SINGLE VARIABLE CALCULUS

MATH 1A, HARVARD COLLEGE 2020

Week 4: Solidification

Healing

1.1. A basic method to compute limits is to **factor out** and **cancel**. One calls this also **healing a function**. What happens often is that we end up with a function which does not have any problems any more in calculating limits. Here are two examples:

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} x + 4 = 8$$

$$\lim_{x \to 1} \frac{x^5 - 1}{x - 1} = \lim_{x \to 1} x^4 + x^3 + x^2 + x + 1 = 5$$

Only look for such cancellations if we really have a problem. In the example

$$\lim_{x \to 1} \frac{x^2 + 1}{x^2 - 3} = -4$$

there was no problem at all because at $x = 1$ the denominator is not zero. We could just plug in the number and get -4.

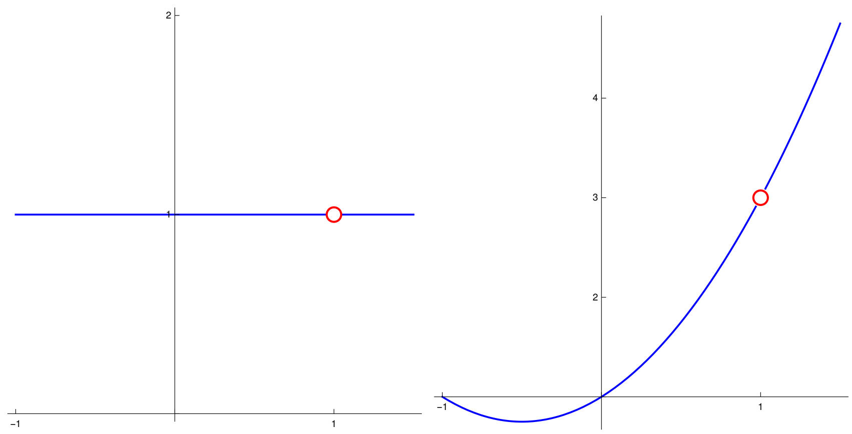

FIGURE 1. To the left, we see the **silly function** $f(x) = (x-1)/(x-1)$ which is $f(x) = 1$. To the right we see the graph of $(x^3 - 1)/(x - 1)$. The healed version $f(x) = 1 + x + x^2$ makes also sense at $x = 1$. In both cases, if $f(1) = 1$ we have a continuous functions.

Strange examples

1.2. We have seen already the **devil function** $f(x) = \sin(1/x)$. This function has a singularity at $x = 0$. It oscillates heavily at $x = 0$ because at every point $x = 1/(\pi n)$, we have a root and at every point $x = 2/(\pi + 4n\pi)$, the function is equal to 1 and at every point $x = 2/(-\pi + 4n\pi)$ the function is equal to -1. The graph of f is also known as the **devil comb**.

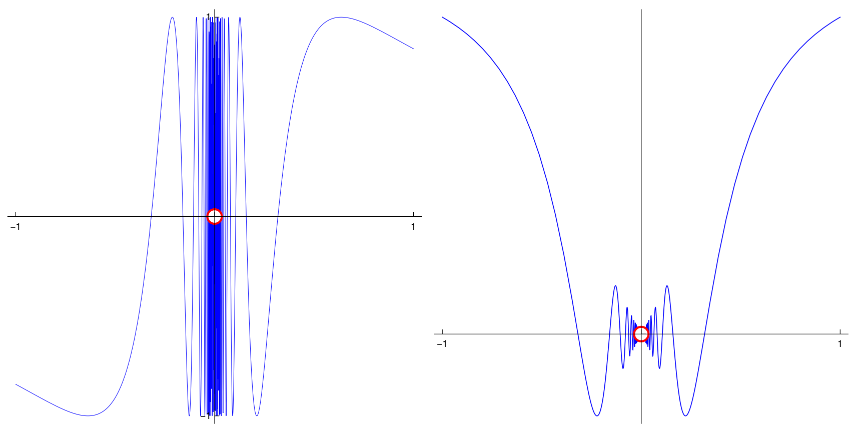

Figure 2. The devil comb $y = \sin(1/x)$ and the tamed devil $f(x) = x\sin(1/x)$. The first one is discontinuous at 0 the second one can be completed to a continuous at 0 when setting $f(0) = 0$.

