Week 3: Limits and continuity

The derivative as a function

1.1. We so far have mostly computed the derivative $f'(x)$ at a specific point to see how the function changes at x. But we can also look $x \to f'(x)$ as a new function. This is a bit of a different perspective. We have already computed some of these functions. Like for $f(x) = x^3$ where $f'(x) = 3x^2$ or for $f(x) = 1/x$, where $f'(x) = -1/x^2$ or for $f(x) = \sqrt{x}$ where $f'(x) = 1/(2\sqrt{x})$.

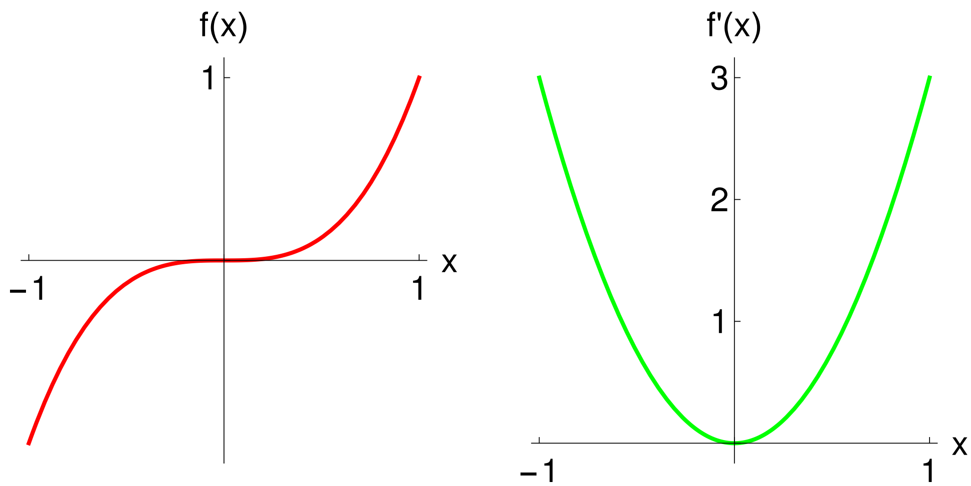

Figure 1. To the left we see the function $f(x) = x^3$. To the right we see its derivative function $f'(x) = 3x^2$. Indeed, $f(x+h) - f(x) = (x+h)^3 - x^3 = h[3x^2 + 3xh + h^2]$ so that after dividing by h and setting $h \to 0$ gives $f'(x) = 3x^2$.

1.2. We can now repeat the process and compute the derivative of the derivative. This is called the **second derivative**. It is familiar to us: if $f(t)$ is the position at time t, then $f'(t)$ is the **velocity** at time t and $f''(t)$ is the **acceleration** at time t.

1.3. Here is a riddle: can you see which of the three functions is the function f, which is the derivative f' and which is the second derivative f''?

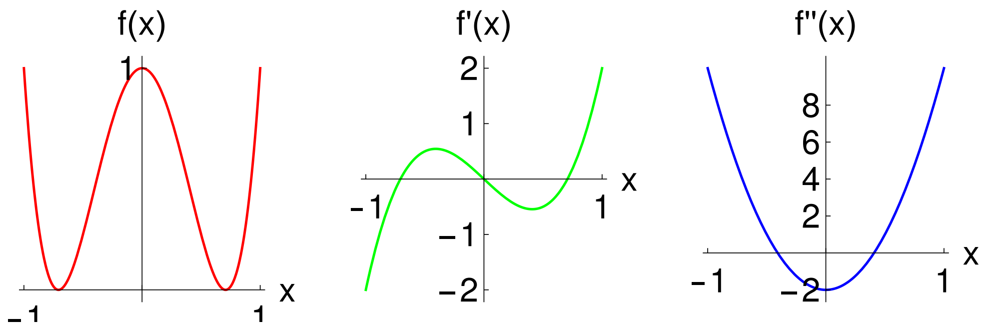

FIGURE 2. We see the function $f(x) = x^4 - x^2 + 1$, then its first derivative $4x^3 - 2x$ and finally the derivative of that derivative, which is $12x^2 - 2$. We will see that the second derivative is positive, where the first derivative increases and that the second derivative is negative where the first derivative decreases. Positive second derivative corresponds to concave up parts of the function f.

