SOME ILLUSTRATIONS TO VECTOR BUNDLES

September 14, 2009

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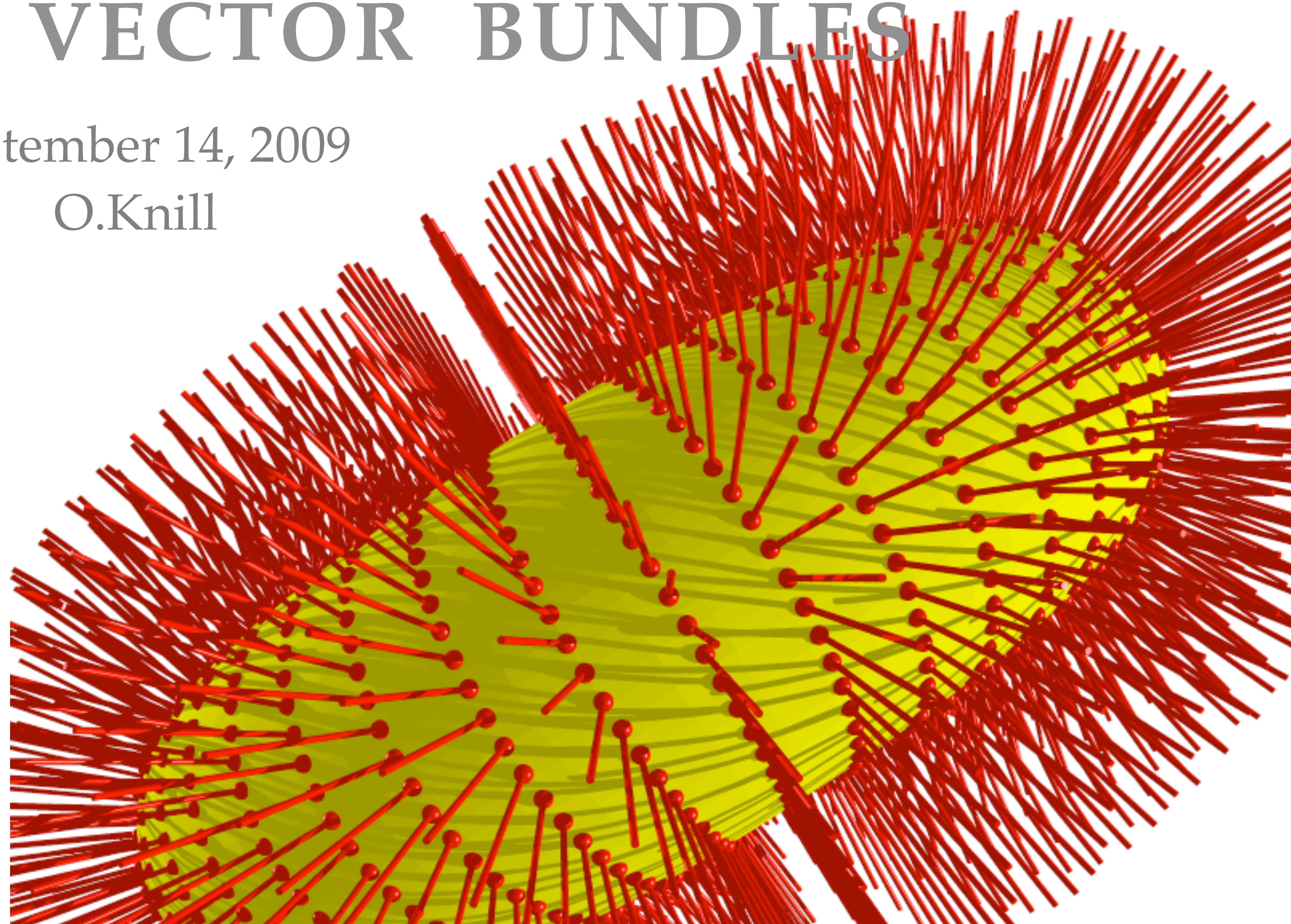
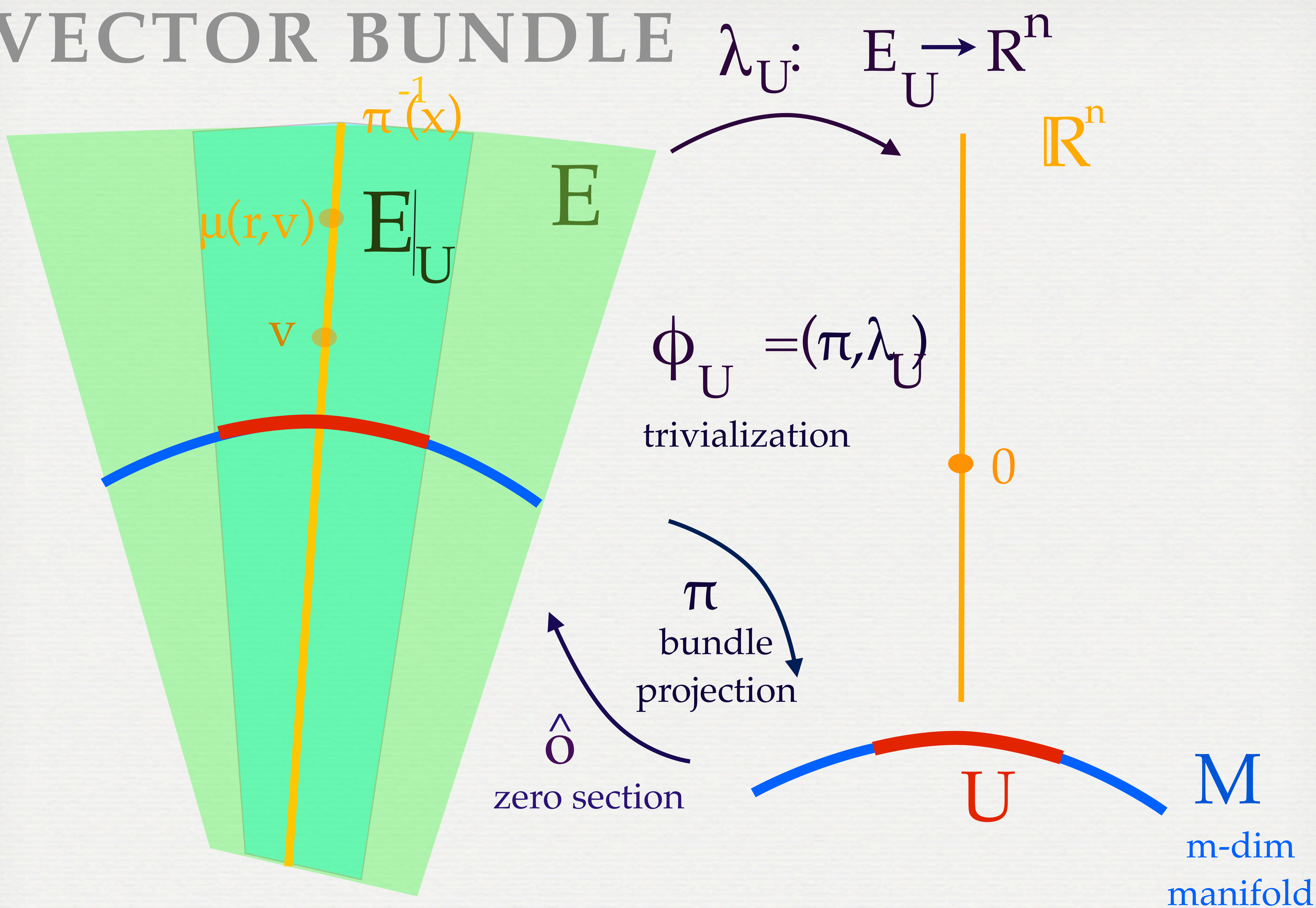


