LECTURE 34

WRAPPING UP

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PLAN

1. Poll

2. How to review

2. Motion, Continuity, Functions, Music

3. Extrema, Catastrophes

4. Data

5. Statistics, Infinite integrals

6. Beyond Calculus

POLL

What was the best topic in this course for you?

A Continuity

B Extrema

C Limits

D Related Rates

E Differentiation

F Integration

POLL

What was the most beautiful formula for you?

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C$$

SUBSTITUTION

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

THE FTC

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{xn} = e^x$$

EXPONENTIAL

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

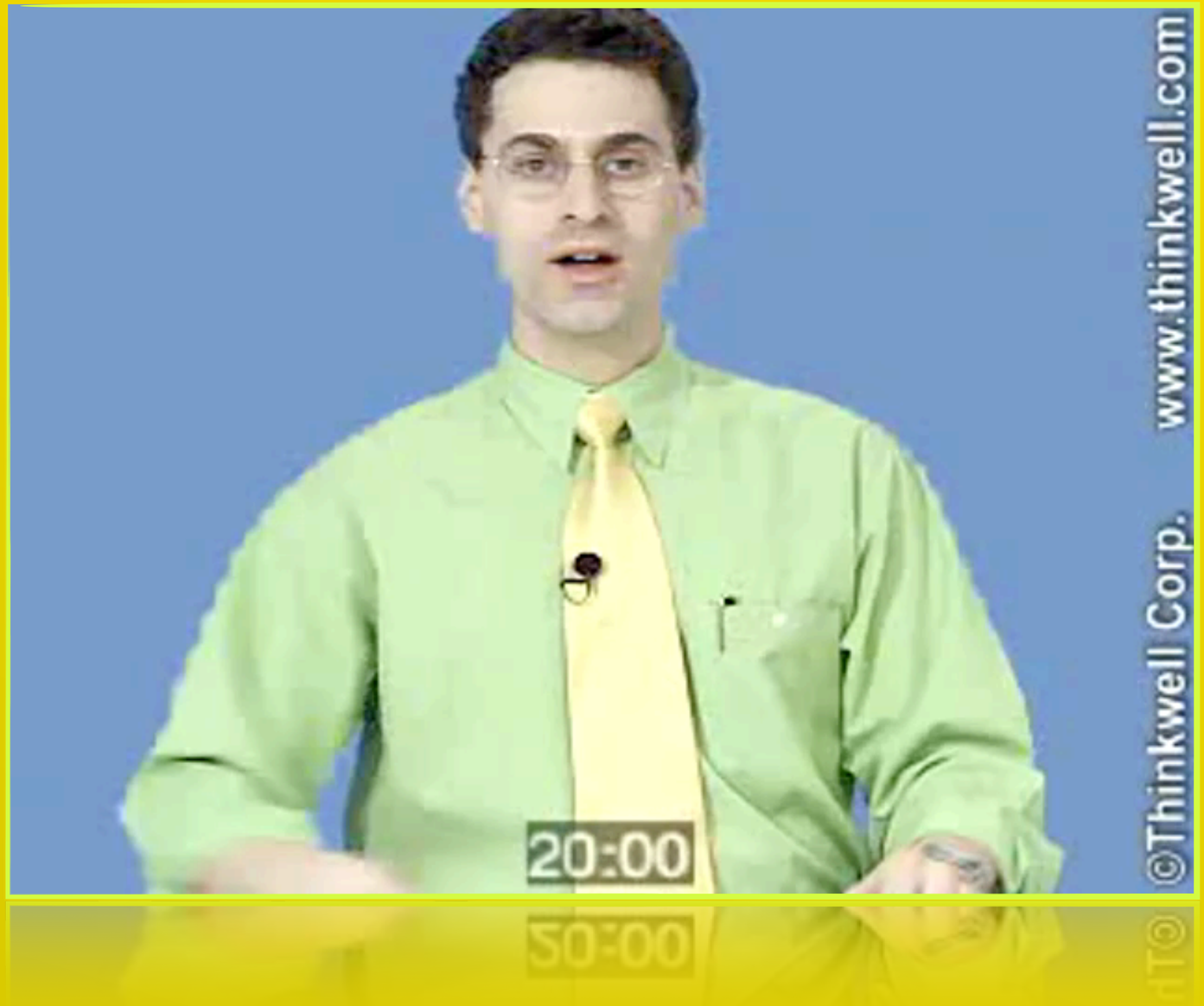
HOSPITAL

$$(f(g))' = f'(g)g'$$

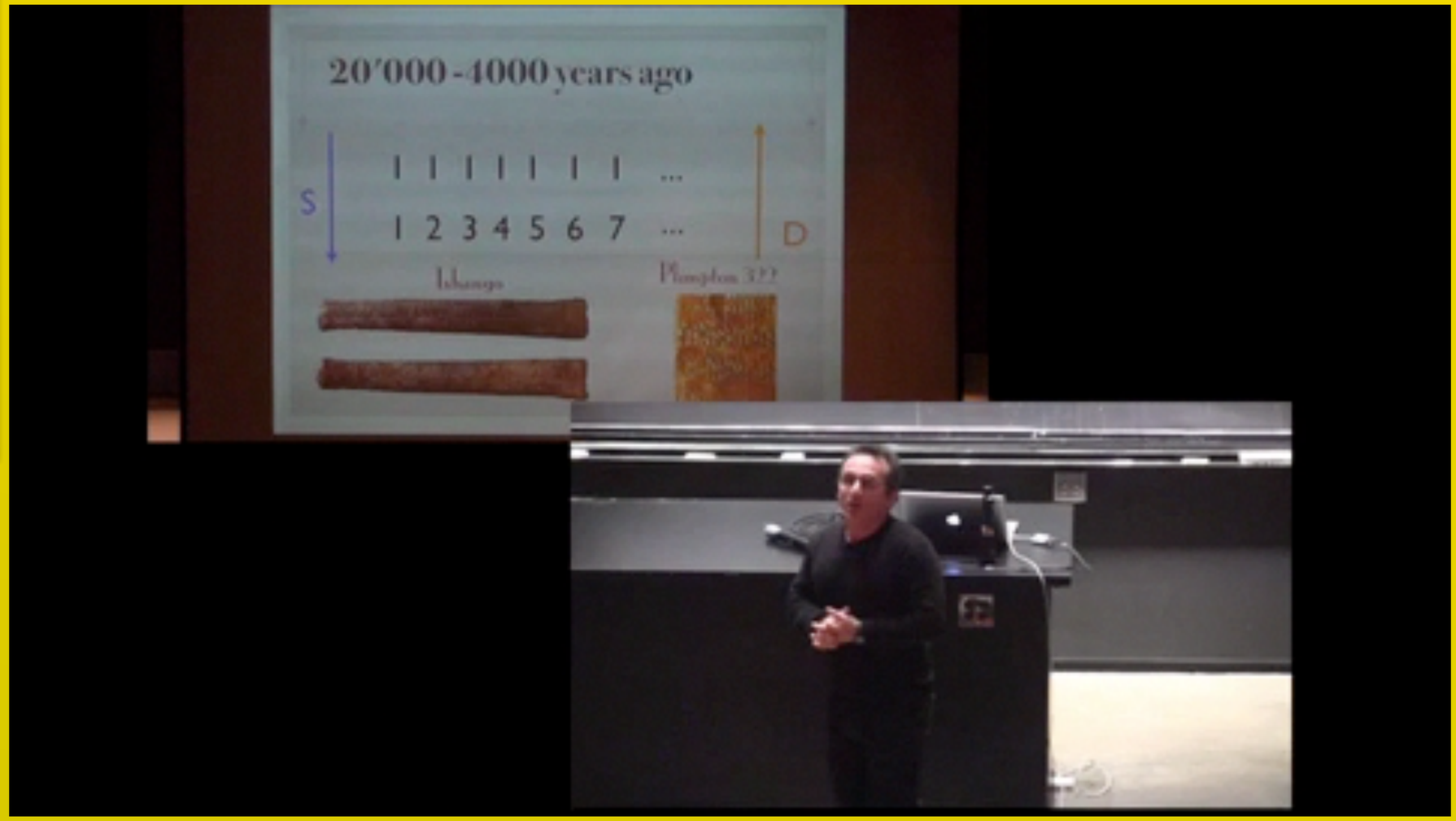
CHAIN

HOW TO REVIEW

IN 10 MINUTES

