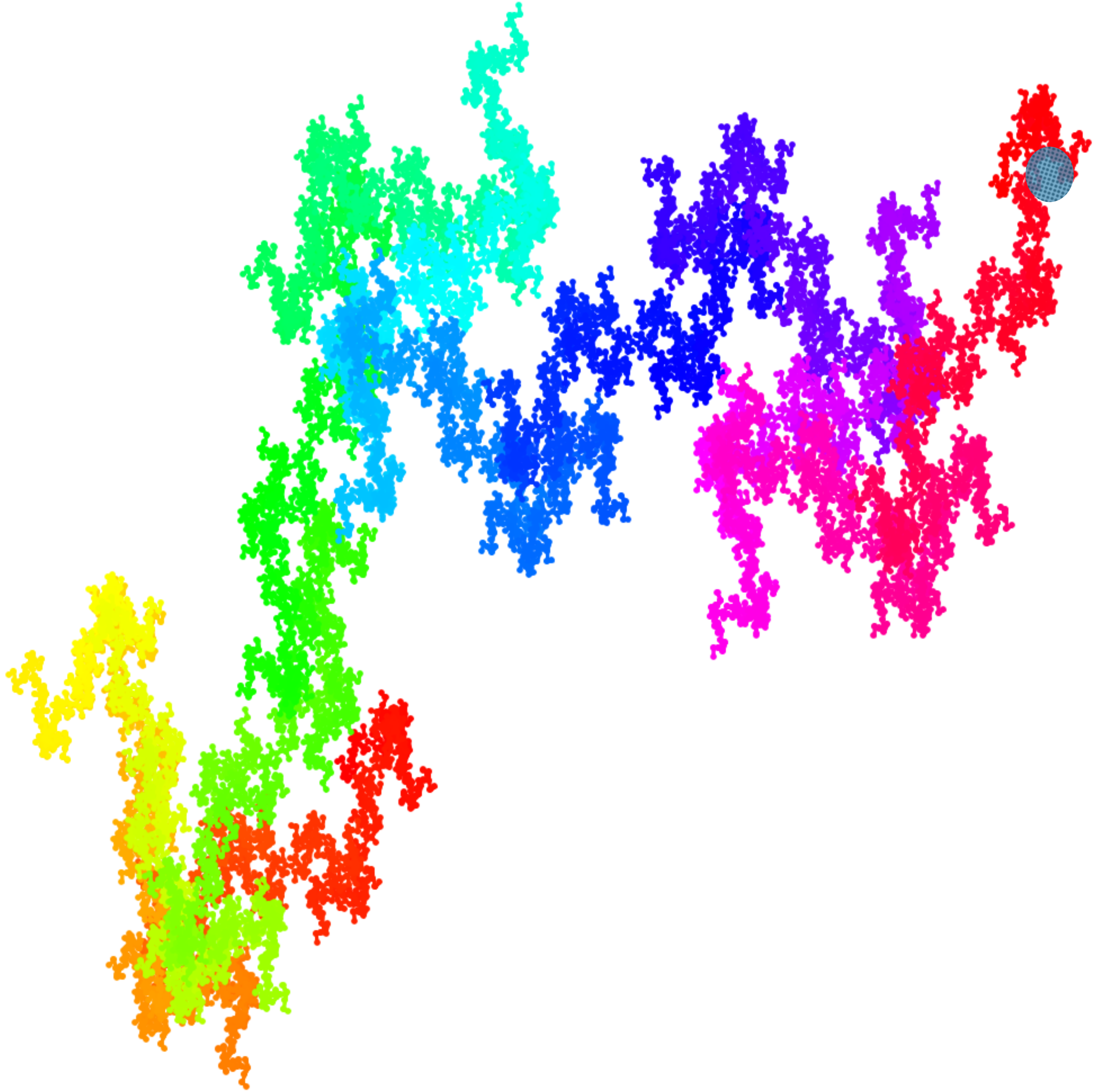












$$\sum_{k \in Z} q^{k^2} = \theta(q)$$

$$\theta^2(q) = \sum_{k \in Z} \sec(\pi k \alpha)$$

modular form R

LESIEUTRE-KNILL

$$\zeta_n(s) = \sum_{k=1}^{n} \frac{X_k}{k^s}$$

has entire analytic continuation, if g is real analytic and α is Diophantine

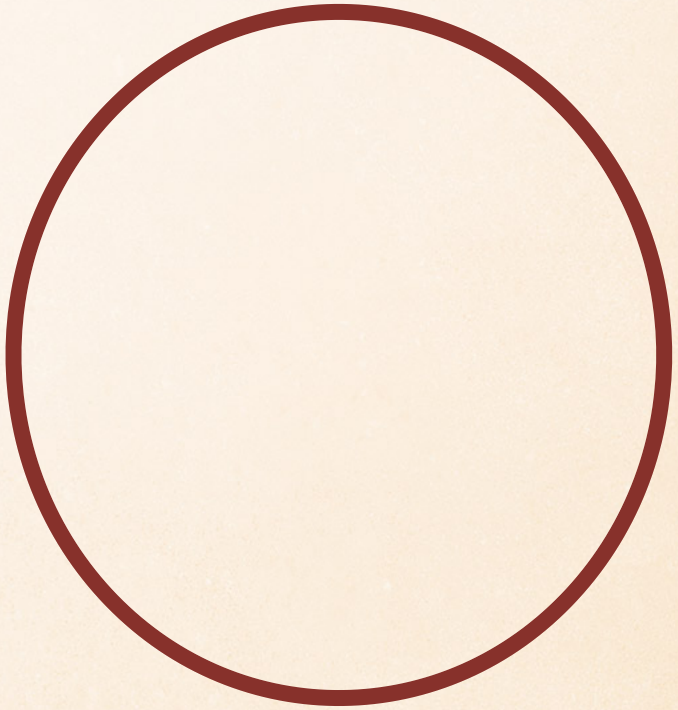
ZETA FUNCTION

$$\zeta_n(s) = \sum_{k=1}^{n} \frac{X_k}{k^s}$$