VECTOR BUNDLE



some notation from Cliff Taubes notes for vector bundles

Propo: The smooth m+n dimensional manifold E has a vector space structure on each fibre $\pi^{-1}(p)$

Proof:

1) on each fiber, there is a vector space structure inherited from diffeo λ_U. Could depend on U however.

2) the R action μ on each fibre is compatible with the scaling multiplication on R^n. The zero is $\hat{o}(p)$.

3) linearity of any map ψ on R^n follows from $t\,\psi(v)= \psi(tv)$. (linear algebra lemma)

4) apply 3) to $\psi = \lambda_{U'} \circ \lambda_U^{-1}$ to see that the vector space structure does not depend on the chart of the atlas.

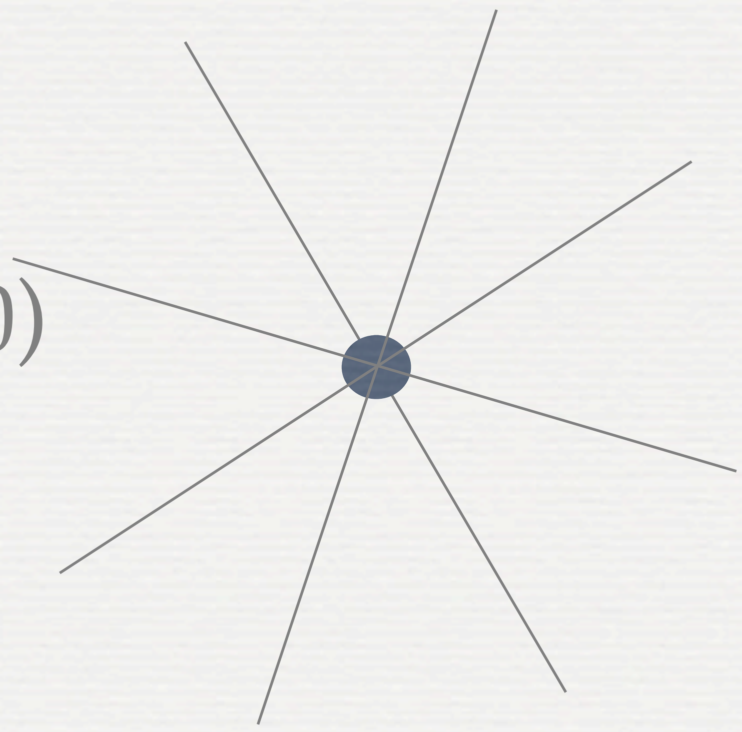
Lemma: A smooth map ψ on R^n satisfying
$t\,\psi(v) = \psi(tv)$ for all t,v is linear.

Proof: differentiate $t\,\psi(v) = \psi(tv)$

to see that

$$\psi(v) = \psi_*(tv) \cdot v$$

is independent of t. Call $A = \psi_*(0)$

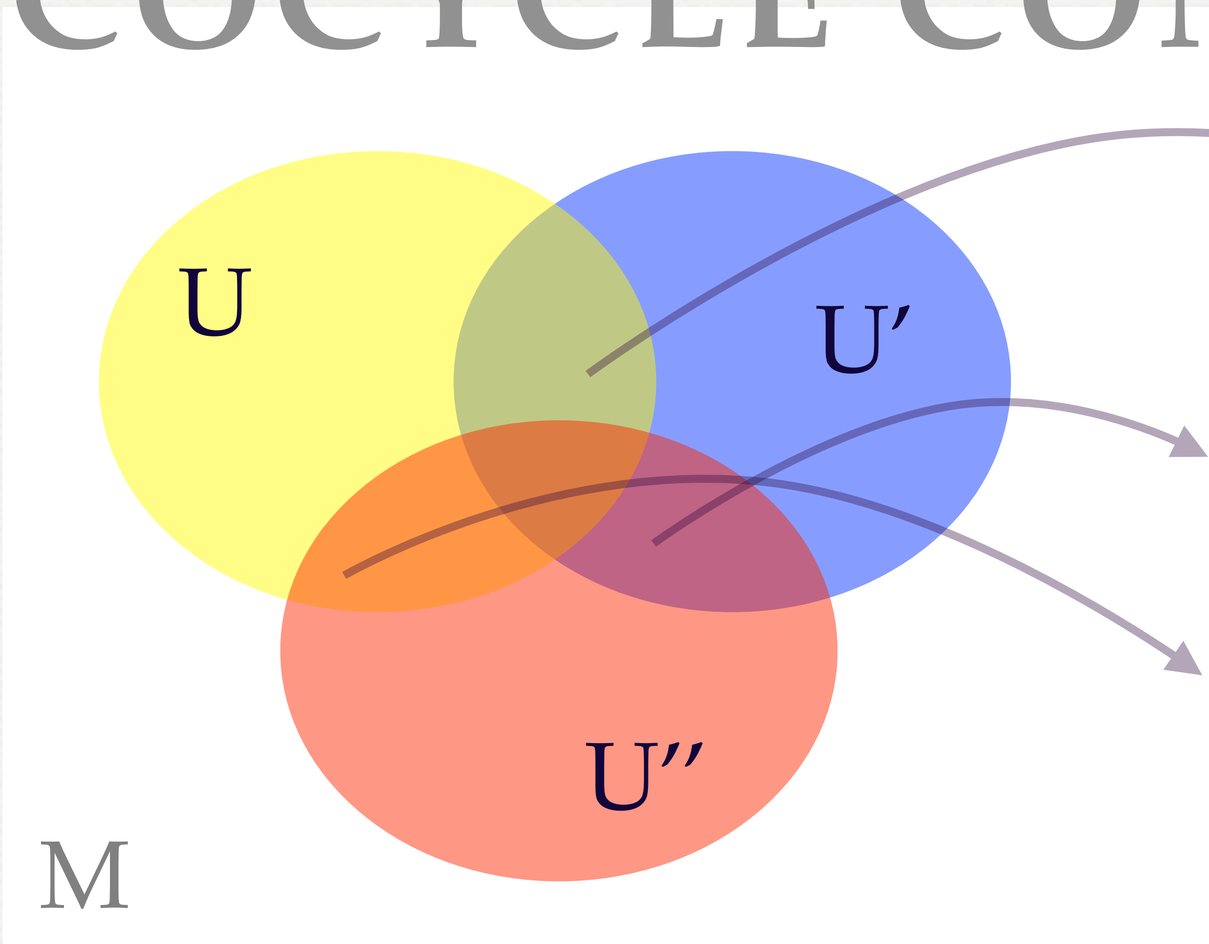


$$\psi(v) = Av$$

This is linear and A does not depend on v

Remark: not true between general vector spaces $V \longrightarrow W$ or in infinite dimensions $V \longrightarrow V$