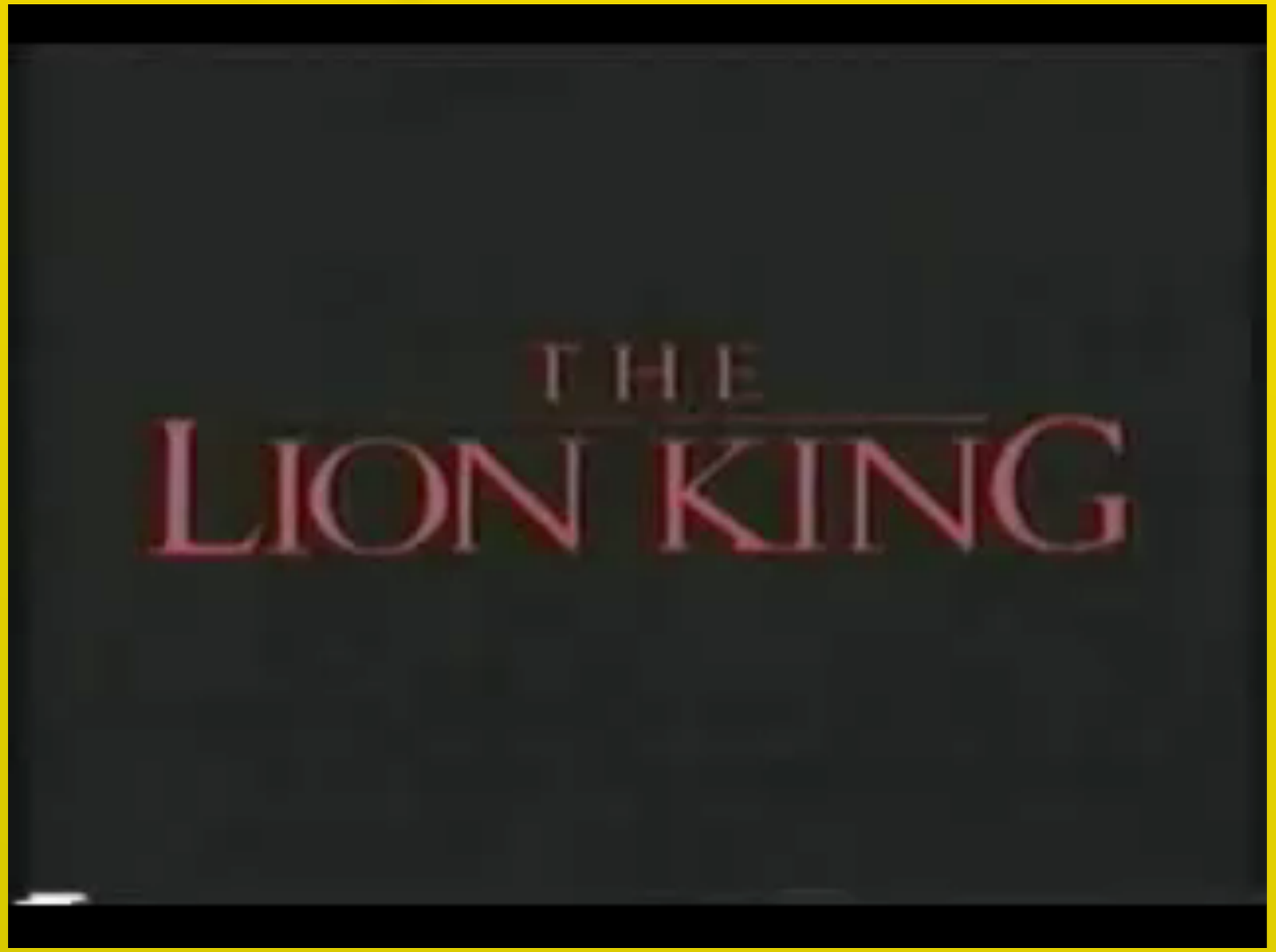


IN 400 SECONDS

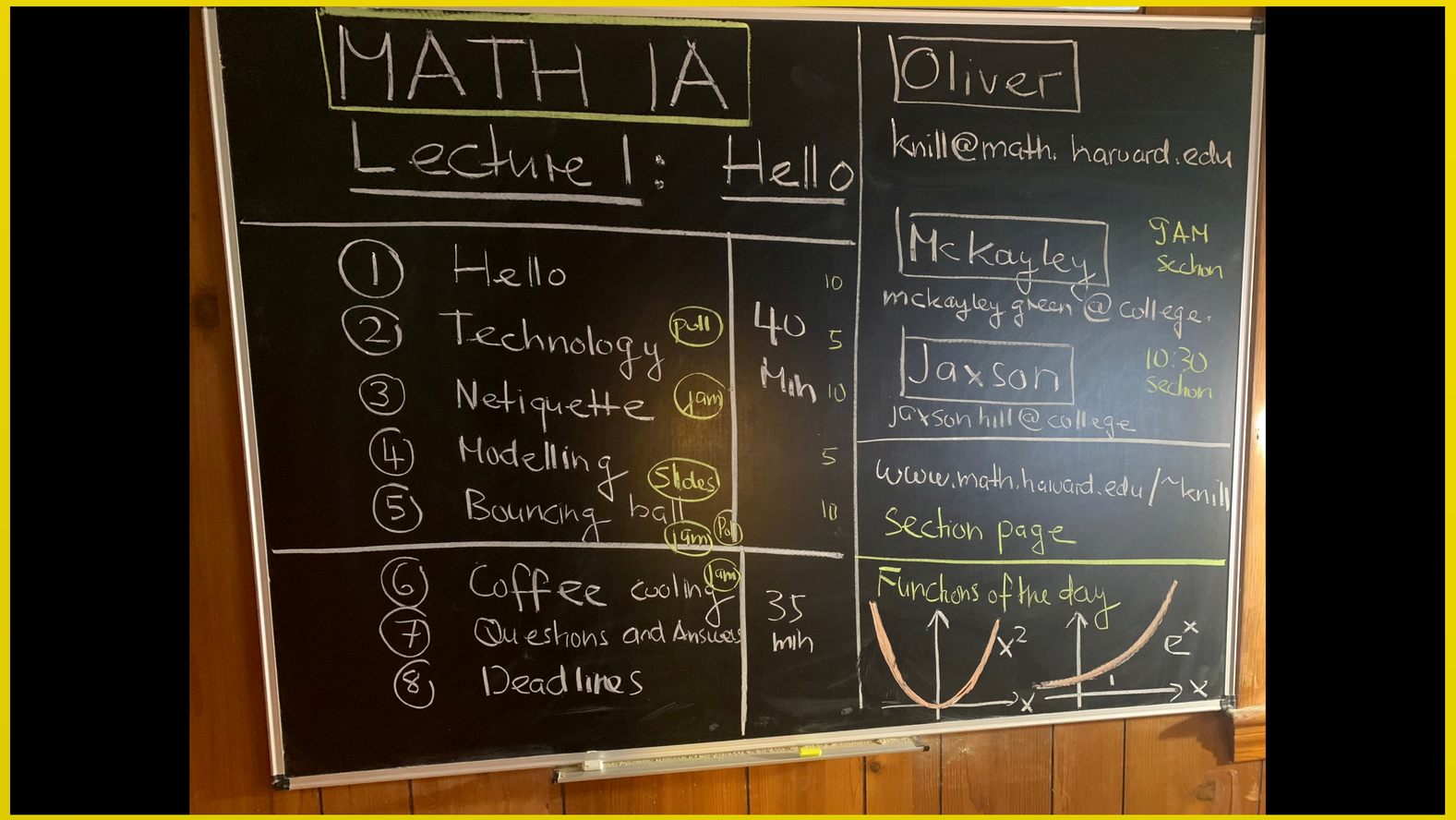


IN 5 SECONDS

IN 5 SECONDS



IN 20 SECONDS

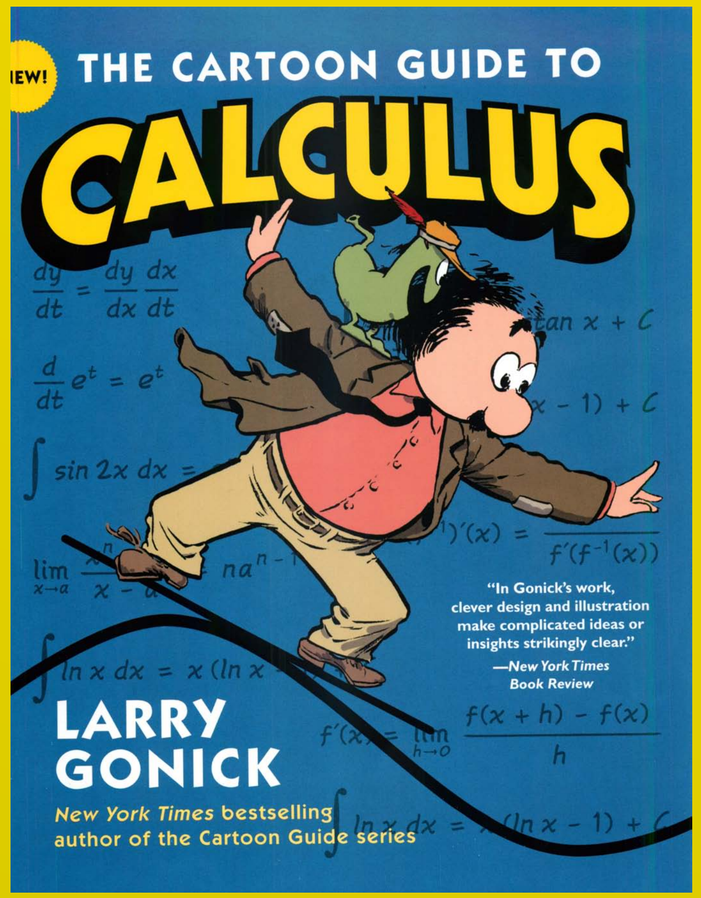


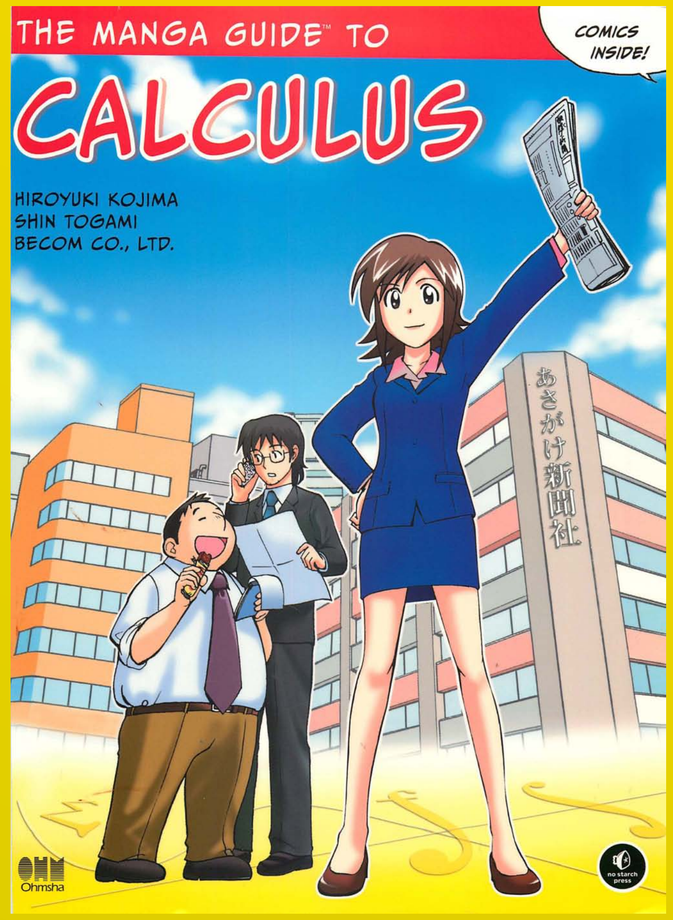
SHORT SUMMARIES

Twitter summary:

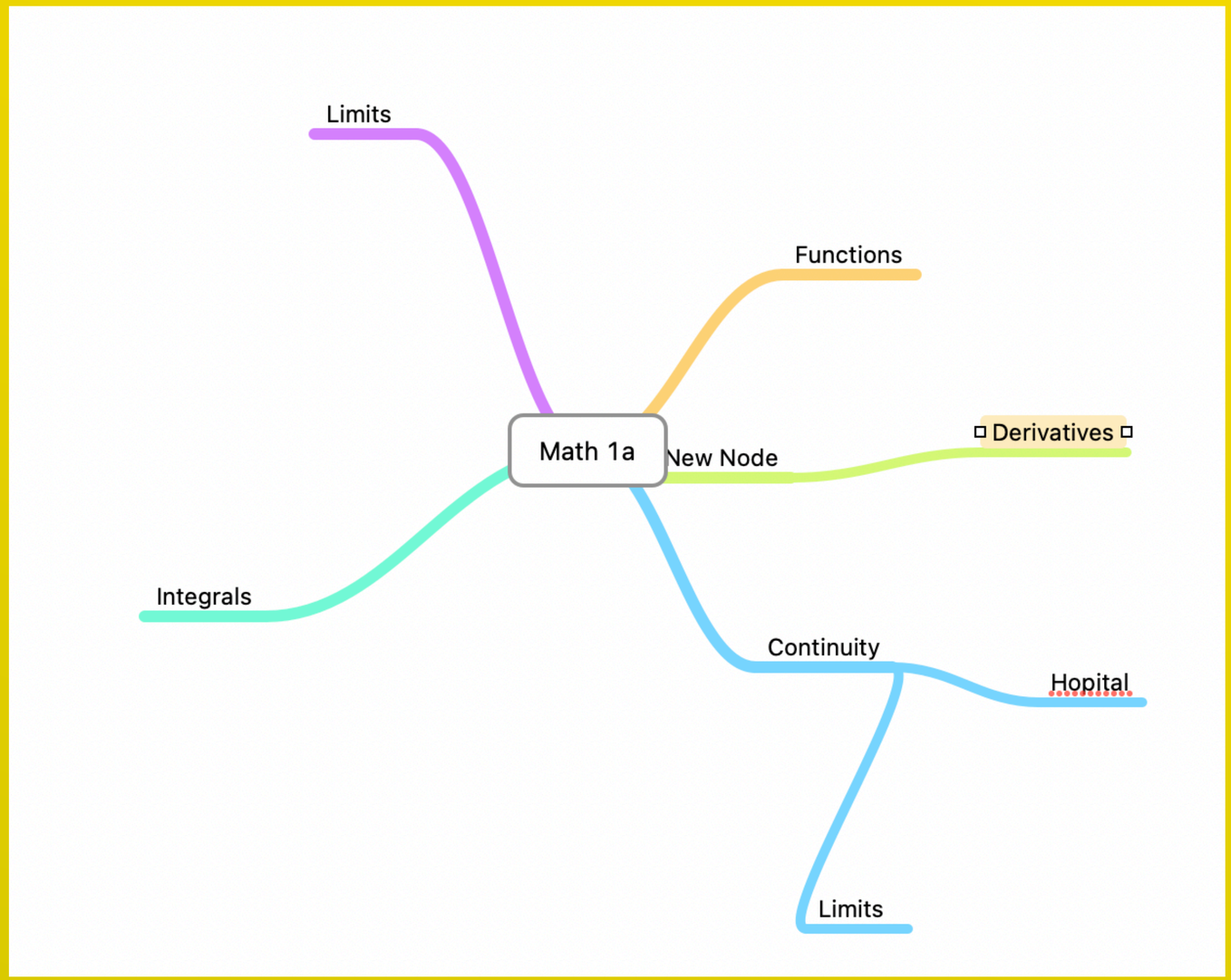
Calculus links the rate of change of a function with the area under the curve. The derivative F’ of the integral F is equal to f.