$$X_k = g(T^k(t))$$

RIEMANN ZETA

$$\zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}$$



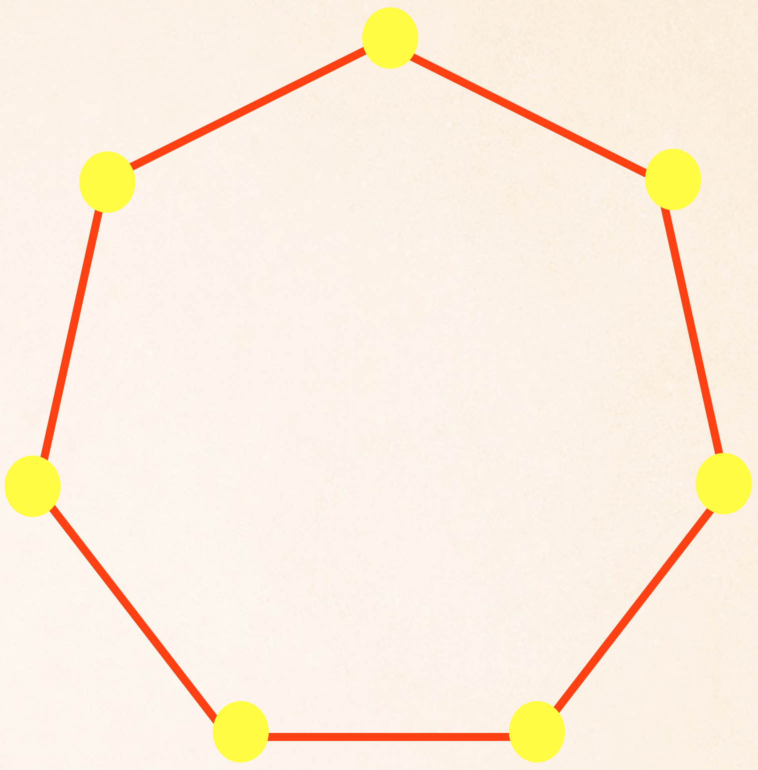
analytic

$$D = -i\,\partial/\partial x$$

has positive integer eigenvalues

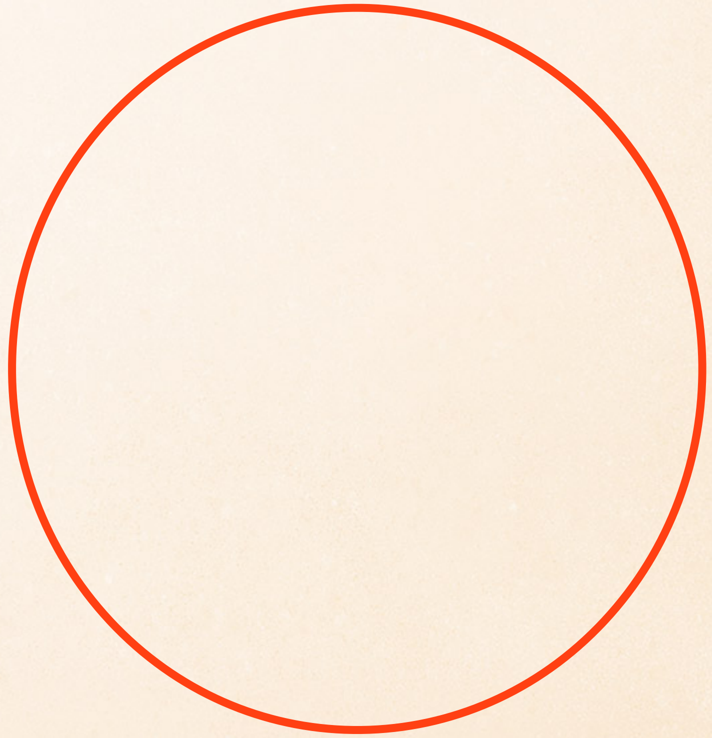
A=Dirac matrix of graph C_n

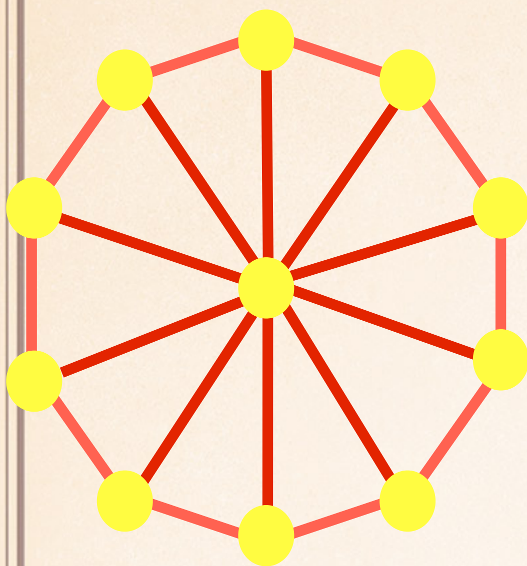
$$\zeta_n(s) = \sum_{k=1}^{n} \sin(\pi k/n)^{-s}$$