COCYCLE CONSTRUCTION



$g_{U,U'} : U' \cap U \rightarrow Gl(n,R)$

$g_{U',U''} : U'' \cap U' \rightarrow Gl(n,R)$

$g_{U,U''} : U'' \cap U \rightarrow Gl(n,R)$

bundle transition functions

$$g_{U',U} \circ g_{U,U''} \circ g_{U'',U} = \iota$$

cocycle constraint

Define the bundle as an equivalence relation on the disjoint union of $U \times R^n$

$(p,v) \sim (p',v')$ if $p=p'$ and $v'=g_{U',U}(p)\, v$

Propo: The cocycle definition leads to the same vector bundle E as discussed before.

Proof: Assume the cocycle condition:

$$\pi(\overline{(p,v)}) = p$$

$$\mu(r, \overline{(p,v)}) = \overline{(p,rv)}$$

$$\lambda(\overline{(p,v)}) = v$$

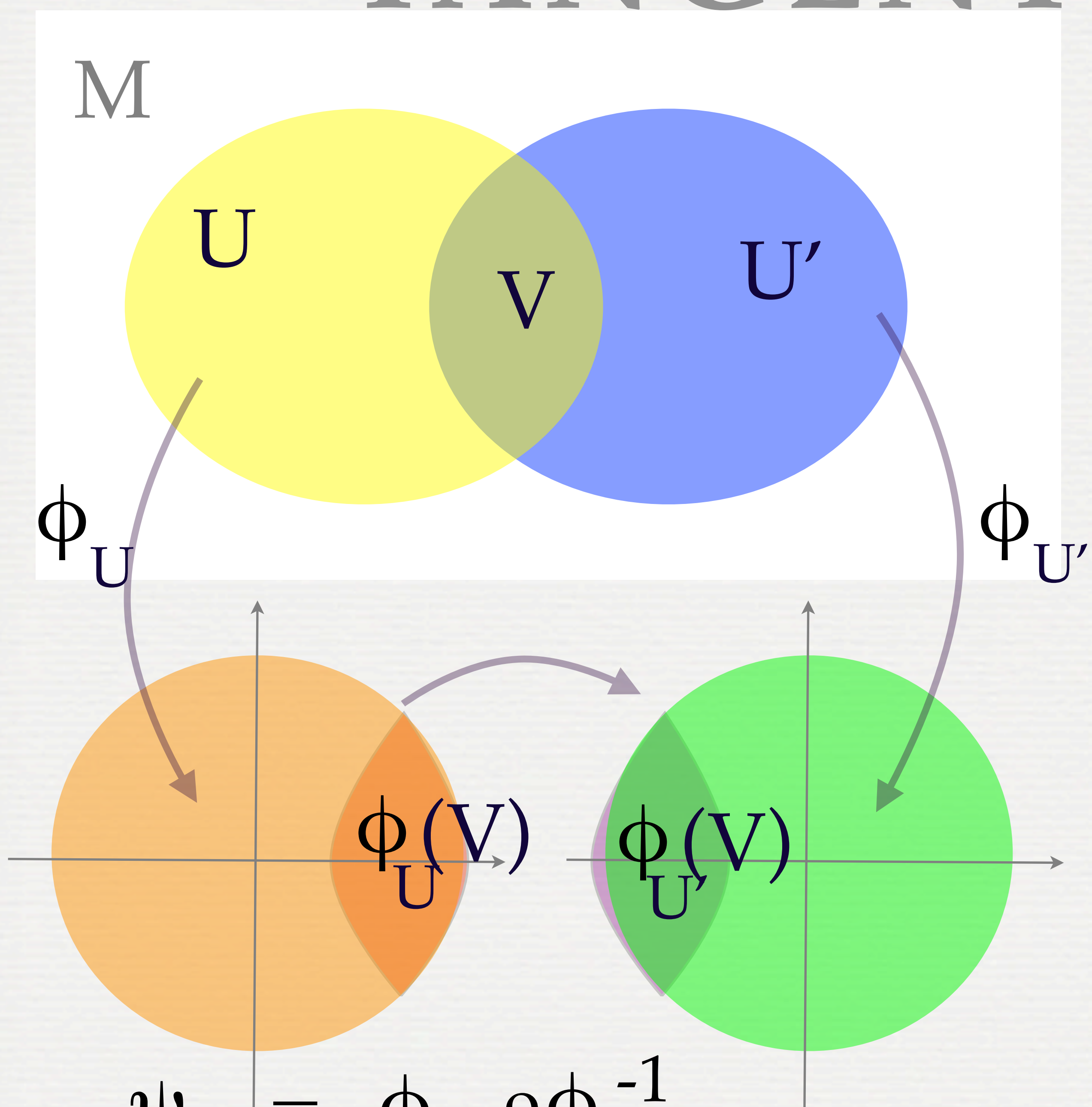
$E|_U$ form an atlas of E

with coordinate functions

$$(\phi_U, \lambda_U) : E|_U \longrightarrow R^m \times R^n$$

and transition functions: $g_{U,U'}(x)$

TANGENT BUNDLE



$$\psi_{U',U} = \phi_{U'} \circ \phi_U^{-1}$$

$$g_{U',U} : U' \cap U \rightarrow Gl(n,R)$$

defined by

$$g_{U', U} = \psi_{U',U'\,*}$$

$$g_{U',U} \circ g_{U,U''} \circ g_{U'',U} = \iota$$

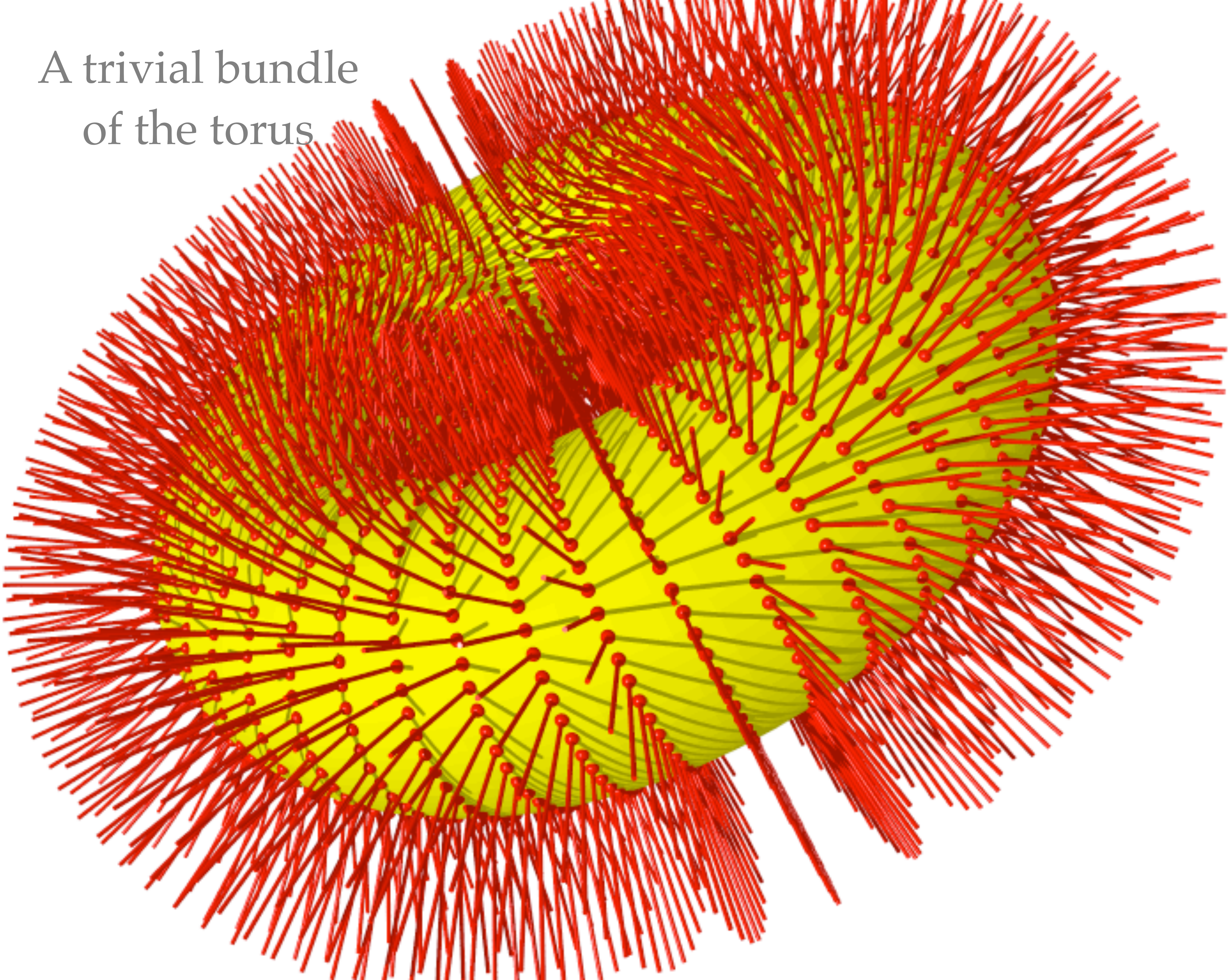
cocycle constraint
by chain rule

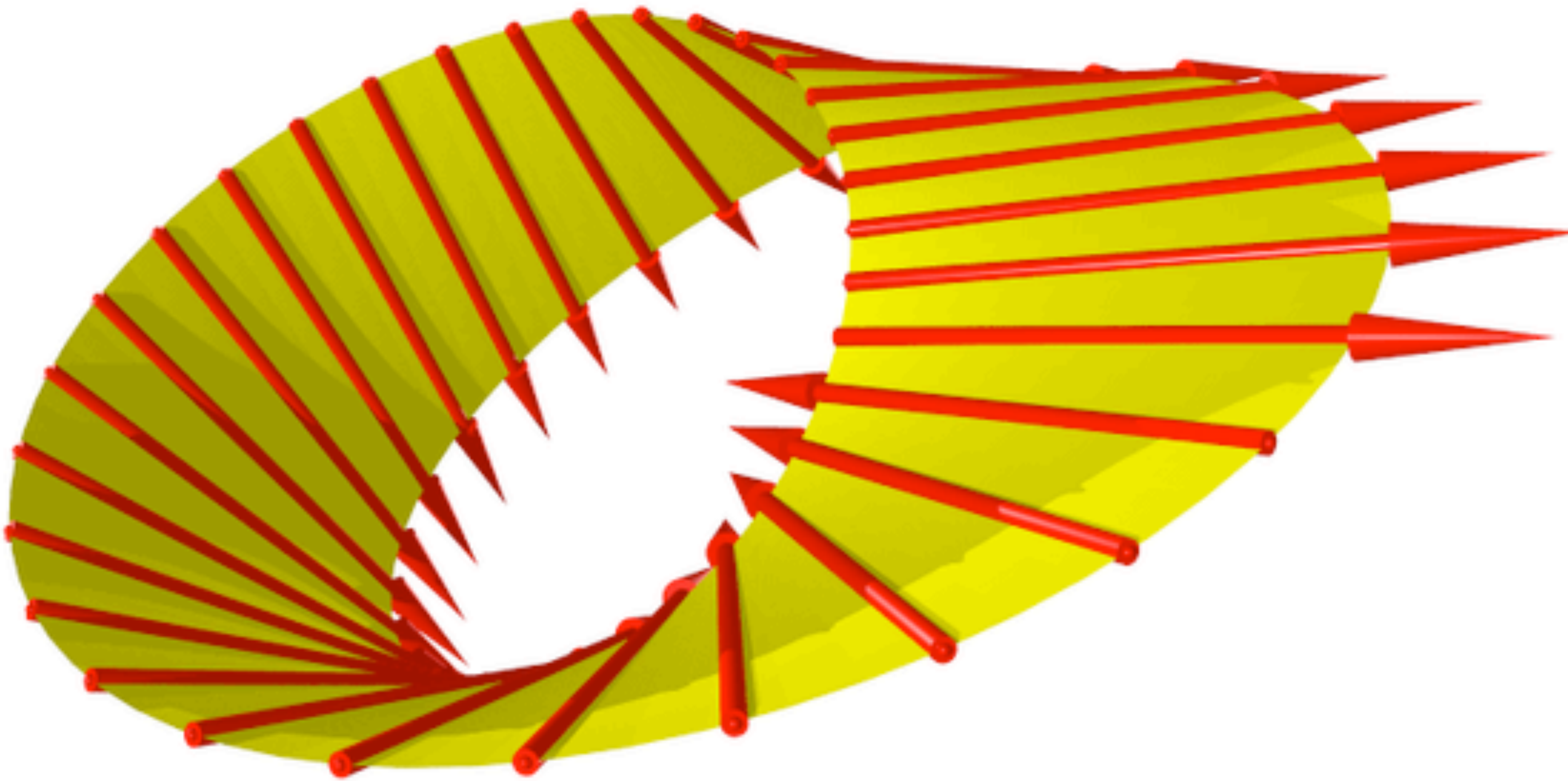
$$\psi_{U,U'} \circ \psi_{U',U''} \circ \psi_{U'',U} = \iota$$