DIFFERENT ANGLES

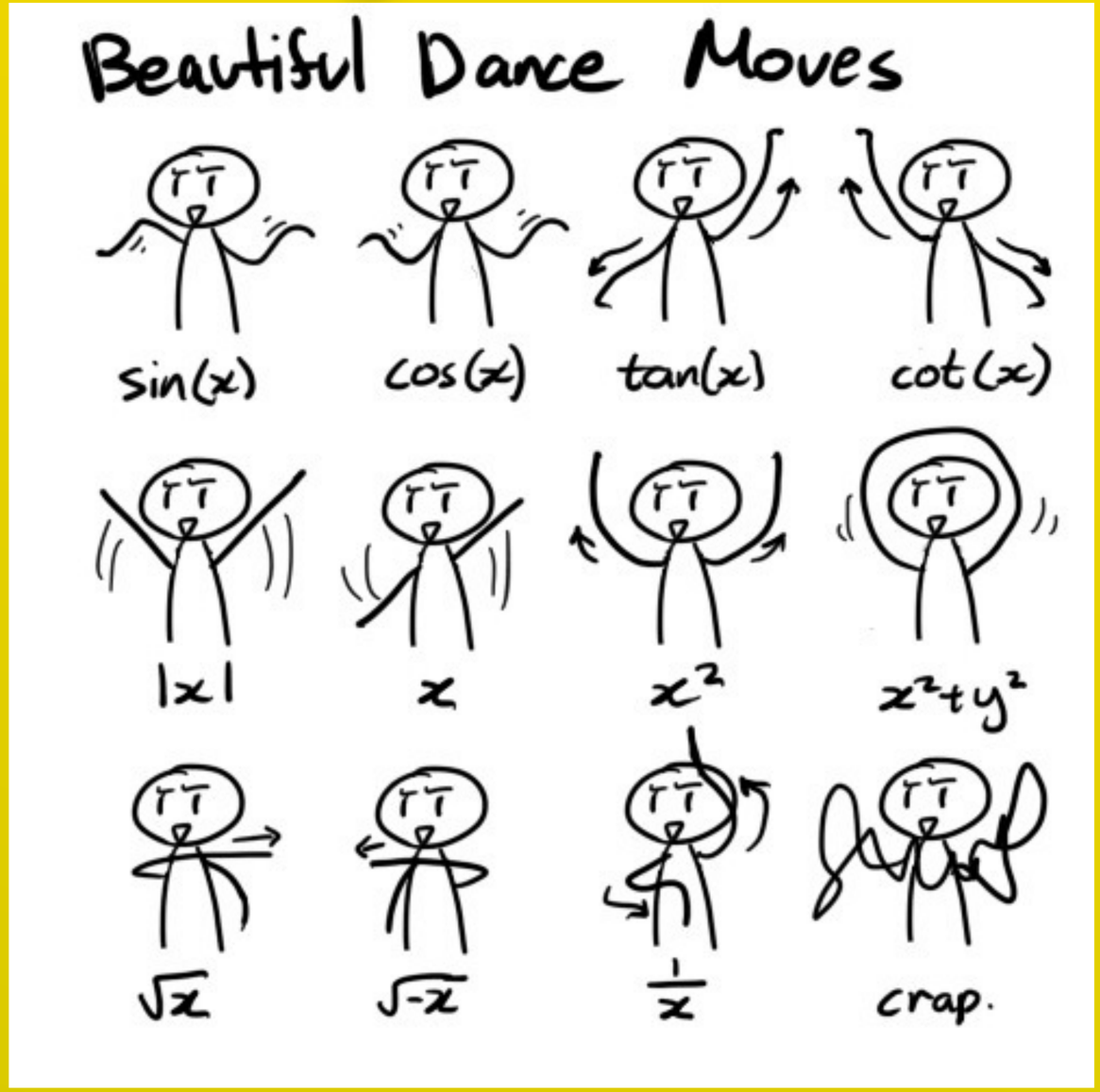




MINDMAP

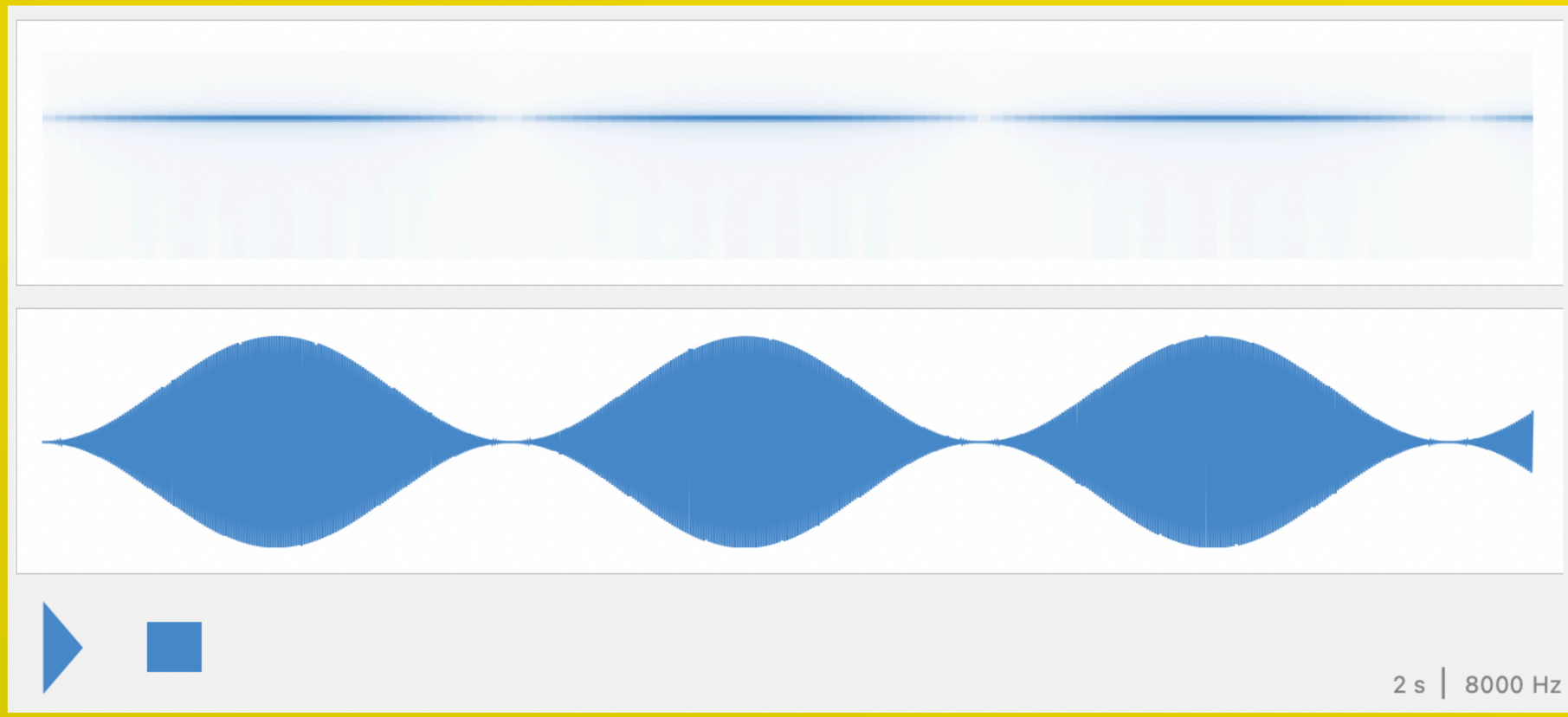

Limits
Functions
Math 1a
New Node
Derivatives
Integrals
Continuity
Hopital
Limits