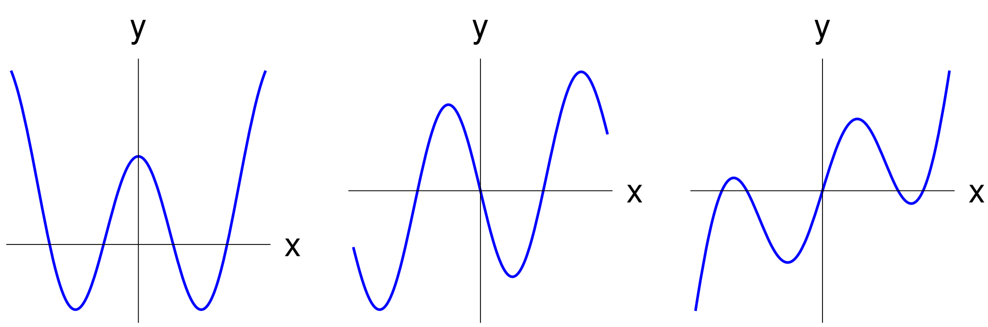

FIGURE 3. Three graphs.

LIMITS

1.4. Given a function $f(x)$ and a point a, one can look at the **limit** $\lim_{x\to a} f(x)$. There are two ways to approach a, we can come from the left and write $\lim_{x\to a^-} f(x)$ or from the right, where we write $\lim_{x\to a^+} f(x)$. In general both these limit exist and agree with the value $f(a)$. So why look at limits at all? One reason is that one can sometimes not see whether what the function value is at some point. An other reason is that we also want to see the limit when $x \to \infty$ or $x \to -\infty$. We therefore define $\lim_{x\to a} f(x) = b$ if however we approach a, we have $|f(x) - b| \to 0$.

1.5. Lets look at the example $f(x) = \frac{x^2-1}{x-1}$. This function is defined everywhere except at $x = 1$, where we divide 0 over 0. It is a task for limits. One of the ways to see the limit is to **heal** the function first (the function is broken at $x = 1$, so that we have to try to mend it). In this case we can factor out the top to get $x^2 - 1 = (x - 1)(x + 1)$, then divide out $x - 1$ if x is not 1 to get $f(x) = x + 1$ whenever x is not equal to 1. But now there is no problem: we can evaluate the function also at $x = 1$ and see the limit $f(1) = 2$.

1.6. An important example is $f(x) = \text{sinc}(x) = \sin(x)/x$. We can not just plug in $x = 0$ because we divide "0" over "0". But we have $\text{sinc}(0.01) = \sin(0.01)/0.01 = 0.999983$ and $\sin(0.001)/0.001$ is already to 7 digits in agreement with 1. The function $\text{sinc}(x) = \sin(x)/x$ is extremely important, like in signal processing. We come back to why the limit actually exists.

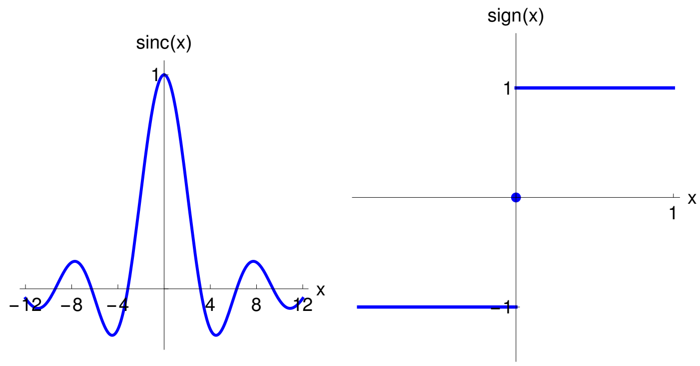

FIGURE 4. The sinc function and the sign function.

