SINGLE VARIABLE CALCULUS

SOME NOTES OF OLIVER KNILL, HARVARD COLLEGE FALL, 2020

Week 1: Modeling

Functions

1.1. A **function** is a rule which assigns to a number x a new number $f(x)$. For example, $f(x) = x^2$ assigns to the number 7 the result 49. The **graph** of a function is a drawing of the curve we get when looking at all points $(x, f(x))$. If $f(x)$ is the height of a ball released at time $x = 0$ from a height of $f(0) = 5$ meters, then at time x, then we can see the graph as the actual trajectory of the ball. It is known to be a **parabola** $f(x) = 5-5x^2$. This parabola has a **root** at $x = 1$. This means that the ball hits the ground at $x = 0$. The process of assigning of assigning to a phenomenon an explicit function is called **modeling with functions**. We all know how a stone falls in principle but assume you throw it from high up. How does the height $h(x)$ depend on time exactly? What happens if you throw a stone out of an airplane? What happens after it hits the ground? The picture to the left in figure 1 was computed by actually let the computer simulate a stone falling in air. The parabolic parts are actually not parabola because friction flattens them out. After reaching a certain velocity, the stone falls with constant speed.

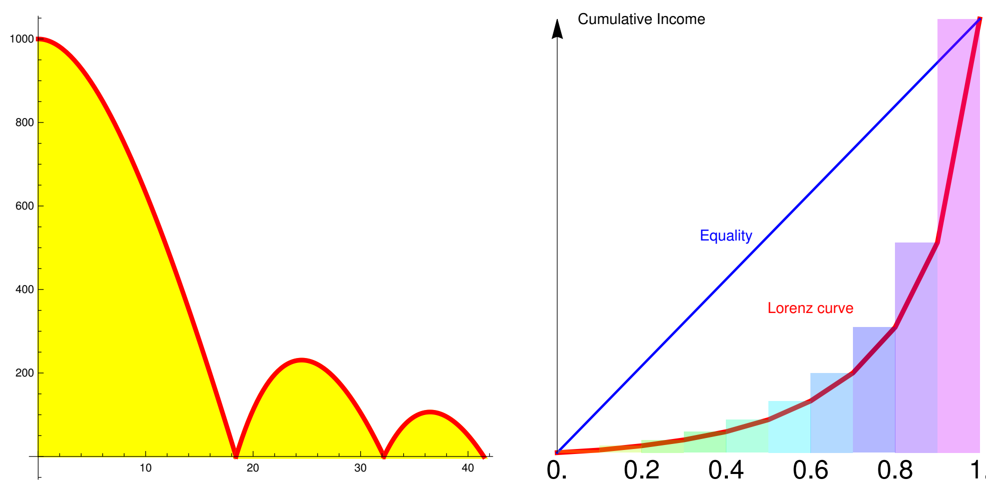


FIGURE 1. Bouncing of a stone and Lorenz curve.

1.2. We can assign **properties** to functions like **increasing**, **decreasing**, **constant**, **being concave up** or **being concave down**. We often associate the **graph** of the function with the function itself. Sometimes, functions are given by **data points**. In a project you look at the **Lorenz curve**. This curve is an interpolation of the data points which lists the income of each **decile** of the population. These are 10 data points $(f(0.0), f(0.1), \dots, f(0.9), f(1.0)$. The **Gini index** is a property which measures the deviation from equal income. It is twice the area between the graphs of $y = x$ and the **Lorenz curve**. One also looks at the **20:20 ratio** which is the area under the Lorentz curve above $(0.8, 1.0)$ divided by the area under the Lorentz curve above $(0.0, 0.2)$.

1.3. A function is called **increasing**, if $f(y) > f(x)$ whenever $y > x$. We often mean **strictly increasing** when we say increasing. The function $f(x) = 1 + 3x$ therefore is increasing, but the constant $f(x) = 1$ would not be considered increasing. Also the function $f(x) = x^3$ is an example of an increasing function. An other important function is the **exponential function** $f(x) = e^x$. The function $f(x) = \sin(x)$ is the **sine function** it is increasing if x is between 0 and $\pi/2$, but then decreasing from $\pi/2$ to π. The sine function periodically changes from increasing to decreasing.