DANCING FUNCTIONS



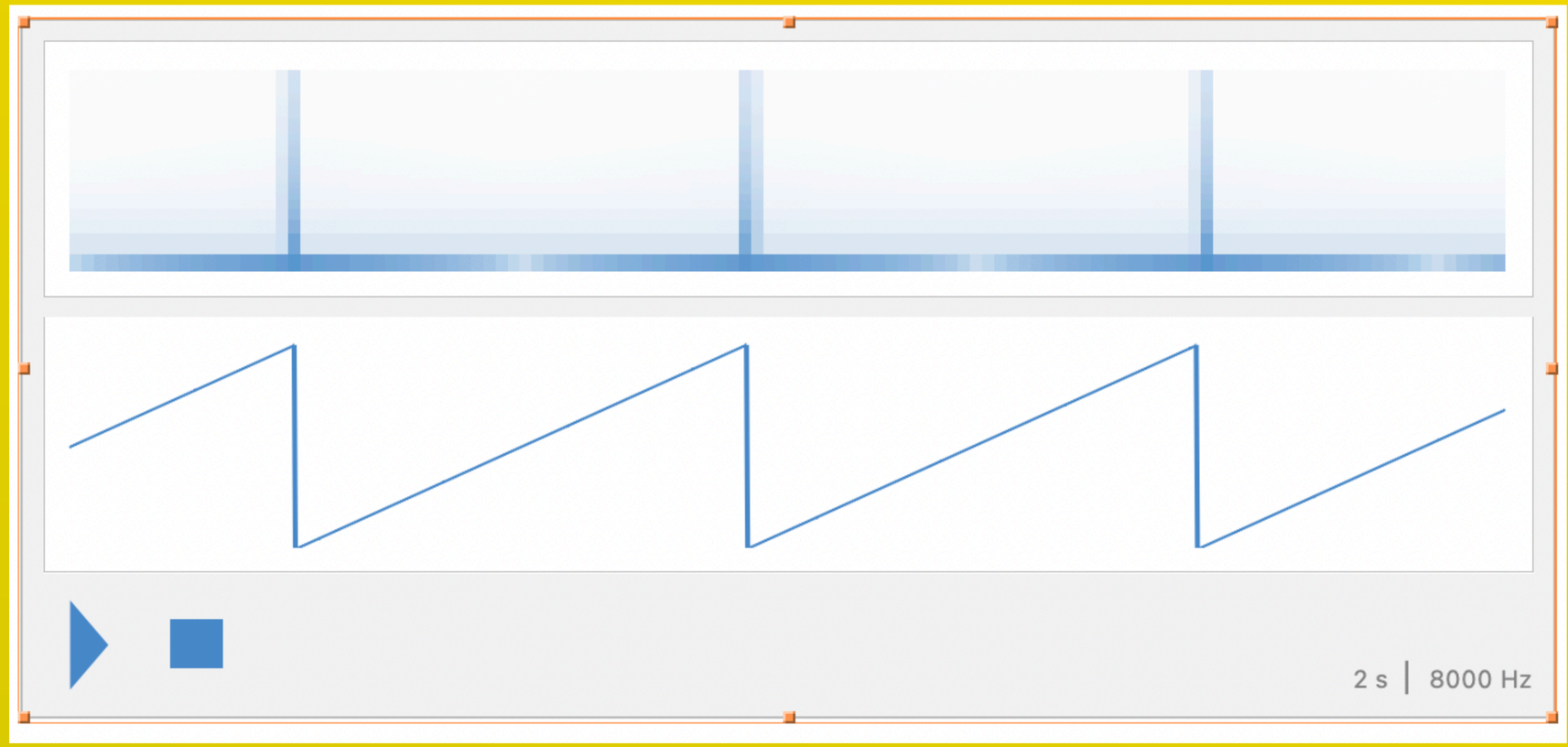
FROM DRAW TO PLAY

```
Play[Sin[5 x]^2 Sin[5200 x], {x, 0, 2}]
```



DISCONTINUITIES

Play[ArcTan[Tan[5 x]], {x, 0, 2}]



Discontinuities
do not sound nice!

EXPLORE FUNCTIONS

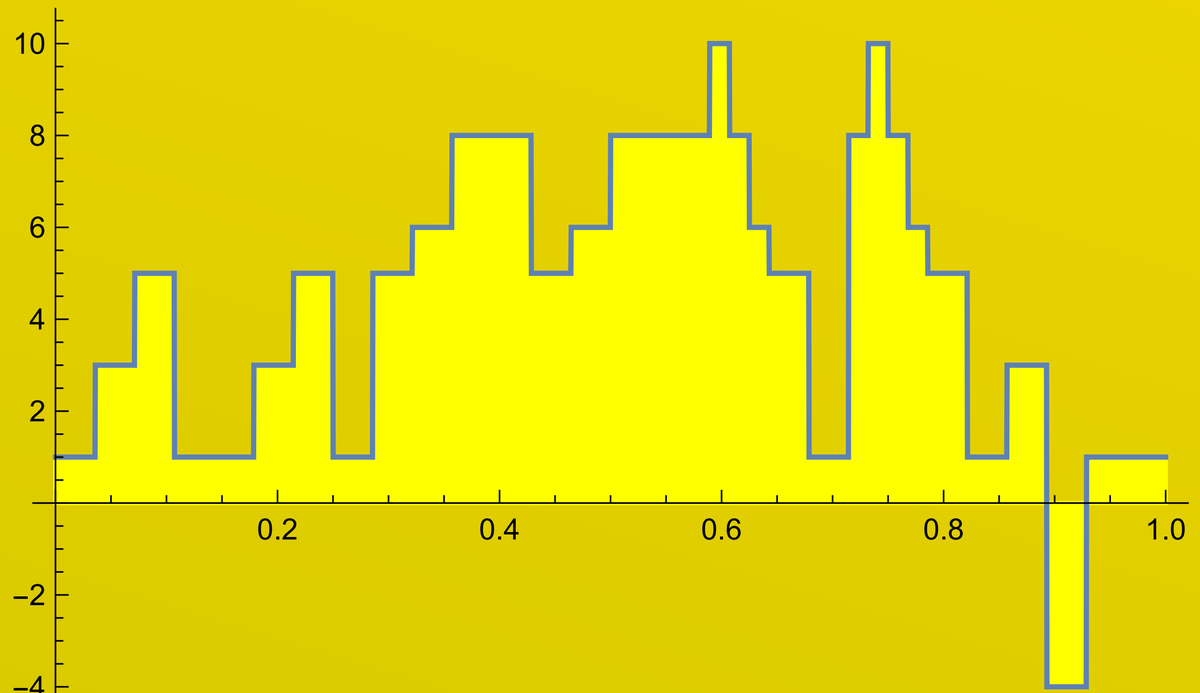
$$f(x) = \sin(10000\sqrt{x})$$

LARGER SCALE


```
T = {1, 2, 4, 8, 16};
f[x_] := SoundNote[x, T[[Mod[x, 5] + 1]]/10, "Oboe"];
A = Table[Floor[13*Sin[k/2]], {k, 20}];
Sound[Map[f, A], {0, 8}]
```

MELODIES

```
f = {1, 1, 3, 3, 5, 5, 1, 1, 1, 1, 3, 3, 5, 5, 1, 1, 5, 5, 6, 6, 8, 8, 8, 8, 5, 5, 6, 6,
8, 8, 8, 8, 8, 10, 8, 6, 5, 5, 1, 1, 8, 10, 8, 6, 5, 5, 1, 1, 3, 3, -4, -4, 1, 1, 1, 1};
A = Map[SoundNote, f];
s1 = Sound[{"Piano", A},                    {0, 24}];
s2 = Sound[{"Flute", RotateRight[A, 8]},    {0, 24}];
s3 = Sound[{"Guitar", RotateRight[A, 16]}, {0, 24}];
Sound[{s1, s2, s3}, {0, 12}]
```