$A = i\partial/\partial x$

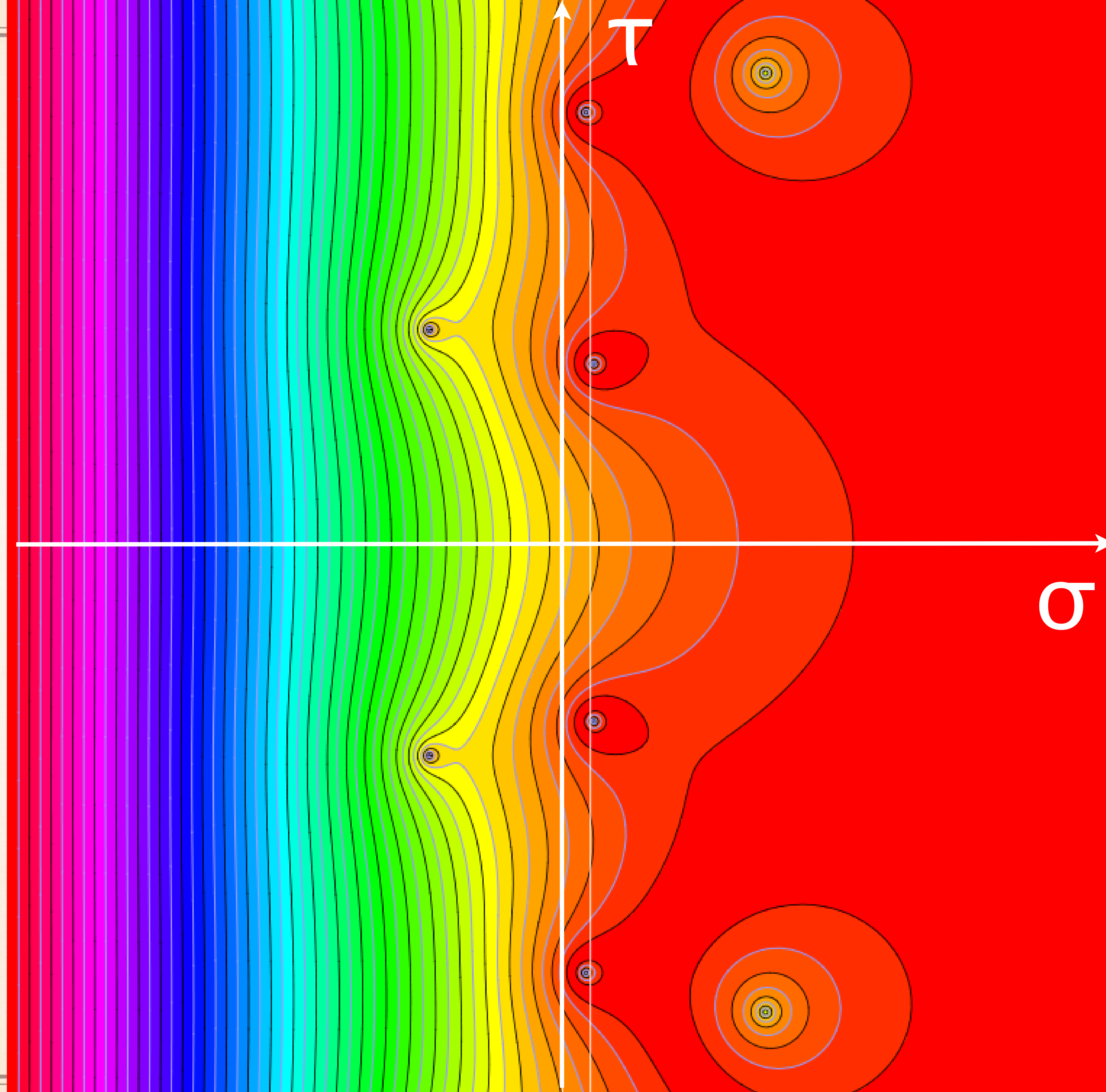
$$\zeta(s) = \sum_