1.3. The **tamed devil** is the function $f(x) = x\sin(1/x)$. In this case the function value is bound by $|f(x)| \leq |x|$ so that for $x \to 0$, the function goes to zero. This function is continuous everywhere despite the fact that we have heavy oscillations.

1.4. The function $f(x) = \exp(1/x)$ is a function you have seen in the lesson problems. In this case we have to distinguish the case $x > 0$ and $x < 0$. The function $g(x) = 1/x$ goes to $+\infty$ when $x \to 0^+$ and to $-\infty$ when $x \to 0^-$. So, $\lim_{x\to 0^+} \exp(1/x) = +\infty$ and $\lim_{x\to 0^-} \exp(1/x) = 0$.

1.5. Let us look at the problem to compute $\lim_{x\to 0} \frac{1}{\log|x|}$. We know that $\log|x| \to -\infty$ if $x \to 0$. Therefore, we have $\lim_{x\to 0} \frac{1}{\log|x|} = 0$.

1.6. What happens to $|x|^x = e^{x\log|x|}$ at $x = 0$? This is a case in which two things compete when looking at $x\log|x|$ for $x \to 0$. We have $x \to 0$ and $\log|x| \to -\infty$. We will learn tools to decide who wins but we can experiment. Take $x = e^{-100}$, then $\log|x| = -100$ and $x\log|x| = -100/e^{100}$ which is very close to 0. Clearly the exponential function wins and $x\log|x| \to 0$ so that $e^{x\log|x|} \to 1$.

1.7. What happens for $f(x) = 1/\sin(1/x)$ at $x = 0$? We have seen that $\sin(1/x)$ is zero for arbitrary small x. This means that $f(x)$ is ∞ and $-\infty$ arbitrarily close to 0. This is a devil which is singular arbitrary close to 0.

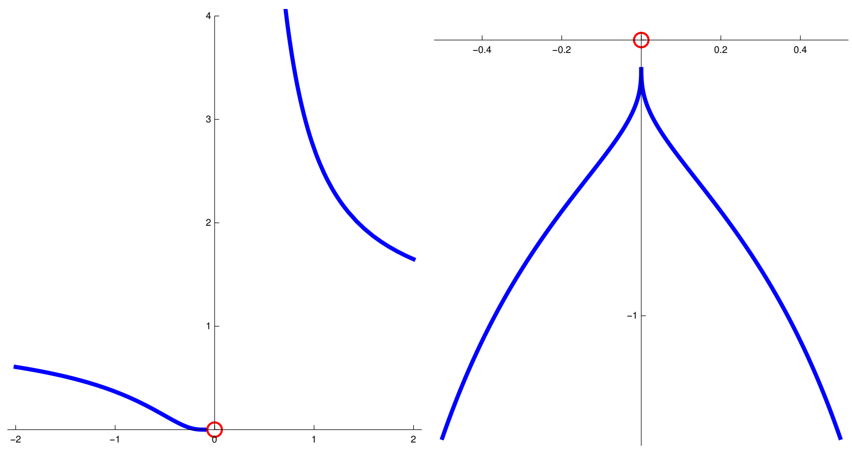

FIGURE 3. The function $\exp(1/x)$, and the function $1/\log|x|$. The first one is discontinuous at $x = 0$ and can not be saved. The second one can be completed to be continuous at 0 by setting $f(0) = 0$.