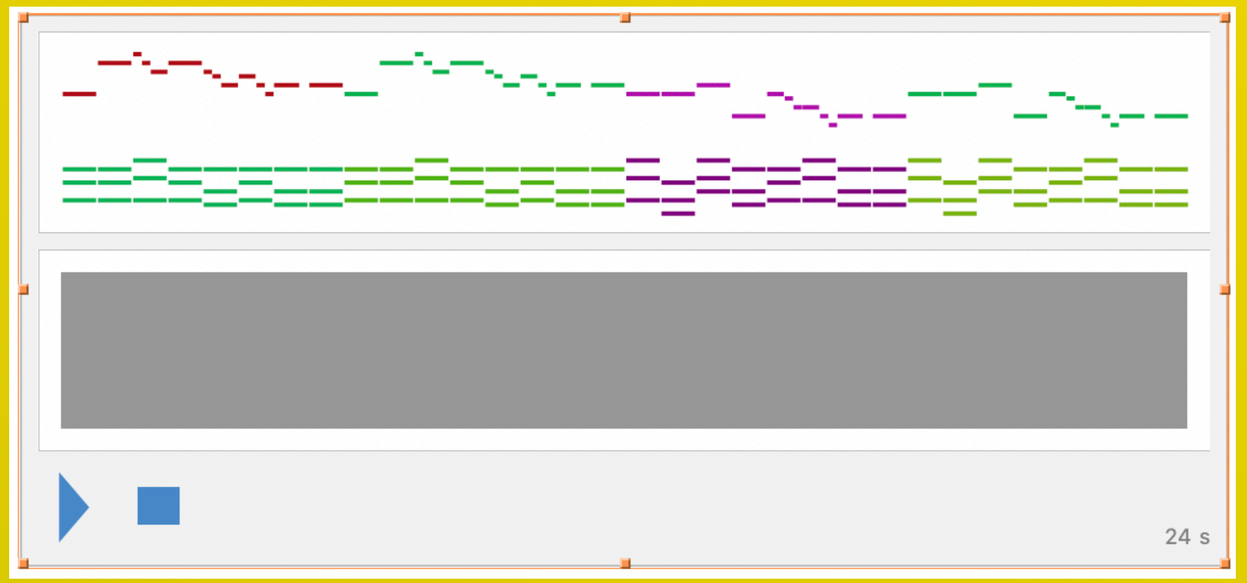


1, 1, 3, 3, 5, 5, 1, 1

Bro-ther-Ja-cob

SONGS

```
rise[x_] := Table[{x[[k, 1]] + 12, x[[k, 2]]}, {k, Length[x]}];
down[x_] := Table[{x[[k, 1]] - 12, x[[k, 2]]}, {k, Length[x]}];
SN[{x_, t_}] := SoundNote[x, t]; T = 6;
d = {1, 5, 8}; g = {1, 6, 10}; a = {0, 3, 8}; h = {-2, 1, 5}; e = {3, 6, 10};
a1 = {{{1}, 4}, {{8}, 4}, {{10}, 1}, {{8}, 1}, {{6}, 2}, {{8}, 4}, {{6}, 1}, {{5}, 1}, {{3}, 2}, {{5}, 2}, {{3}, 1}, {{1}, 1}, {{3}, 3}, {{}, 1}, {{3}, 4}};
b1 = {{{1}, 4}, {{1}, 4}, {{3}, 4}, {{-4}, 4}, {{1}, 2}, {{0}, 1}, {{-2}, 1}, {{-2}, 2}, {{-4}, 1}, {{-6}, 1}, {{-4}, 3}, {{}, 1}, {{-4}, 4}};
a2 = {{d, 4}, {d, 4}, {g, 4}, {d, 4}, {a, 4}, {d, 4}, {a, 4}, {a, 4}};
b2 = {{g, 4}, {h, 4}, {e, 4}, {a, 4}, {d, 4}, {g, 4}, {a, 4}, {a, 4}};
s1 = Sound[{"Flute", SN /@ rise[a1]}, {0 T, 1 T}];
s2 = Sound[{"Guitar", SN /@ down[a2]}, {0 T, 1 T}];
s3 = Sound[{"Viola", SN /@ rise[a1]}, {1 T, 2 T}];
s4 = Sound[{"Harp", SN /@ down[a2]}, {1 T, 2 T}];
s5 = Sound[{"Violin", SN /@ rise[b1]}, {2 T, 3 T}];
s6 = Sound[{"Piano", SN /@ down[b2]}, {2 T, 3 T}];
s7 = Sound[{"Viola", SN /@ rise[b1]}, {3 T, 4 T}];
s8 = Sound[{"Cello", SN /@ down[b2]}, {3 T, 4 T}];
S = Sound[{s1, s2, s3, s4, s5, s6, s7, s8}]
```



SONGS

```
(* Oliver Knill, May 5, 2020                                    *)
R[x_]:=RandomChoice[Permutations[x]];
f[{x_,t_}]:=Module[{a=R[x]},Table[{{a[[k]]},t/3},{k,3}]];
rise[x_]:=Table[{x[[k,1]]+12,x[[k,2]]},{k,Length[x]}];
down[x_]:=Table[{x[[k,1]]-12,x[[k,2]]},{k,Length[x]}];
SN[{x_,t_}]:=SoundNote[x,t];T=3;
d={1,5,8,13};g={1,6,10,13};a={0,3,8,12};
d1={1};e1={3};g1={6};f1={5};k1={8};g2={18};
d2={13};h1={10};c1={12};e2={15};k2={20};f2={17};
a1={{d,1},{d,1},{g,2},{d,2},{d,2},{a,4}};
a2={{a,1},{a,1},{a,2},{a,2},{a,2},{d,4}};
a3={{d,1},{d,1},{d,2},{d,2},{d,2},{g,4}};
a4={{g,1},{g,1},{d,2},{d,2},{a,2},{d,4}};
b1={{k1,1},{k1,1},{h1,2},{k1,2},{d2,2},{c1,4}};
b2={{k1,1},{k1,1},{h1,2},{k1,2},{e2,2},{d2,4}};
b3={{k1,1},{k1,1},{k2,2},{f2,2},{d2,2},{c1,2},{h1,2}};
b4={{g2,1},{g2,1},{f2,2},{d2,2},{e2,2},{d2,4}};
c1=Join[f[{d,1}],f[{d,1}],f[{g,2}],f[{d,2}],f[{d,2}],f[{a,4}]];
c2=Join[f[{a,1}],f[{a,1}],f[{a,2}],f[{a,2}],f[{a,2}],f[{d,4}]];
c3=Join[f[{d,1}],f[{d,1}],f[{d,2}],f[{d,2}],f[{d,2}],f[{g,4}]];
c4=Join[f[{g,1}],f[{g,1}],f[{d,2}],f[{d,2}],f[{a,2}],f[{d,4}]];
s1=Sound[{"Piano" ,SN/@down[a1]},{0T,1T}];
s2=Sound[{"Piano" ,SN/@down[a2]},{1T,2T}];
s3=Sound[{"Piano" ,SN/@down[a3]},{2T,3T}];
s4=Sound[{"Piano" ,SN/@down[a4]},{3T,4T}];
t1=Sound[{"Flute" ,SN/@rise[b1]},{0T,1T}];
t2=Sound[{"Flute" ,SN/@rise[b2]},{1T,2T}];
t3=Sound[{"Flute" ,SN/@rise[b3]},{2T,3T}];
t4=Sound[{"Flute" ,SN/@rise[b4]},{3T,4T}];
u1=Sound[{"Guitar",SN/@rise[c1]},{0T,1T}];
u2=Sound[{"Guitar",SN/@rise[c2]},{1T,2T}];
u3=Sound[{"Guitar",SN/@rise[c3]},{2T,3T}];
u4=Sound[{"Guitar",SN/@rise[c4]},{3T,4T}];
S=Sound[{s1,s2,s3,s4,t1,t2,t3,t4,u1,u2,u3,u4}];
Export["birthday.mp3",S,"MP3"]
```

With random components.

CREATIVITY

```
A = Table[RandomChoice[Range[80] - 40], {30}, {6}];
f[x_] := SoundNote[x, RandomChoice[{1, 2}]];
Sound[{"Piano", Map[f, A]}, {0, 6}]
```

TRY IT OUT!

WolframAlpha computational intelligence.

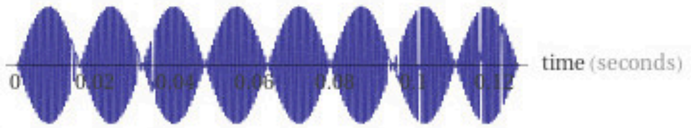
play sin(200 t) sin(10000 t)

Extended Keyboard | Upload | Examples | Random

Input interpretation:

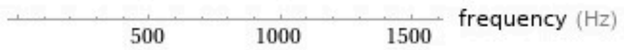
waveform | $\sin(200\,t)\,\sin(10\,000\,t)$

Waveform: Play Sound



(audible to the human ear)

Frequency power spectrum:



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POWERED BY THE WOLFRAM LANGUAGE

