SINGLE VARIABLE CALCULUS

MATH 1A, HARVARD COLLEGE 2020

Week 6: Extremization

Local extrema

1.1. Assume a function f is defined on an open interval (a, b). A point x in that interval is called a **local minimum** if $f(x) \leq f(y)$ for all y in some neighborhood of x. It is called a **local minimum** if $f(x) \geq f(y)$ for all y in some neighborhood of x. Calculus allows to list points which are candidates for local maxima and minima. There are two possibilities on an open interval:

1.2. A point x is called a **critical point** of f if $f'(x) = 0$. Critical points are candidates for **local maxima** or **local minima**. An example is $f(x) = x^2$ which has a local minimum at 0. For the function $f(x) = 1$, all points are critical points. For the function $f(x) = \sin(1/x)$ on $(a, b) = (0, 1)$ there are infinitely many isolated critical points.

1.3. A point is called a **singular point** if f is not differentiable at x. Singular points are other candidates for a local maximum or local minimum. The prototype is $f(x) = |x|$. An other example is $f(x) = 1 - x$ for $x \geq 0$ and $f(x) = x$ for $x < 0$. This function f is obviously not continuous at $x = 0$ and so also not differentiable. But it has a local maximum at $x = 0$.

1.4. On a closed interval, there will be an also **boundary points** which can be candidates for maxima or minima. We will look at this in the next lecture. The realization that one can use calculus to find critical points is due to Fermat:

Theorem: If f is differentiable and x is a local maximum, then $f'(x) = 0$.

1.5. The reason is that if $f'(x) > 0$, then $(f(x+h) - f(x))/h > 0$ for small enough h. Therefore $f(x+h) > f(x)$ for small h so that x is not a maximum. If $f'(x) < 0$, then $(f(x-h) - f(x))/h > 0$ for small enough h and $f(x-h) > f(x)$ for small h.

1.6. To the terminology. When we say critical "point", we mean x. If we say critical "value" we mean $f(x)$ where x is a critical point. Some calculus textbooks call x a critical place and the pair $(x, f(x))$ a critical point and rename x a critical "place". Everywhere else in mathematics one uses the terminology "critical point" however.

First and Second derivative test

1.7. The first derivative test

Theorem: If $f'(x) = 0$ or x is an isolated singular point and $f'(y)$ is negative for y smaller than x and $f'(y)$ is positive for y larger than x, then x is a local minimum.

Similarly, if $f'(y)$ is positive for y smaller than x and $f'(y)$ is negative for y larger than x, then x is a local maximum. This is sometimes called the first derivative test. The test can be useful in two situations. It can be useful in cases where the second derivative is zero at the point like for $f(x) = x^4$. It can also be useful in the case of a singular point like for $f(x) = |x|$, where the derivative is negative for $y < x$ and positive for $y > x$. The first derivative test can also be useful if the function is only differentiable but not twice differentiable.

1.8. The second derivative test uses the second derivative.

Theorem: If $f'(x) = 0$ and $f''(x) > 0$, then x is a local minimum. If $f'(x) = 0$ and $f''(x) < 0$, then x is a local maximum.

The test does not say anything in the case when $f''(x) = 0$. For $f(x) = x^3$ for example, there is no minimum and no maximum. For $f(x) = x^4$ we have a minimum and for $f(x) = -x^4$ we have a maximum.

Global extrema

1.9. When a continuous function is defined on a closed interval $[a, b]$, the maxima or minima can occur either in the interior or then at the boundary. In order to find a **global maximum** we have to look at local maxima in the interior (a, b) and then at points on the boundary.

1.10. Note that for points on the boundary, we do not necessarily have to have $f'(x) = 0$ even if the function is smooth everywhere.

Some math notes of Oliver Knill, knill@math.harvard.edu, 2020