

SINGLE VARIABLE CALCULUS

MATH 1A, HARVARD COLLEGE 2020

Week 9: Related Rates

Related rates problems deal with a relation for variables. Differentiation gives a relation between the derivatives (rate of change). In all these problems, we have an **equation** and a **rate**. You can then solve for the rate which is asked for.

Hydrophilic **water gel spheres** have volume $V(r(t)) = 4\pi r(t)^3/3$ and expand at a rate $V' = 30$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.



A **wine glass** has a shape $y = x^2$ and volume $V(y) = y^2\pi/2$. Assume we slurp the wine with constant rate $V' = -0.1$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi y y'(t)$ so that $y'(t) = -1/(\pi y)$.



A **ladder** has length 1. Assume slips on the ground away with constant speed $x' = 2$. What is the speed of the top part of the ladder sliding down the wall at the time when $x = y$ if $x^2(t) + y^2(t) = 1$. Differentiation gives $2x(t)x'(t) + 2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

A **kid slides** down a slide of the shape $y = 2/x$. Assume $y' = -7$. What is $x'(t)$? Evaluate it at $x = 1$. **Solution:** differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.



Image source: <http://www.dmfco.com>

A **canister of oil** releases oil so that the area grows at a constant rate $A' = 5$. With what rate does the radius increase? **Solution.** We have $A(r) = r^2\pi$ and so $5 = A'(r) = 2rr'\pi$. Solving for r' gives $r' = 5/(2r\pi)$.

Related rates problems link quantities by a rule. These quantities can depend on time. To solve a related rates problem, differentiate the rule with respect to time use the given rate of change and solve for the unknown rate of change. Since related change problems are often difficult to parse. We have the rule and given rate of change boxed.

The **ideal gas law** $pV = T$ relates pressure p and volume V and temperature T . Assume the temperature $T = 50$ is fixed and $V' = -5$. Find the rate p' with which the pressure increases if $V = 10$ and $p = 5$.



Solution: $-5 = V' = d/dt(T/p) = -(T/p^2)p'$ so hat $p' = 5p^2/50 = 10$. Assume the **total production rate** P of an Oculus Rift is constant $P = 100$ and given by the **Cobb-Douglas formula** $P = L^{1/3}K^{2/3}$. Assume labor is increased at a rate $L' = 2$. What is the cost change K' ? Evaluate this at $K = 125$ and $L = 64$.



Solution: $0 = P' = (1/3)L^{-2/3}K^{2/3}K' + (2/3)L^{1/3}K^{-1/3}2$. The answer is

$$K' = -\frac{(2/3)L^{1/3}K^{-1/3}}{(1/3)L^{-2/3}K^{2/3}}.$$

This is $K' = -125/64$.

Problem: You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\phi' = 5\pi/180 = \pi/36$ radians per second (which means 5 degrees per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\tan(\phi) = x/h$ to get ϕ for $x = 0$.



Problem: An **isosceles triangle** with base $2a$ and height h has fixed area $A = ah = 1$. Assume the height is decreased by a rate $h' = -2$. With what rate does a increase if $h = 1/2$?

