SINGLE VARIABLE CALCULUS

MATH 1A, HARVARD COLLEGE 2020

Week 5: Some Differentiation rules

Linearity

1.1. You have all already used the linearity of the derivative. If we multiply a function by a constant c, then the average rate of change $(f(x+h)-f(x))/h$ also gets multiplied by c. We can pass to the limit and see

Theorem:

$$(cf)' = cf'$$

1.2. For example, for $f(x) = 7\sqrt{x}$, the derivative is $7(\sqrt{x})' = 7/(2\sqrt{x})$.

1.3. Also, if we take the sum of two functions $f + g$, this is a new function, whose derivative is the sum of the derivatives of f and g

Theorem:

$$(f + g)' = f' + g'$$

The two properties together show that the process of going from f to f' is linear.

The power rule

1.4. We have also already seen that we can compute the derivative of $f(x) = x^n$ by using the binomial expansion

$$(x + h)^n = x^n + nx^{n-1}h + ... + h^n$$

The magic is that subtracting x^n gives an expression which is a multiple of h so that we can divide out h and get

Theorem:

$$(x^n)' = nx^{n-1}$$

This especially implies that the derivative of a constant function is 0, something you can see immediately from the fact that all average rate of changes are already zero $f(x + h) - f(x) = c - c = 0$.

1.5. Together with linearity, we can now differentiate polynomials like

$$f(x) = 5x^3 + 17x - 15 \ .$$

The derivative is $f'(x) = 15x^2 + 17$.

Product rule

1.6. The product rule for differentiation follows from the identity

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)] \ .$$

When dividing by h we get on the left hand side the average rate of change of fg on $[x, x+h]$ and on the right the average rate of change of f times $g(x+h)$ plus f times the average rate of change of g. For $h \to 0$, this is

Theorem:
$$(fg)' = f'g + fg'$$

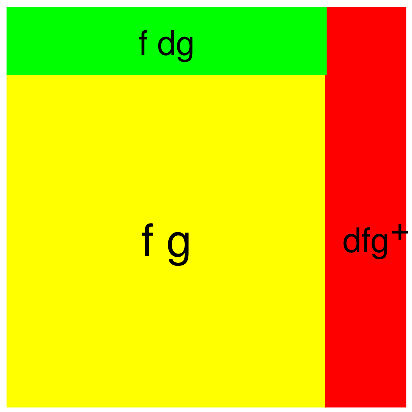

Figure 1. The product rule

1.7. The product rule allows to reduce the problem to compute the derivative of x^n to the problem for x^{n-1}: Assume we know $(x^{n-1})' = (n-1)x^{n-2}$ and $x' = 1$, then $(x \cdot x^{n-1})' = x^{n-1} + (n-1)x^{n-2}x = nx^{n-1}$.

Graphing functions

1.8. Knowing the derivative of a function allows to get more information about the function:
First of all we can look at the function values themselves and ask questions like:

- What is the domain of f?
- Where are the roots of the function?
- Where does the graph hit the y axes?
- Are there horizontal or vertical asymptotes?

Then we can look at the derivative $f'(x)$.

- Where is f flat (the derivative is zero)
- Where does f increase?
- Where does f decrease?
- Where is f maximal or minimal?

Finally, we can look at the second derivative $f''(x)$ and answer questions like

- Where is f concave up?

- Where is f concave down?
- Where are inflection points, points where f'' changes sign.

Examples

1.9. Example: $f(x) = (x^2 + x)/x = x + 1/x$.

1.10. Example: (from Jam) $f(x) = x - \sqrt{x}$.

1.11. Example: (from Jam) $f(x) = \frac{e^{2x}+4e^x-17}{e^{3x}-1}$.

The exponential function

1.12. You should know all the **compound interest formula**

$$(1 + h)^n$$

which tells your fortune after n years, if the interest rate is h. If $[x/h]$ is the closest integer to x/h, then

$$\exp_h(x) = (1 + h)^{[x/h]}$$

is a nice function in x which for h going to infinity converges to the exponential function $\exp(x)$.

1.13. The function $\exp_h(x)$ has the nice property that the average rate of change on an interval of length h is again the function:

Theorem:

$$\frac{\exp_h(x + h) - \exp_h(x)}{h} = \exp_h(x)$$

1.14. So, in the limit $h \to 0$, we have

Theorem:

$$\exp'(x) = \exp(x)$$

This is a very important relation. Many textbooks define the exponential function as such, but then one would have to justify that the function exists. What is nice about the function $\exp_h(x)$ is that if we do not go with $h \to 0$, we get still a calculus which looks the same than the calculus of Newton and Leibniz. It is actually formally equivalent but does not involve any limits. It is calculus without limits or **quantum calculus**. You are all quite familiar with that since in the first class of this course we have looked at the average rate of change on an interval. This can be seen as a discrete derivative.

1.15. You should from pre-calculus and previous calculus courses already be familiar with the rules of the exponential function. Here they are

Theorem: $\exp(x)\exp(y) = \exp(x+y)$

With $\log(x)$ denoting the natural log, one has

Theorem: $a^x = x^{x\log(a)}$

The following result follows by using $e^x = a$ and $\log(a) = x$.

Theorem: $(e^x)^y = \exp(xy)$

Prize Elasticity of Demand

2. Prize Elasticity

In economics, one usually uses different variables. Here x is the quantity and y is the prize. The demand curve $p(q)$ is a function which assigns to the quantity sold the prize p. Now by increasing the quantity sold from q_1 to q_2 then the increased revenue is $(q_2 - q_1)p_1$. If increasing the prize then one expects less to be sold. The lost revenue is $q_1(p_2 - p_1)$. The ration $(q_2 - q_1)p_2/((p_2 - p_1)q_1)$ is called the prize elasticity of the demand. In the limit when the change becomes smaller and smaller one gets $q'p/q$ or $p'q/p$. The elastisticy is unitary if this coefficient is -1. This means $p'/p = -q$. We will learn how to integrate that once the chain rule has been used on $\log(p)$ as its derivative is p'/p. It means $\log(p) = -\log(q) = \log(1/q)$. After taking exponentials we get $p = 1/q$.

Theorem: We have unitary elasticity if $p(q) = 1/q$.

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