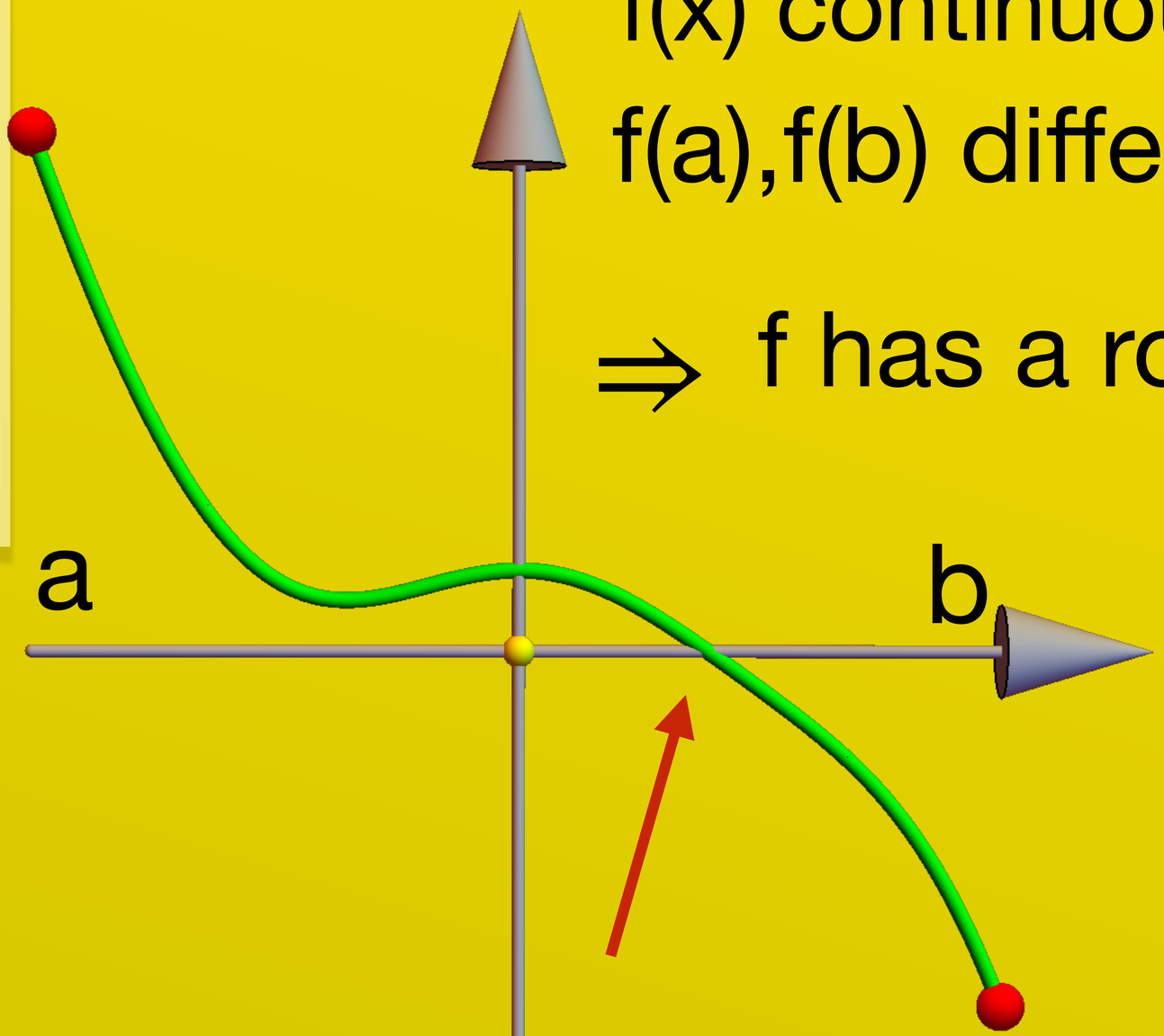
MOTION

INTERMEDIATE VALUE

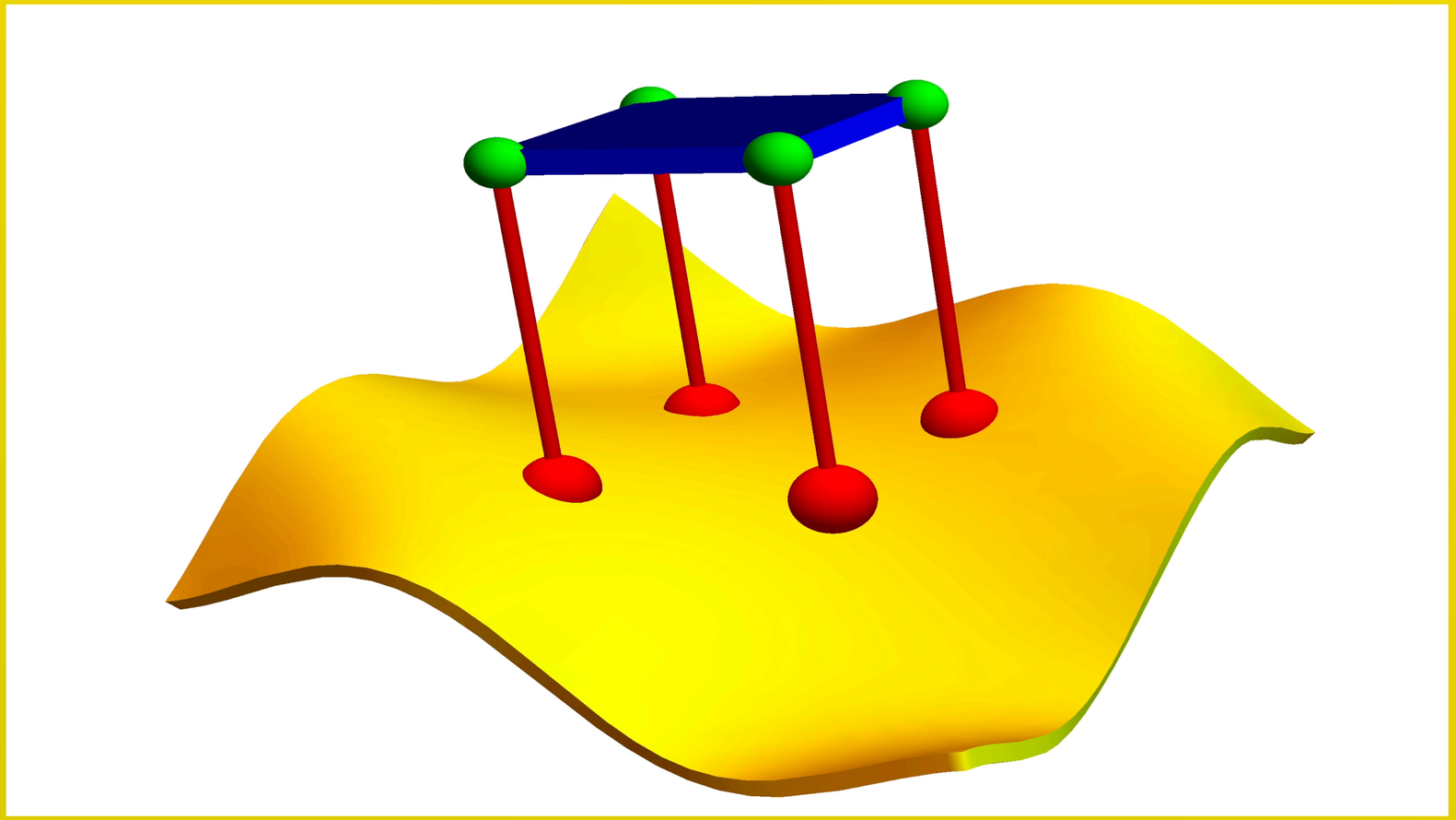
f(x) continuous on [a,b]

f(a),f(b) different signs

$\Rightarrow$ f has a root in [a,b]



