Week 8: Implicit differentiation

Extrema problems

This is a review lecture where we extremize things with more tools.
Remember the derivative rules:
Remember that when looking for extrema of a function, we need not only to look at critical points but also look at boundary points. Sometimes, this requires first to figure out, where the function is defined.

Implicit differentiation

Here is a problem: The curve

$$y^2 = x^3 - x + 1$$

is an example of an **elliptic curve**. Find the tangent line at $(1, -1)$ and intersect it with the curve.

To get the slope at the point $(1, -1)$, differentiate and treat y as a function of x:

$$2yy' = 3x^2 - 1 \ .$$

Now plug in the point $x = 1, y = -1$ to get $-2y' = 3 - 1 = 2$ so that the slope m is equal to -1. The equation

$$y = mx + b$$

of the tangent line can be obtained by plugging in the point $x = 1, y = -1$ and $m = -1$ We see $b = 0$. Now, if we plug in $y = -x$ into the equation we get $x^3 - x^2 - x + 1$ which is solved not only by $x = 1$ but also by $x = -1$. The point $(-1, 1)$ is the intersection point.

Elliptic curves are a pretty cool set-up in cryptology. The reason is that one can do computations on elliptic curves similarly as we can do computations with numbers. What the above problem actually did was to compute the addition of $(1, -1)$ with itself. It gave the point $(-1, 1)$. We can say $(1, -1) \oplus (1, -1) = (-1, 1)$. Note that the $\oplus$ operation is not just the addition of the coordinates. An other example in that curve is $(-1, 1) \oplus (0, 1) = (1, 1)$. The reason is that the line segment through the two points $(-1, 1)$ and $(0, 1)$ intersects the curve again in $(1, 1)$. The addition rule is done geometrically.

Logarithmic derivative

The method of logarithmic differentiation is best explained in examples. The idea is to take the logarithm of the function to be differentiated. If $y = f(x)$, then $\log(y) = \log(f)$ and differentation gives $y'/y = d/dx \log(f(x))$. But unlike implicit differentiation which really gives advantages, logarithmic differentiation does not give any advantages. Sometimes, it can be of advantage however to see y'/y as the derivative of $\log(y)$. We will see this when integrating y'/y. Having heared of logarithmic differentiation can help when doing integration by substitution.

There are three examples

a) $y = x^{1/3}\sqrt{1+x^2}$.

$y'/y = \frac{d}{dx}(\log(x)/3 + \log 1 + x^2/2$ This gives $y' = y(1/(3x)) + x/(1+x^2))$. Now fill in
b) $y = x^x$

The direct method is to rewrite this as $e^{x\log(x)}$ and now differentiate.
c) $y = x^{(x^x)}$.

Also this is better done by rewriting it as $e^{x^x \log(x)}$ then use b).

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