{k>0} k^{-s}$$





W_{10}

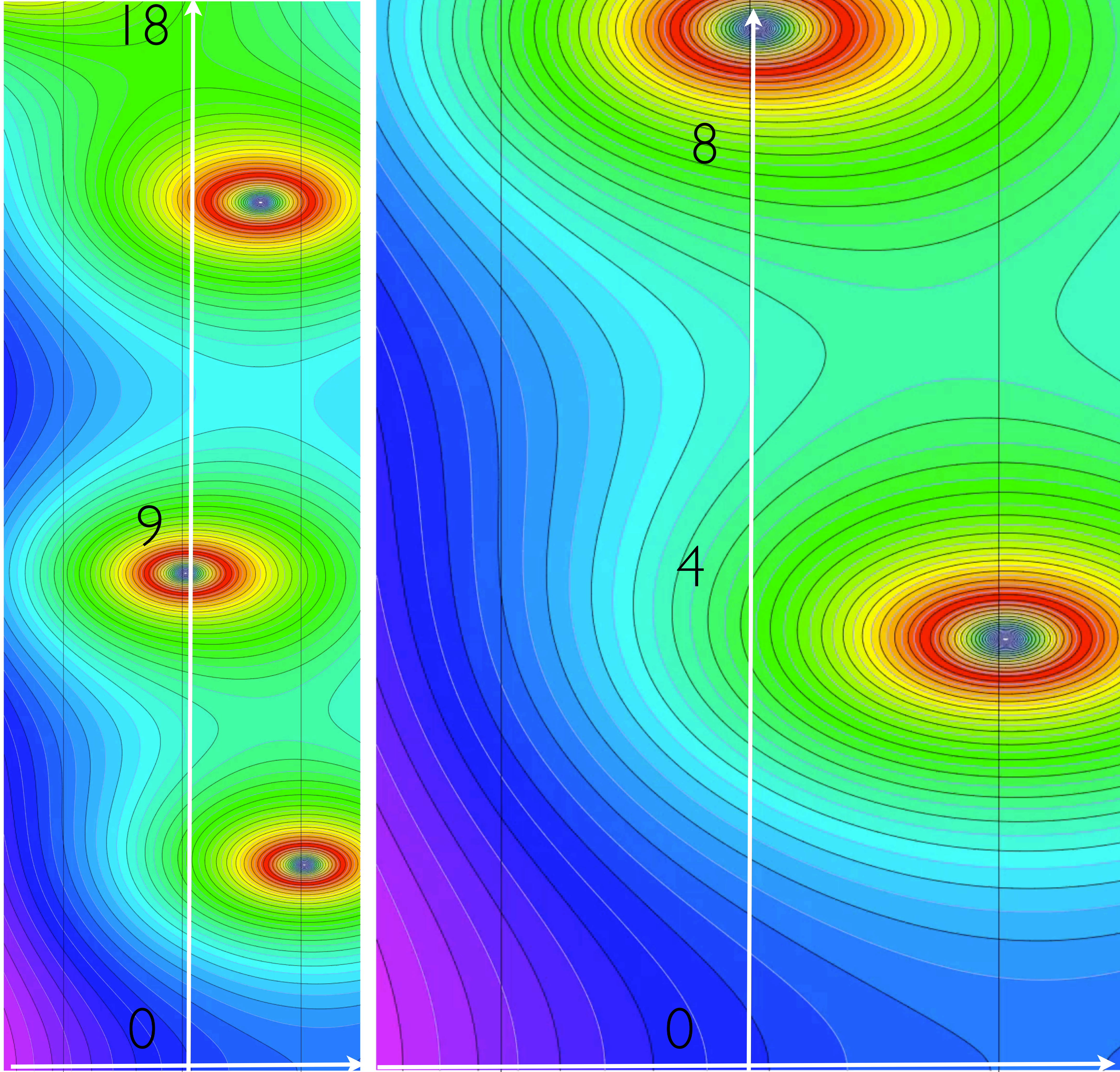
A = Dirac operator