1.7. An other example is the function $f(x) = \text{sign}(x)$ which is defined to be 1 of $x > 0$ and 0 for $x = 0$ and equal to -1 for $x < 0$. In this case, the limit at $x = 0$ does not exist. If we approach 0 from the right, then the limiting value is 1. If we approach 0 from the left, then the limiting value is -1.

1.8. An other example is the function $f(x) = \frac{1-\cos(x)}{x^2}$ it is not so clear what happens at $x = 0$. We can not plug in $x = 0$ because then we divide 0 over 0. But we can evaluate the function for x values close to $x = 0$ and see what happens. Lets see:

$$\frac{1-\cos(0.1)}{0.1^2} = 0.499583$$

$$\frac{1-\cos(0.01)}{0.01^2} = 0.499996.$$

What actually happens is that

$$\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = 1/2 .$$

We will come back to this when we have more tools to compute limits.

1.9. A third example is the function $f(x) = 1/x^2$. This function converges to infinity at $x = 0$. We also say just that the limit does not exist because ∞ is not a number. The function has no finite limit at $x = 0$. We usually understand a limit to be finite. We also say that the function has a **pole** at $x = 0$ or a vertical asymptote. We can here also look at the limit $x \to \infty$ and write $\lim_{x\to\infty} 1/x^2 = 0$.

1.10. The next example is the function $f(x) = \sin(1/x)x$. In this case, because $|f(x)| \leq |x|$, the function $f(x)$ converges to 0 for $x \to 0$. This is the case both from the left or from the right. Indeed if $x > 0$, then $0 \leq \sin(1/x) \leq 1$ and $0 \leq f(x) \leq x$. Similary, if $x < 0$, then $-x \leq f(x) \leq 0$ The limit at $x = 0$ exist.

1.11. Finally, lets look at the function $f(x) = \sin(1/x)$. This function takes values 1 arbitrarily close to $x = 0$ and also takes values -1 arbitrarily close to $x = 0$. Indeed for $x = 2/(\pi n)$ we have the value 1 for odd n and -1 for even n. The function therefore has no limit at $x = 0$. The graph of the function is sometimes called the **devil's comb**.

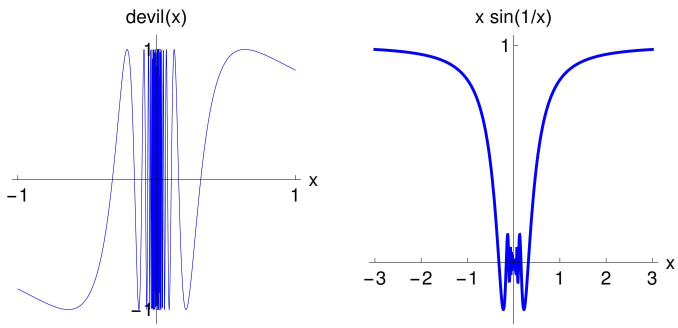

FIGURE 5. The devil function $f(x) = \sin(1/x)$ and the **tamed devil function** $f(x) = x\sin(x/x)$.

CONTINUITY

1.12. A function $f(x)$ is called **continuous** at $x = a$ if there exists a finite value $f(a)$ such that for any choice of approaching $x \to a$, the function values $f(x)$ converge to $f(a)$. A function is called **continuous on some interval** $[a, b]$ if at every point x in that interval the function is continuous. Most functions are continuous everywhere. Polynomials are continuous everywhere, the sine and cosine functions and the exponential function are continuous everywhere. Sums and products or compositions of continuous functions are continuous.

1.13. The function $f(x) = \log|x|$ is continuous everywhere except at $x = 0$. As $f(x)$ goes to minus infinity at $x = 0$, the function is not continuous there. One speaks of a **logarithmic singularity**. The discontinuity at the point $x = 0$ is due to the fact that the function values go to minus infinity there. There is no finite value we can attach to $x = 0$.