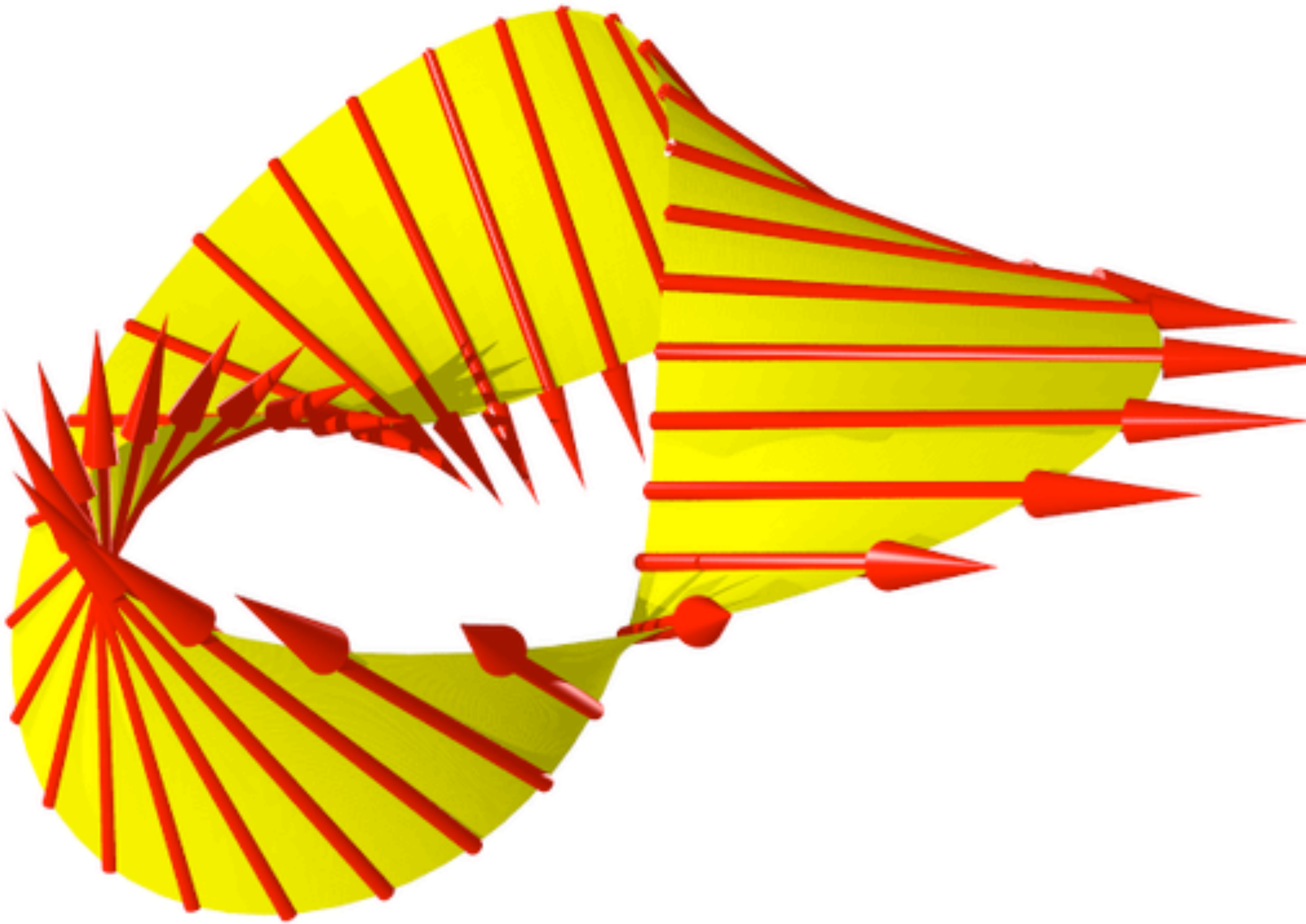
VECTOR BUNDLE EXAMPLES

- Trivial bundle
- Moebius bundle
- Tangent bundle
- Cotangent bundle
- Normal bundle
- Tautological bundle
- Matric cocycle bundle