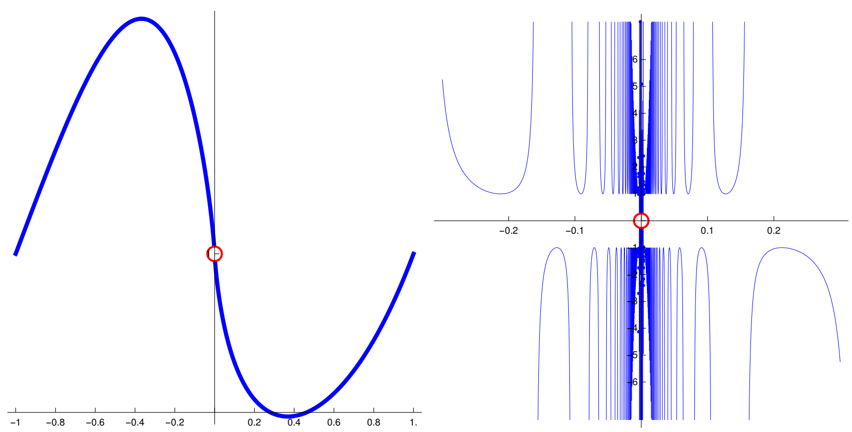

FIGURE 4. The function $|x|^x$ is continuous at 0 if we complete it and say $f(0) = 1$. The super devil function $1/\sin(1/x)$ both oscillates and also escapes to infinity infinitely often close to 0. This function can not be made continuous at $x = 0$.

OPERATIONS

1.8. Functions can be added and subtracted, multiplied and divided as well as composed with other functions. Assume the limits $\lim_{x\to a} f(x) = A$ and $\lim_{x\to a} g(x) = B$ exist. Then

Theorem: $\lim_{x\to a} f(x) + g(x) = A + B$

The same compatibility is true when combining limits and subtraction.

Theorem: $\lim_{x\to a} f(x)g(x) = AB$

1.9. What happens under compositions. Assume that $\lim_{x\to A} g(x) = B$ and that $\lim_{x\to a} f(x) = A$, then

Theorem: $\lim_{x\to a} g(f(x)) = B$

We can also deal with the fractions, but then we have to assume $\lim_{x\to a} g(x) = B$ is **not zero**

Theorem: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{A}{B}$

1.10. In general, we can not make any conclusions if one of the limits does not exist. For example, for $f(x) = 1/x$ and $g(x) = -1/x$, we have $f(x) + g(x) = 0$. So, even if both had poles at $x = 0$ the sum is perfectly fine.

Limits to Infinity

1.11. How do we compute

$$\lim_{x\to\infty} \frac{3x^2+1}{4x^2+2x+1} .$$

You might see the answer by plug in in a large number like one million and see what happens, we can also **divide out the highest order term on both sides** to bet

$$\lim_{x\to\infty} \frac{3+1/x^2}{4+2/x+1/x^2} .$$

Now we have a situation discussed in the last hour. The top expression converges to $A = 3$ the bottom expression converges to $B = 4$. The result therefore is A/B.

1.12.

$$\lim_{x\to\infty} \frac{x}{\sqrt{9x^2+1}}$$

for $x \to \infty$. We can divide again both sides by x and get $\frac{1}{\sqrt{9+1/x^2}}$. Again we end up with an expression which for $x \to \infty$ has perfect limits on both sides.

1.13. What happens for $f(x) = \sin(x)$ for $x \to \infty$? We have infinitely large values $x = \pi/2 + 2\pi n$ for which $f(x) = 1$ and infinitely large values $x = -\pi/2 + 2\pi n$ for which $f(x) = -1$.

1.14. What happens with the devil curve $f(x) = \sin(1/x)$ for $x \to infty$? For x larger than $2/\pi$, the function stays positive and goes then slowly to zero. Indeed $\lim_{x\to\infty} \sin(1/x) = 0$.

What grows faster?