1.4. A function is called **concave up** in an interval if the graph is "bent upwards". Technically this means that the function value at the midpoint $f((x+y)/2)$ is smaller than $(f(x) + f(y))$ for any two points. The function is called **concave down** in an interval, if $-f$ is concave up. This means that that midpoint $f((x + y)/2)$ is larger than the average of the end points. For concave up functions, the connection line between $(x, f(x))$ $(y, f(y))$ is always above the graph, for concave down functions, the connection line is always below the graph. The function $f(x) = e^x$ is an example of a concave up function. The function of the falling stone $f(x) = 5 - 5x^2$ is an example of a concave down function.

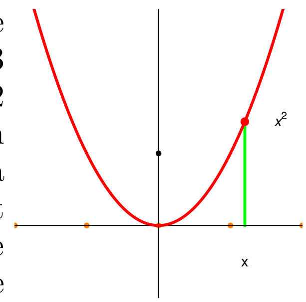
1.5. In nature, we often deal with functions. Sometimes the variable x is **position** like the shape of a hanging rope or the skyline of a city. Sometimes the variable x is **time**. The value of $f(x)$ can also be anything. It can be a position like the height, it can also be a temperature. Assume you have a glass of ice water of 0 degree Celsius and you let it be, how would the temperature of the water behave in time Assume you have a cup of boiling tee of 100 degree Celsius and you let it rest, how would the temperature of the tea behave in time? Assume you throw a piece of wood into water. How does the position of the wood change in time? What about a flat pebble you throw onto a lake.

1.6. Very important type of functions are **linear functions** like $f(x) = 3x + 2$ and especially **constant functions** like $f(x) = 5$, **quadratic functions** like $x^2 + 5x + 6$ or **exponential functions** $f(x) = 2e^{3x}$. Functions can be **added**, **multiplied**, **inverted** or **composed**. For example, we can form $f(x) = x^2 + e^x$ or $(3x + 2)^2 + x$. Not every function can be inverted. The **constant function** $f(x) = 1$ for example has no inverse. We are not able to determine x from the value of f. Here is an example to invert a function. For example, if we know the volume $V(r) = 4\pi r^3/3$ of a sphere, then $r(v) = (3V/(4\pi))^{1/3}$ is the **radius** of the sphere as a function of the volume.

1.7. A function is called **1:1** or **bijective**, if for every y value, there exists exactly one x value such that $f(x) = y$. One uses the notation $x = f^{-1}(x)$. The exponential function $f(x) = e^x$ has the inverse $\log(x) = \ln(x)$, which is called the **logarithm** of x. The function $f(x) = x^2$ has the inverse $\sqrt{x}$ if one assumes x to be non-negative. The function f is not 1:1 on the entire real axes because $f(1) = f(-1)$ both give the same value. But the square function is 1:1 when restricted to the positive real axes.

Polynomial functions

The **constant function** $f(x) = 4$ is a horizontal line. A **linear function** $f(x) = 3x + 5$ draws a line hitting the y axis at $b = 5$ and x axes at $-5/3$. The number $m = 3$ is the **slope**. The **quadratic function** $f(x) = x^2 + 2$ has as a graph a **parabola**. A function which is a sum of **powers** like $f(x) = 3x^5 - 2x^3 + 2x - 4$ is called a **polynomial**. The function $f(x) = 10 - 5x^2$ is the height of a stone thrown from height 10. If one gives the stone the initial forward velocity 1, then the graph of f is the actual trajectory of the stone.