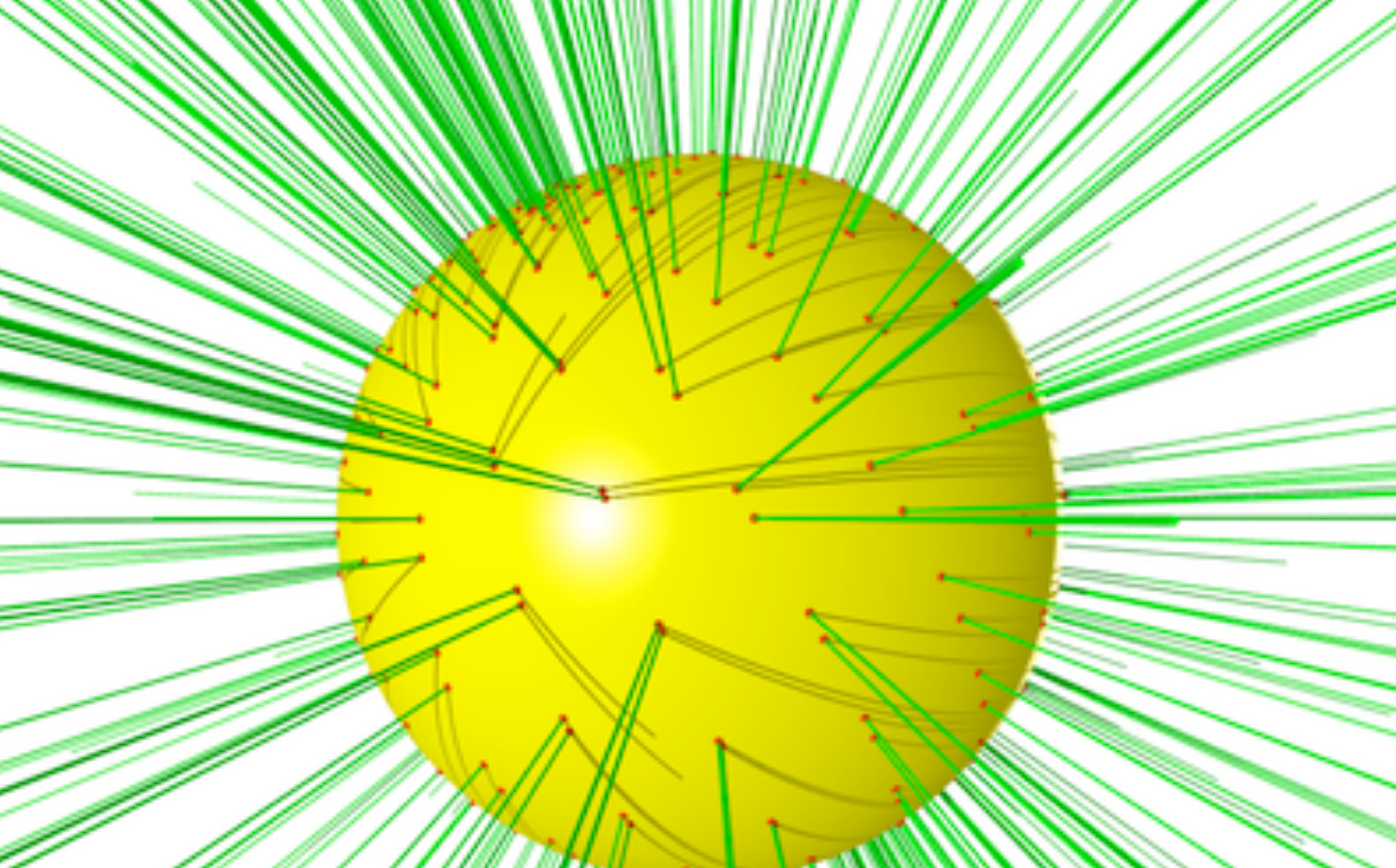
A trivial bundle of the torus



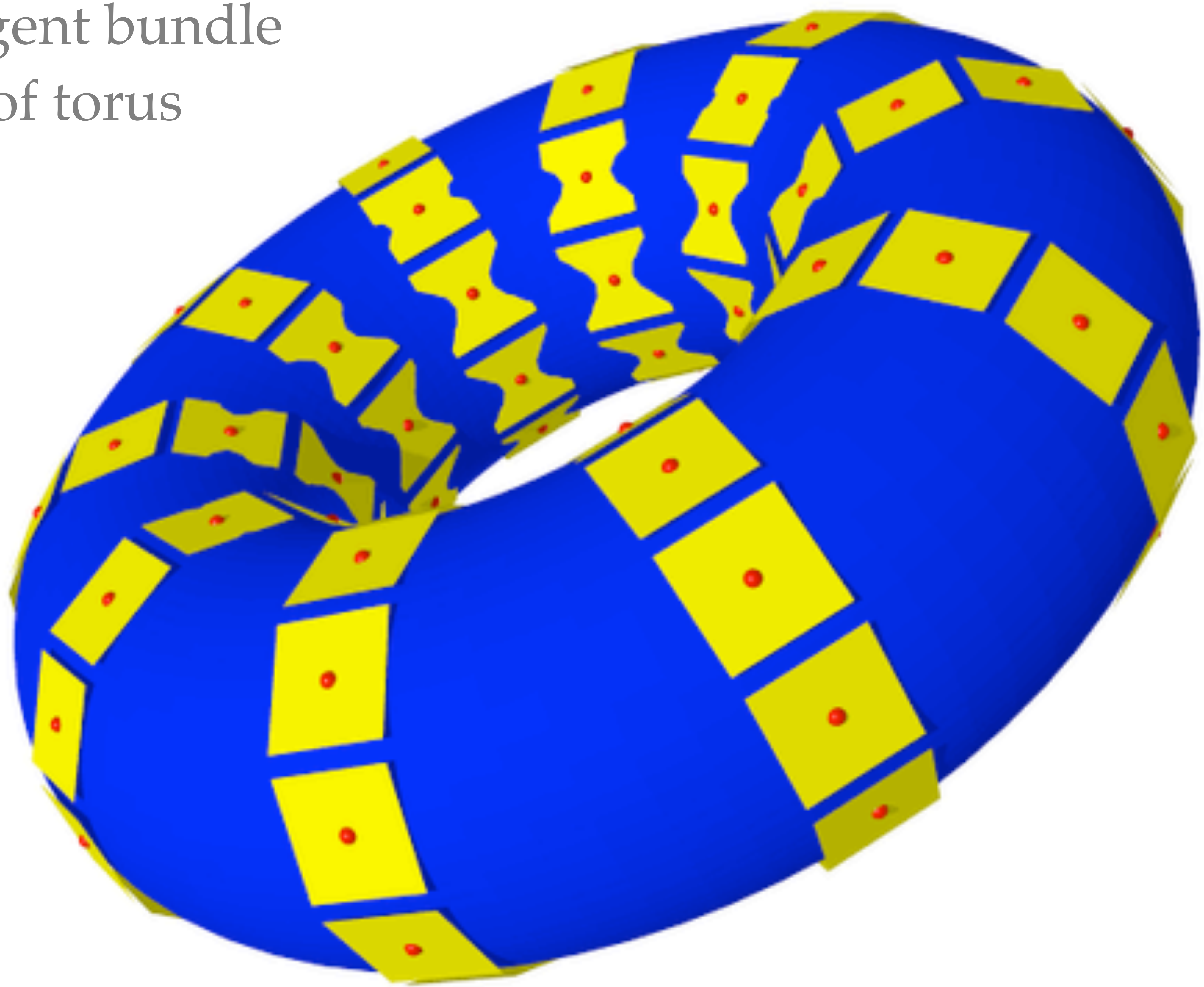




TAUTOLOGICAL BUNDLE