1.14. The function $f(x) = \text{sign}(x) = x/|x|$ is not continuous at $x = 0$. There is no way to fix this because whatever value we attach to $x = 0$, the function either does not converge from the right or then does not converge from the left. Also the compromize $f(0) = 0$ does not solve the problem as now, both limits, from the left and from the right do not work.

1.15. For the function $f(x) = \sin(x)/x$ we can assign the value $f(0) = 1$ and render the function continuous. We can "heal" the point $x = 0$ by assigning to $x = 0$ the value $f(x) = 0$. Indeed the function $\text{sinc}(x) = \sin(x)/x$ is continuous everywhere with that assumption. It is a nice smooth function everywhere.

1.16. Is the function $f(x) = x/\sin(x)$ continuous everywhere? Can we fix the difficulties? No! We can fix things at $x = 0$ but not at $x = \pi$ for example. At $x = \pi$, we have a situation $\pi/0$ which has no chance to be fixed because we have a zero in the denominator without any compensating zero in the nominator. The function $f(x)$ has a **pole singularity** at $x = \pi$. It is not continuous at $x = \pi$.

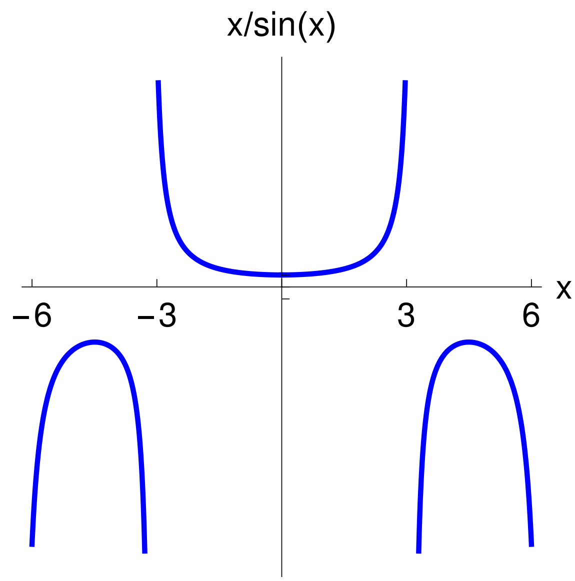

FIGURE 6. The function $f(x) = x/\sin(x)$.

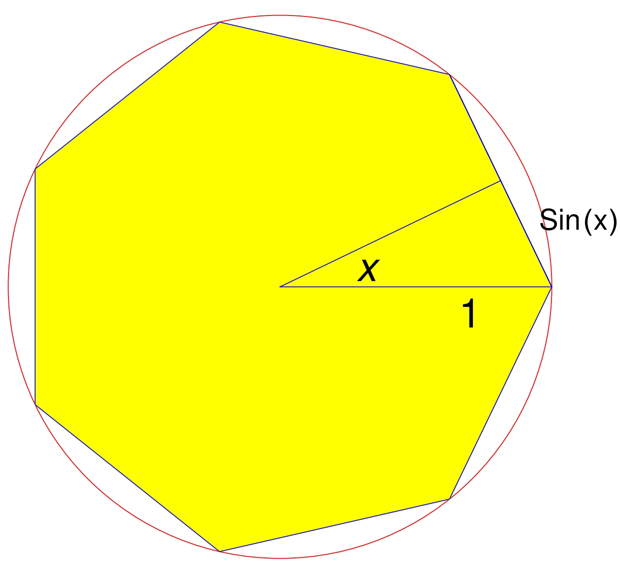

FIGURE 7. Archimedes computed the circumference of the circle by computing the arc length polygons nearby. This amounts of comparing $\sin(x) = \sin(\pi/n)$ with $x = \pi/n$. This is one of the reasons why the function $\sin(x)/x$ is so important.

Multivariable Calculus

Some math notes of Oliver Knill, knill@math.harvard.edu, 2020