of the graph

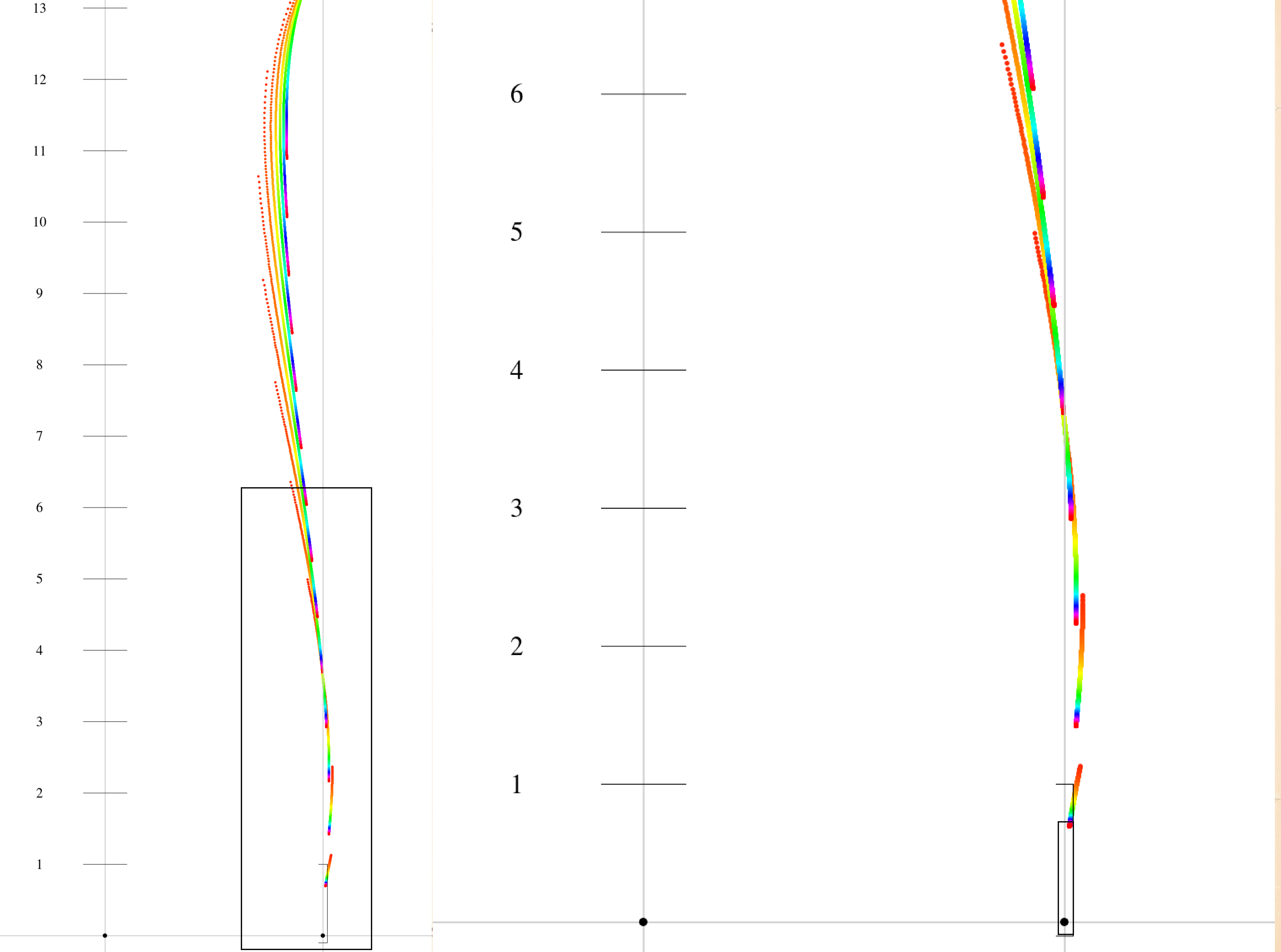


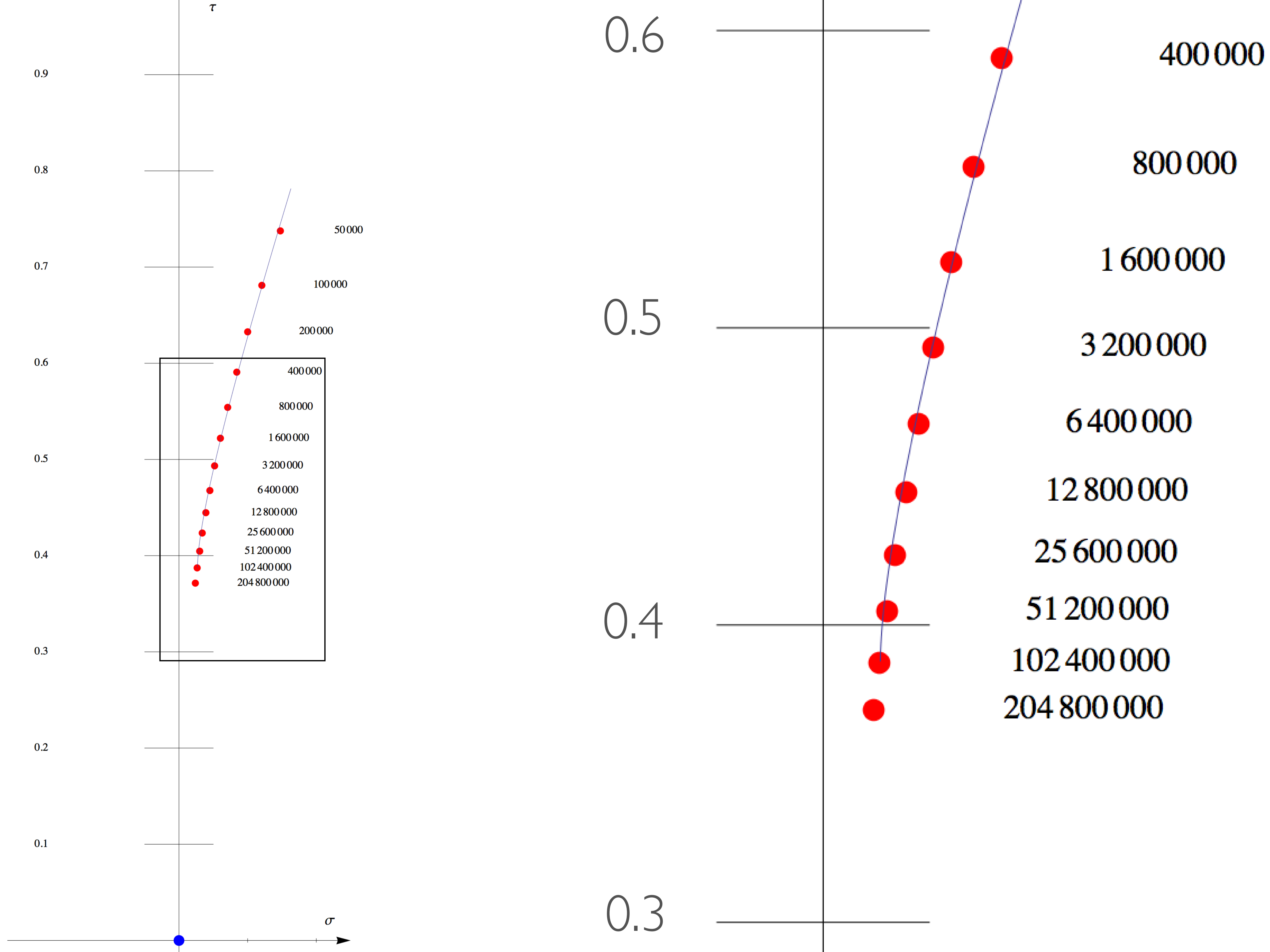
$$
\begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{bmatrix}
$$