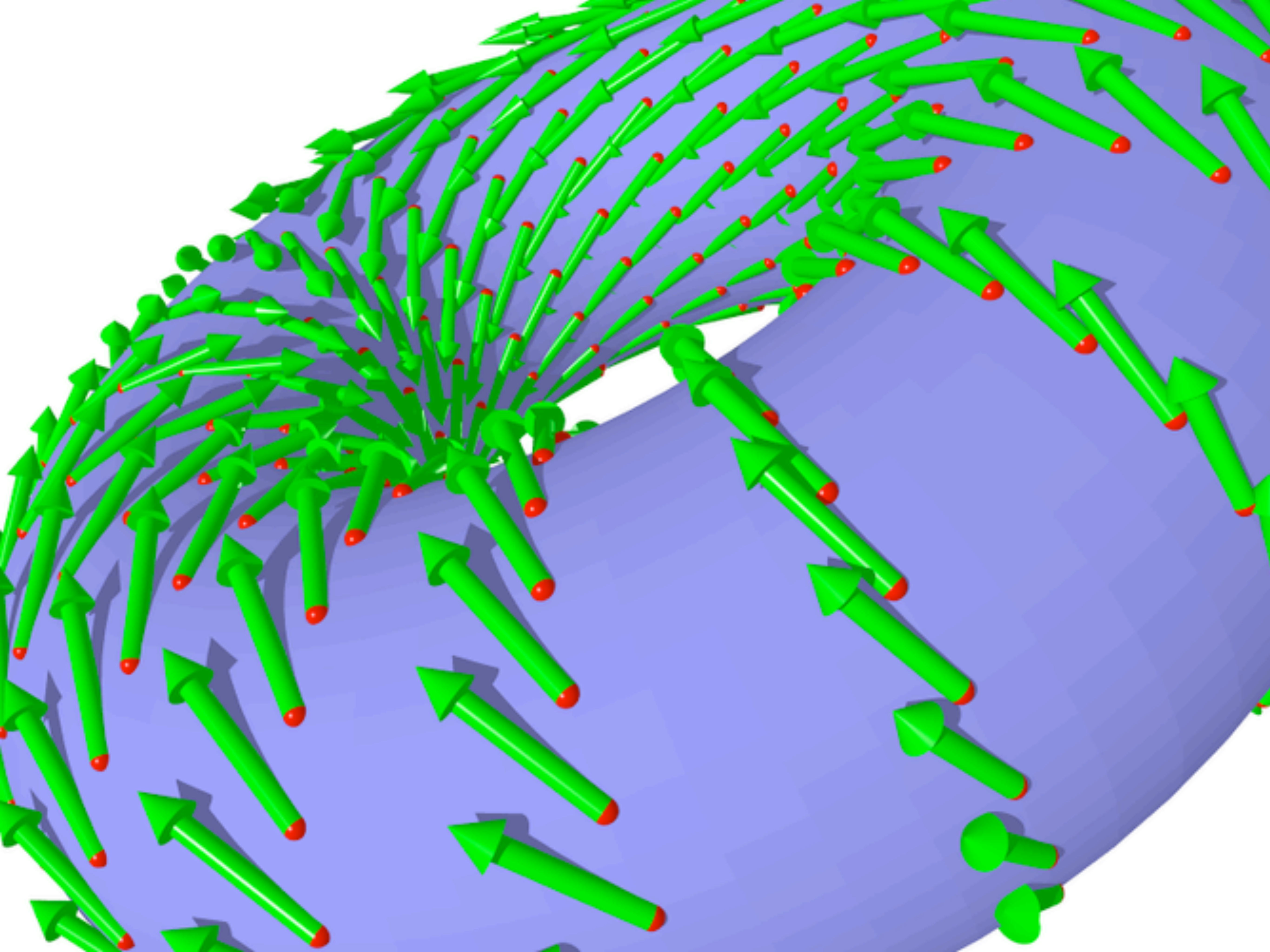


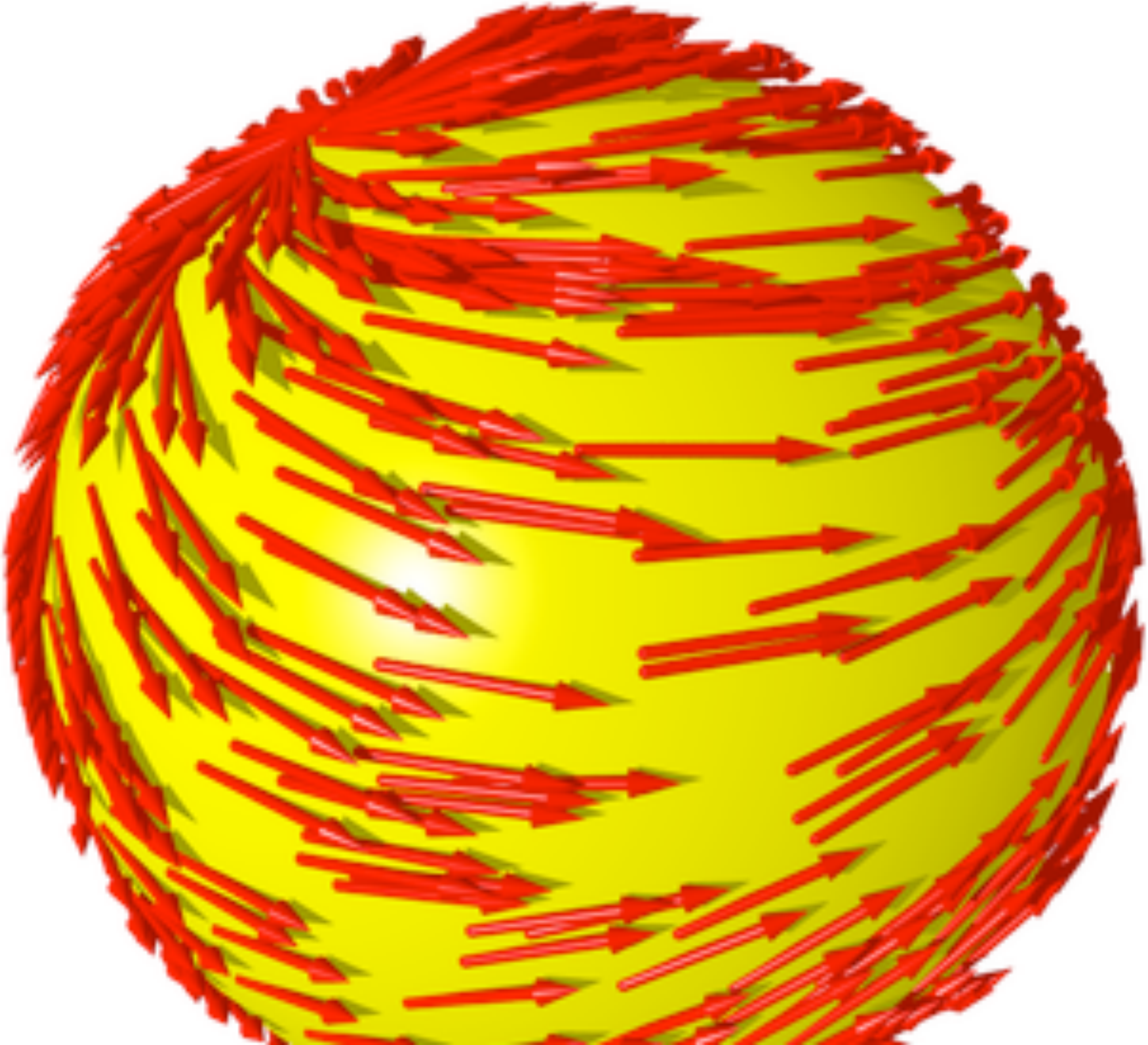
Tangent bundle
of torus



WHY VECTOR BUNDLES?