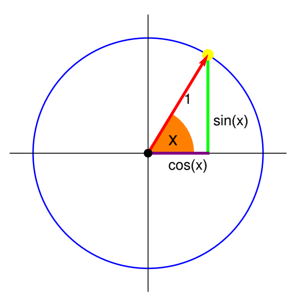


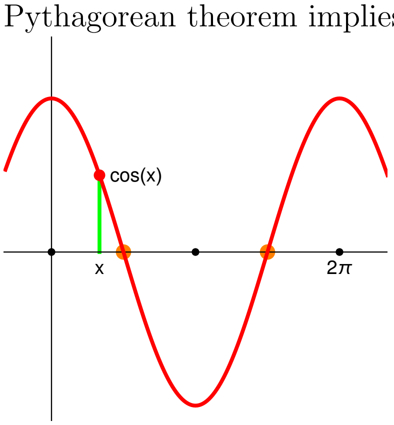
Trigonometric functions

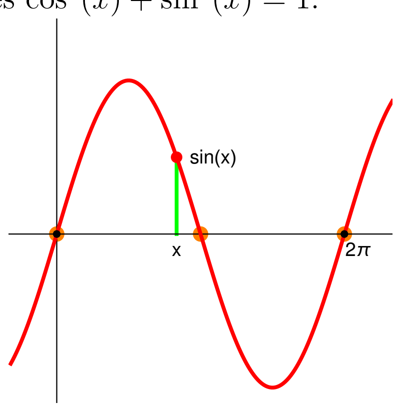
Trigonometric functions model things which change periodically like **daylight time** as a function of the day in the year.

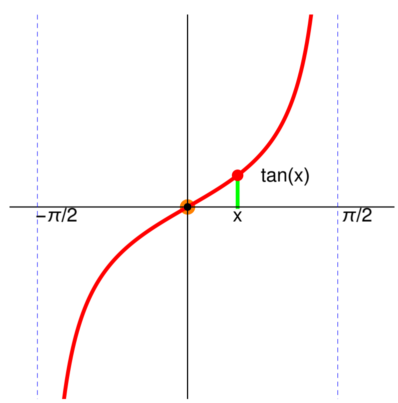
$\sin(x) =$	Opposite/Hypothenuse (SOH)
$\cos(x) =$	Adjacent/Hypothenuse (CAH)
$\tan(x) =$	Opposite/Adjadcent (TOA)

One also defines $\sec(x) = 1/\cos(x), \csc(x) = 1/\sin(x)$. The Pythagorean theorem implies $\cos^2(x) + \sin^2(x) = 1$.



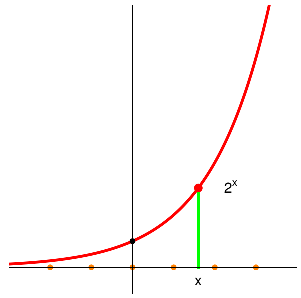






The exponential function

The exponential function e^{ax} models **exponential growth** if $a > 0$ or **exponential decay** if $a < 0$. The number of bacteria on an agar plate grows exponentially at first, the temperature of a coffee cup cools exponentially in time. One can write all exponential functions using the exponential function $a^x = e^{\log(a)x}$ where $\log(x) = \ln(x)$ is the **natural log**, the inverse of exp. One can take any positive base b and define the exponential b^x like 10^x. One has always $b^0 = 1$ $b^{x+y} = b^x \cdot b^y$. For example $10^6 = 10^2 \cdot 10^4$.



Modeling

1.8. Making a **mathematical model** of a phenomenon or process means to find a mathematical description or law which captures the situation. It can mean to find a function which describes some data points.

1.9. Here are example of situations where functions appear:
Volumes of a cube as a function of width
Bank account value as a function of time
Atmospheric Pressure as a function of height
Temperature as a function of time
The number of people living within a center of a town
The income as a function of the percentage of income rank
Time of a sprinter to to as a function of distance
tax rate as a function of income
probability that an error is within an interval
the height of a bungee jumper in time.
The radius of a sphere as a function of the volume
The gravitational force of the moon as a function of the distance

1.10. In order to be able to model, one needs a good **library of functions**. Here are different classes of functions. We will try to make a bit order during class and look at examples. The lesson problems done before class prepare a bit.

Constant	$y = 1$
Linear	$y = x$
Quadratic	$y = x^2$
Cubic	$y = x^3$
Exponential	$y = e^x$
Logarithmic	$y = \log(x)$
Rational	$1/x$
Trigonometric	$\sin(x)$
Step function	$\text{floor}(x)$
Absolute	$\|x\|$
Normal	e^{-x^2}

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