$\zeta(C_n)$

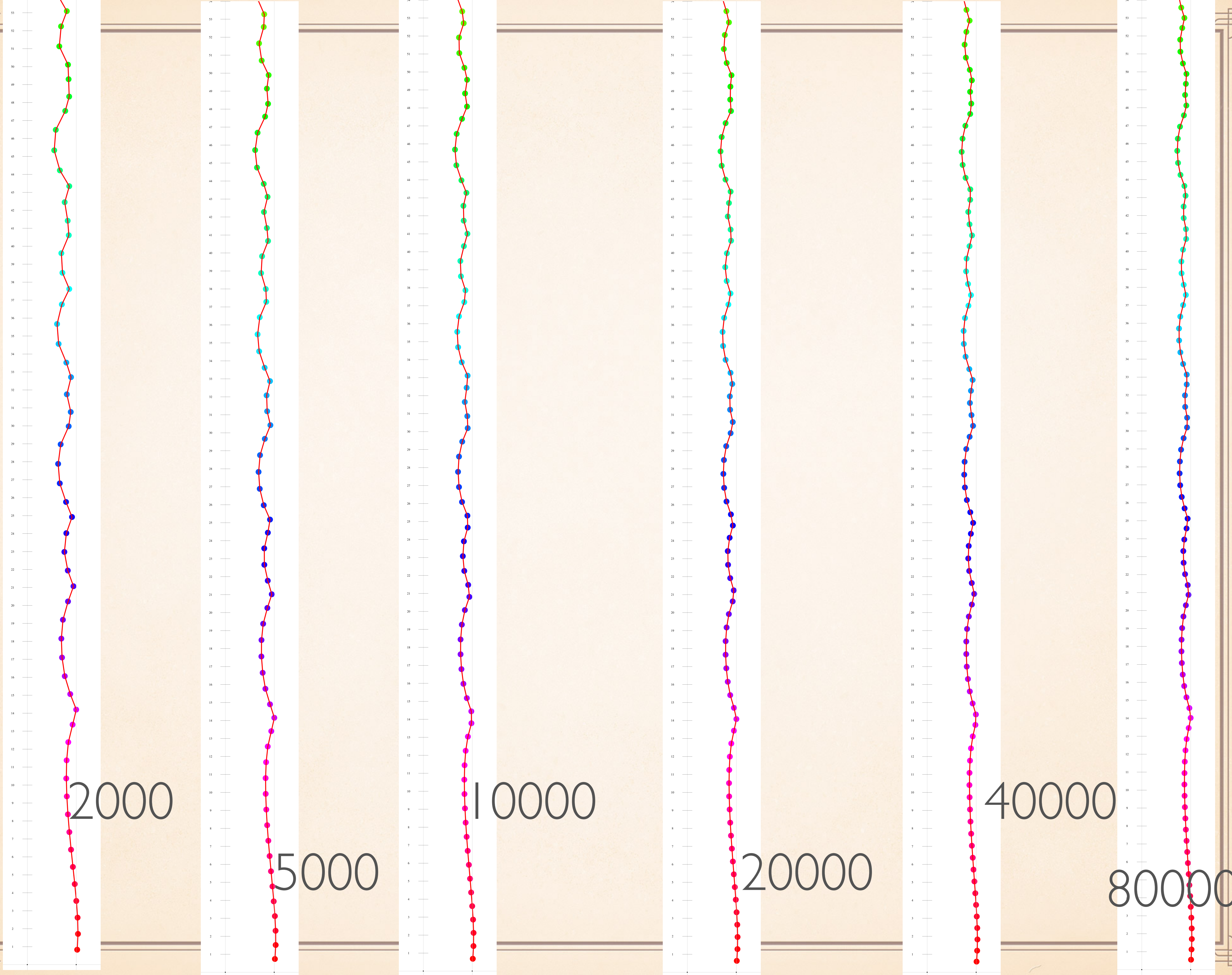
n=10 to 5000

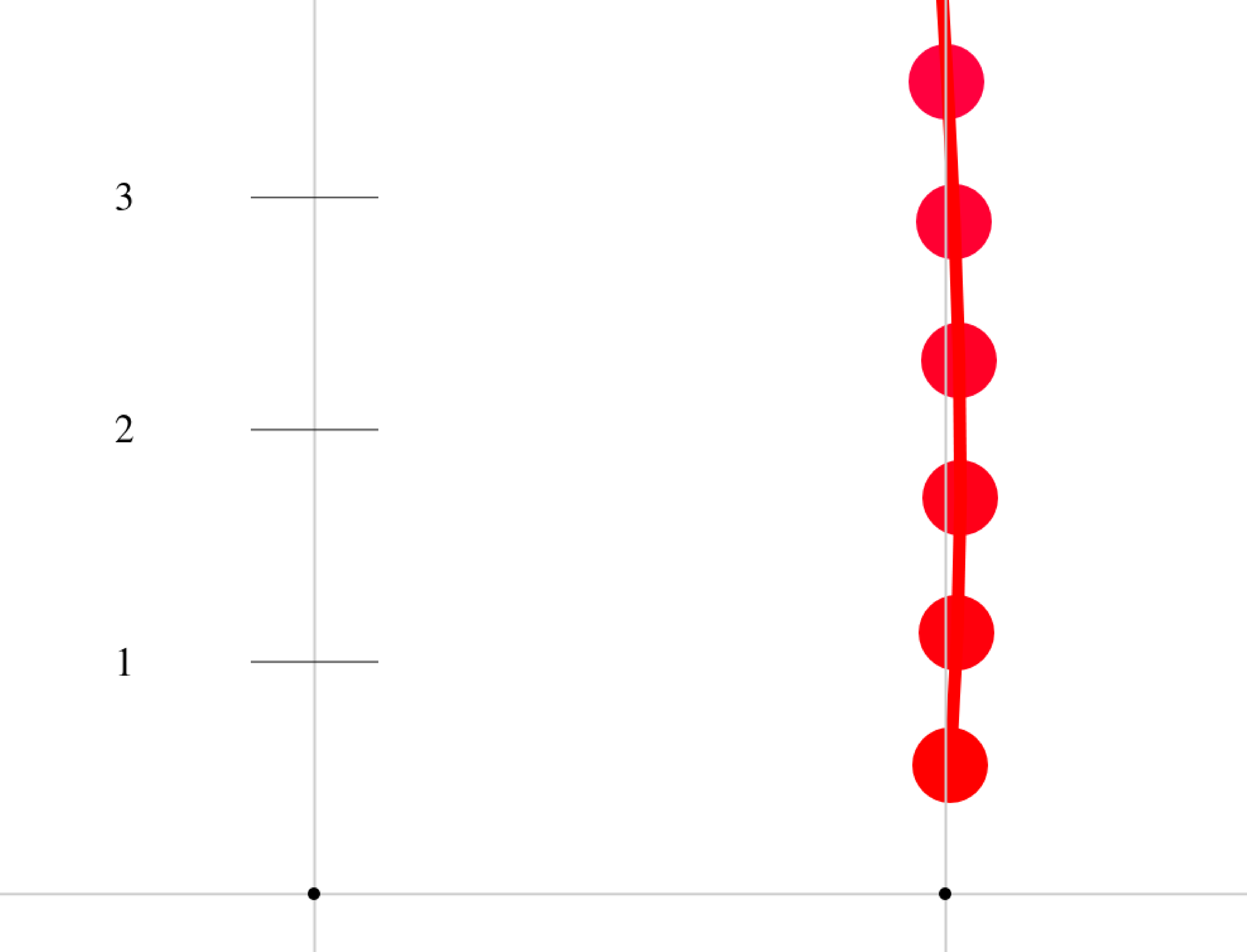



13
12
11
10
9
8
7
6
5
4
3
2
1
6
5
4
3
2
1



τ
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
σ
50 000
100 000
200 000
400 000
800 000
1 600 000
3 200 000
6 400 000
12 800 000
25 600 000
51 200 000
102 400 000
204 800 000
0.6
0.5
0.4
0.3
400 000
800 000
1 600 000
3 200 000
6 400 000
12 800 000
25 600 000
51 200 000
102 400 000
204 800 000


2000
5000
10000
20000
40000
80000


3
2
1

MANY PROBLEMS

How is finite Birkhoff sum growth related with infinite sum for α in upper half plane?

Is there a teachable proof of KAM: $q(x+\alpha)-2q(x)+q(x-\alpha)=c\sin(q(x))$

Relations of roots of Baby Riemann with actual Riemann zeta?

THE END

Knill, Lesieutre, Complex Analysis and Operator Theory, 2012

Knill, Tangerman, Nonlinearity, 2011

Knill, Golden Graph
http://arxiv.org/abs/1412.6985