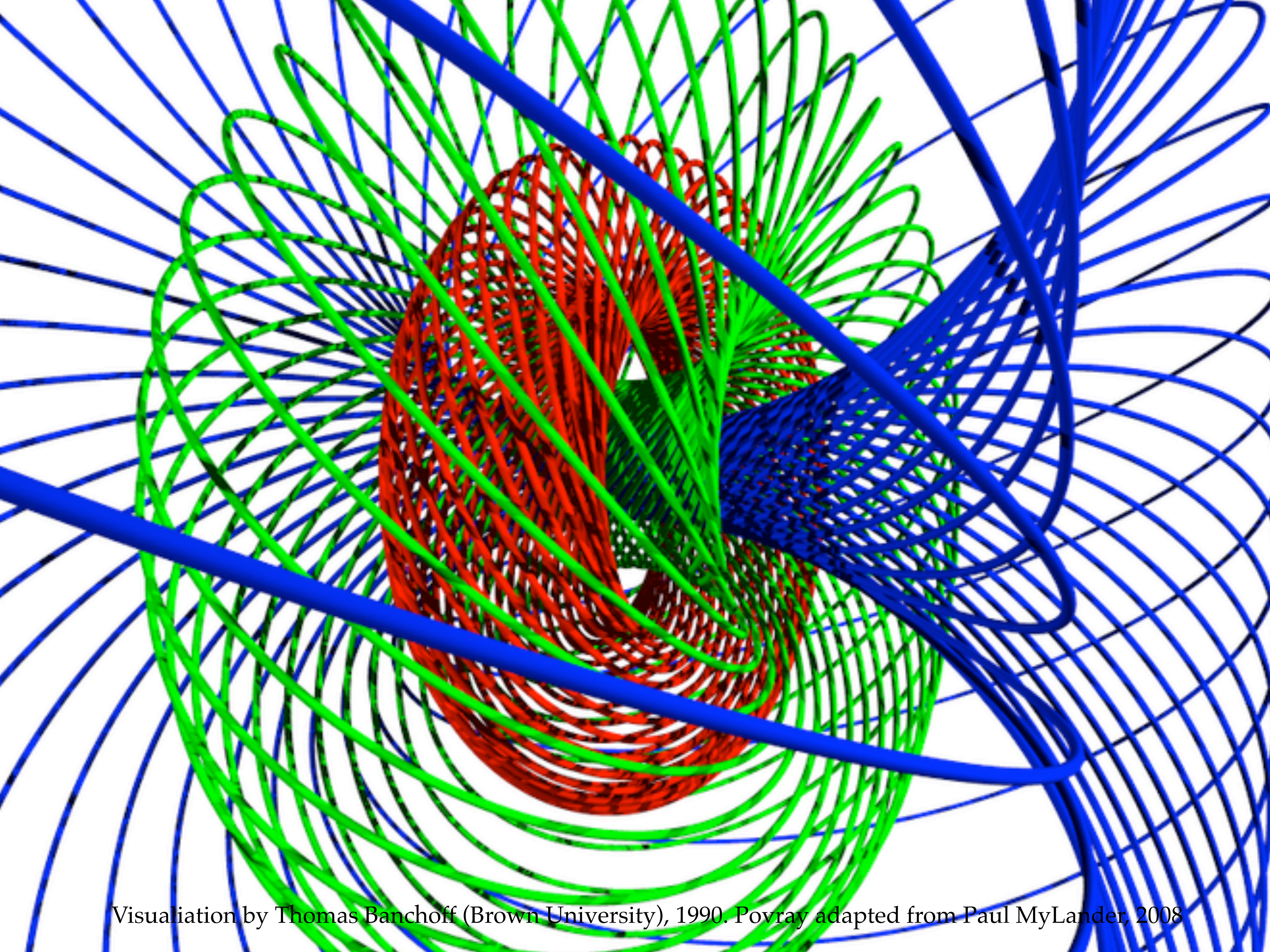
- “Bring Vector space back in picture” (Cliff)
- Construct manifolds from given manifolds
- Understand manifolds (i.e. find invariants by imposing more structure)
- Important in physics: i.e. vector fields, dynamics, geodesic flow, particle physics



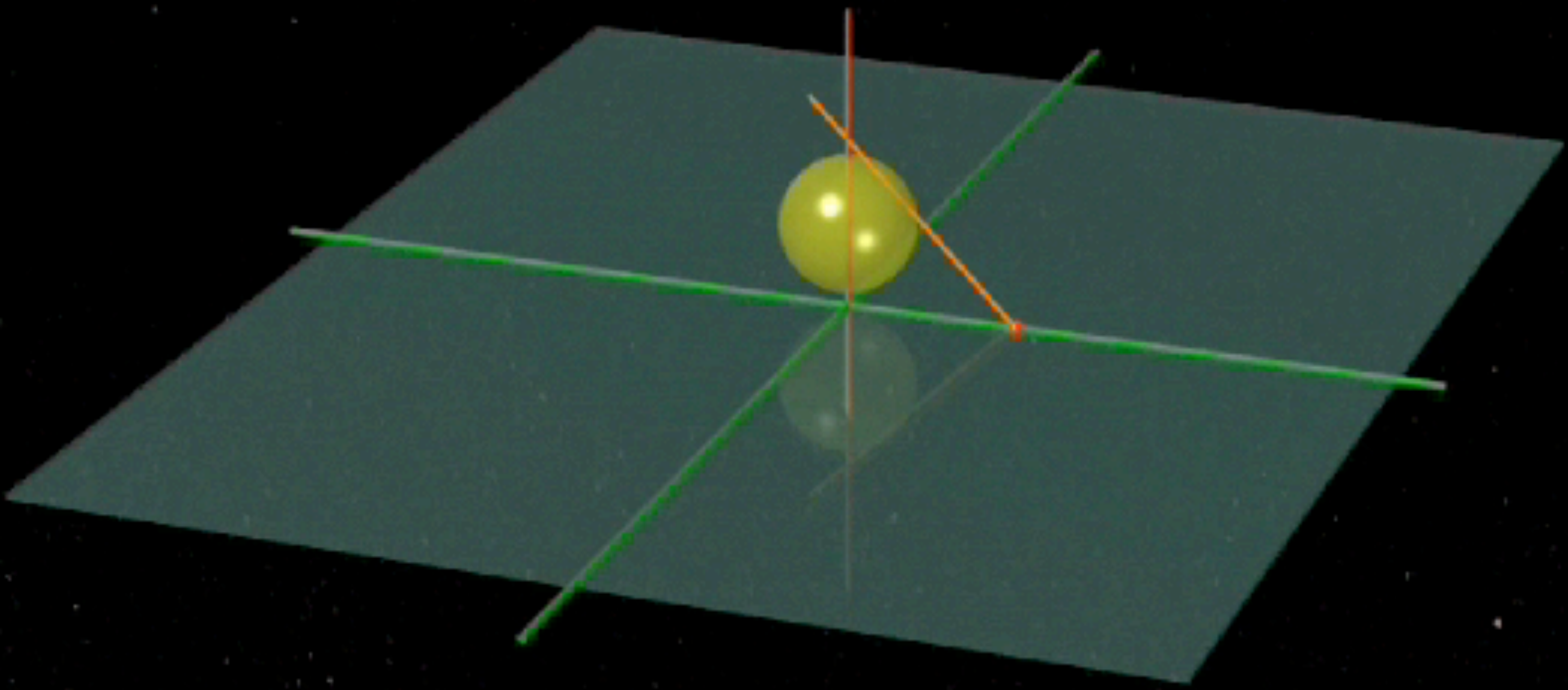


FIBER BUNDLE EXAMPLES

- Unit tangent bundle (needs a metric on each fibre)
- Hopf fibration
- Billiard phase space
- Frame bundles
- Cocycles
- Klein bottle (circle bundle over circle)



Visualiation by Thomas Banchoff (Brown University), 1990. Povray adapted from Paul MyLander, 2008



- Dimensions-math.org