1.15. Lets make a race and see what happens with various functions if x goes to $+\infty$. Our contestants are:

- Logan $\log(x)$
- Ruth $\sqrt{x}$
- Express $\exp(x)$
- Primus $x/\log(x)$ (is important for prime numbers)
- Cubby x^3.
- Squirly $\sqrt{x}/\log(x)$.

Can you order them according how fast they grow? You might want to experiment with the computer.

1.16. Remember that a function f is differentiable at a point if the derivative $f'(x)$ is defined at the point. This requires that a limit $\lim_{h\to 0}(f(x+h)-f(x))/h$ exists from both sides. A function is continuous at x, if $\lim_{h\to 0} f(x+h)=f(x)$.

1.17. The function $f(x)=|x|$ is a prototype of a function that is continuous but not everywhere differentiable. It is not differentiable at $x=0$.

1.18. Also the function $f(x)=x\sin(1/x)$ with the understanding $f(0)=0$ is continuous everywhere but $f'(x)$ is not continuous at $x=0$. Indeed there are points arbitrarily close to $x=0$ where $f'(x)=1$ and points arbitrarily close to $x=0$ where $f'(x)=-1$.

Relating f and f'

1.19. There are various ways how we can relate f with its derivative.

- Points x, where $f'(x)=0$ places where the rate of change is zero. These are places, where f has a horizontal tangent.
- Intervals, where $f'(x)>0$ are places, where f is increasing.
- Intervals where $f'(x)<0$ are places, where f is decreasing.

Relating f and f''

1.20. There are also various ways to relate f with f''.

- Points x, where $f''(x)=0$ can be very flat points or inflection points.
- Intervals, where $f''(x)>0$ are intervals where f is concave up.
- Intervals, where $f''(x)<0$ are intervals where f is concave down.

Getting f from f'

1.21. We will look at the car-trip problem, in which the velocity is given and where we want to find the position.

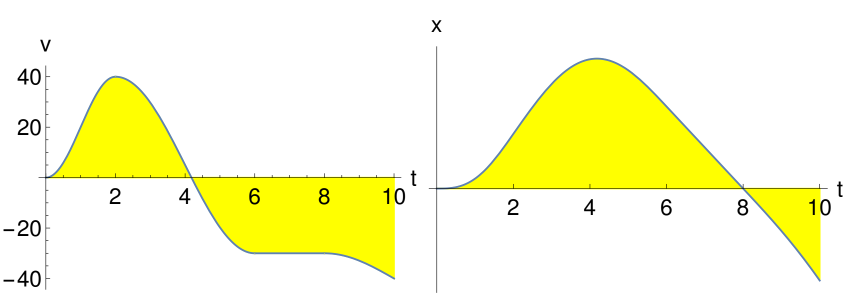

FIGURE 5. The left graph shows the velocity of a car. The right graph shows the position.

Getting f'' from f'

1.22. The opposite, going from f' to f'' is easier. It is the same problem than going from f to f'. Unlike for the position, where we also need to know the position at some time, the acceleration has no choice.

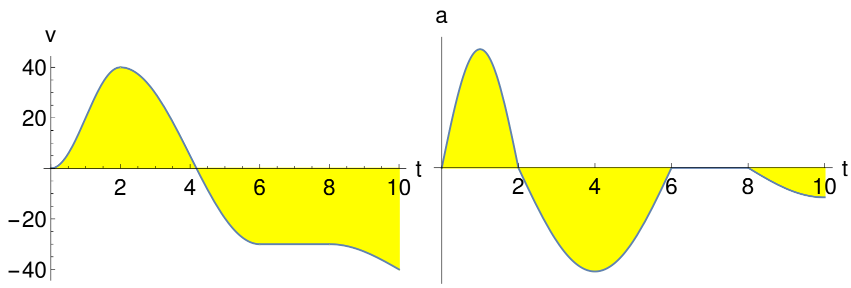

FIGURE 6. The left graph shows the velocity of a car. The right graph shows the acceleration