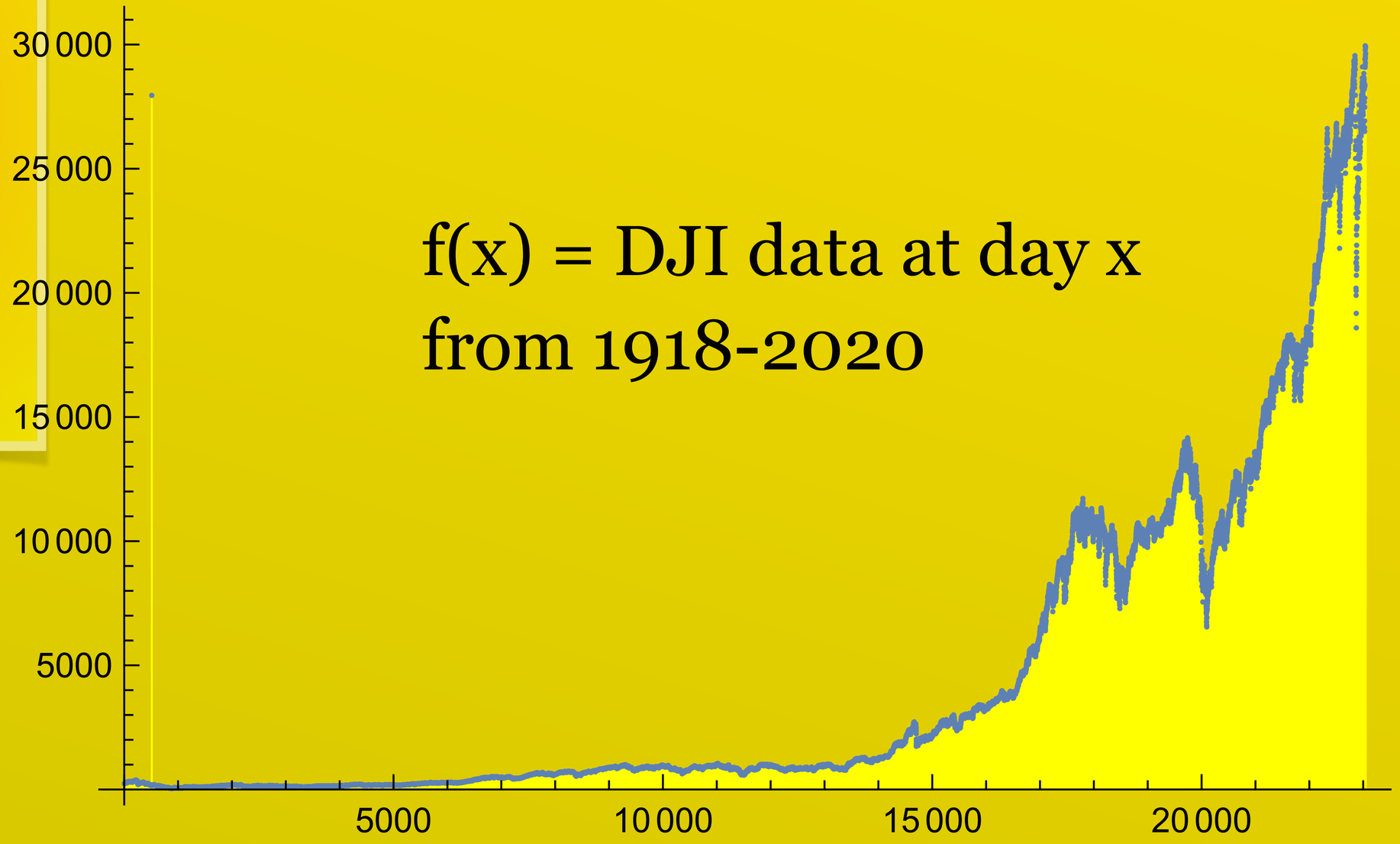
CONTINUITY



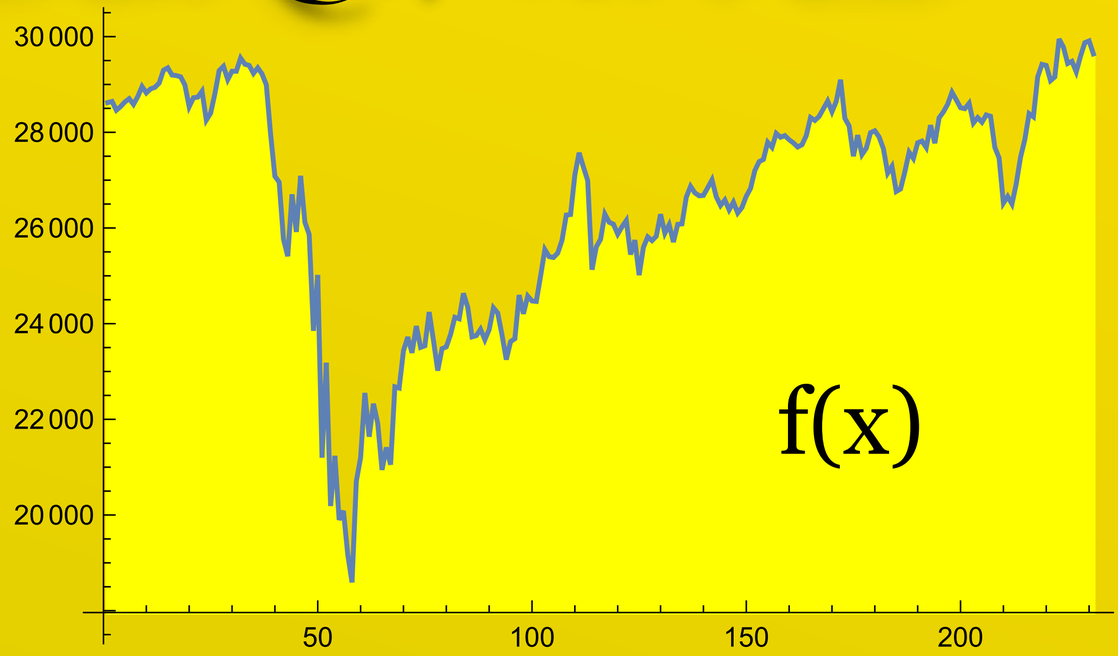
CONTINUITY

DERIVATIVES

f(x) = DJI data at day x
from 1918-2020



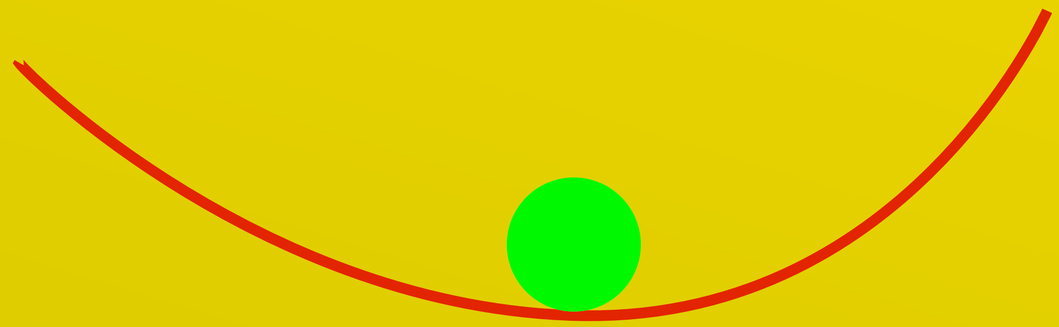
DERIVATIVES



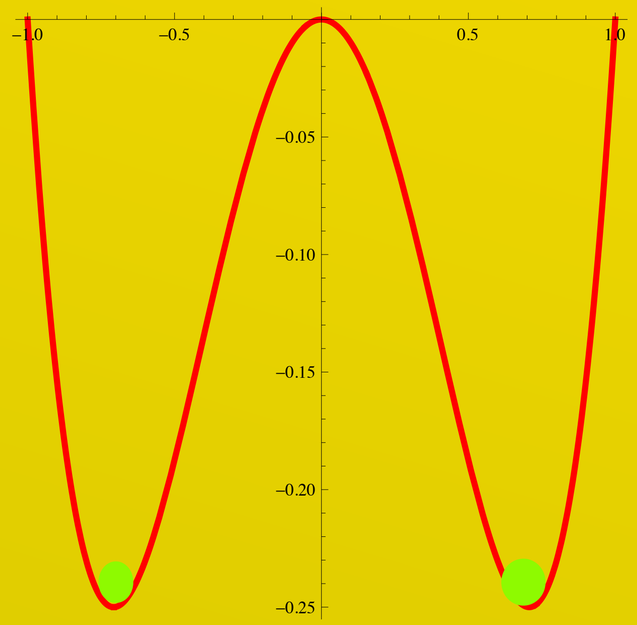
DJI change last year, 230 trading days so far

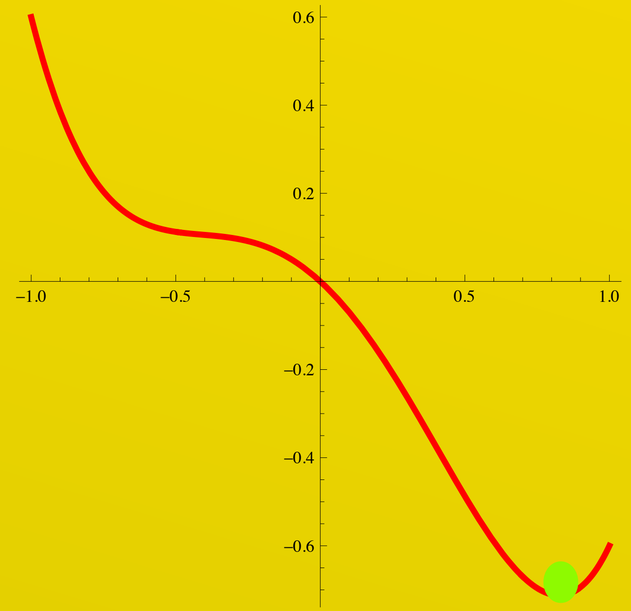
EQUILIBRIA

Basic Principle: A system wants to be in a minimal energy state.



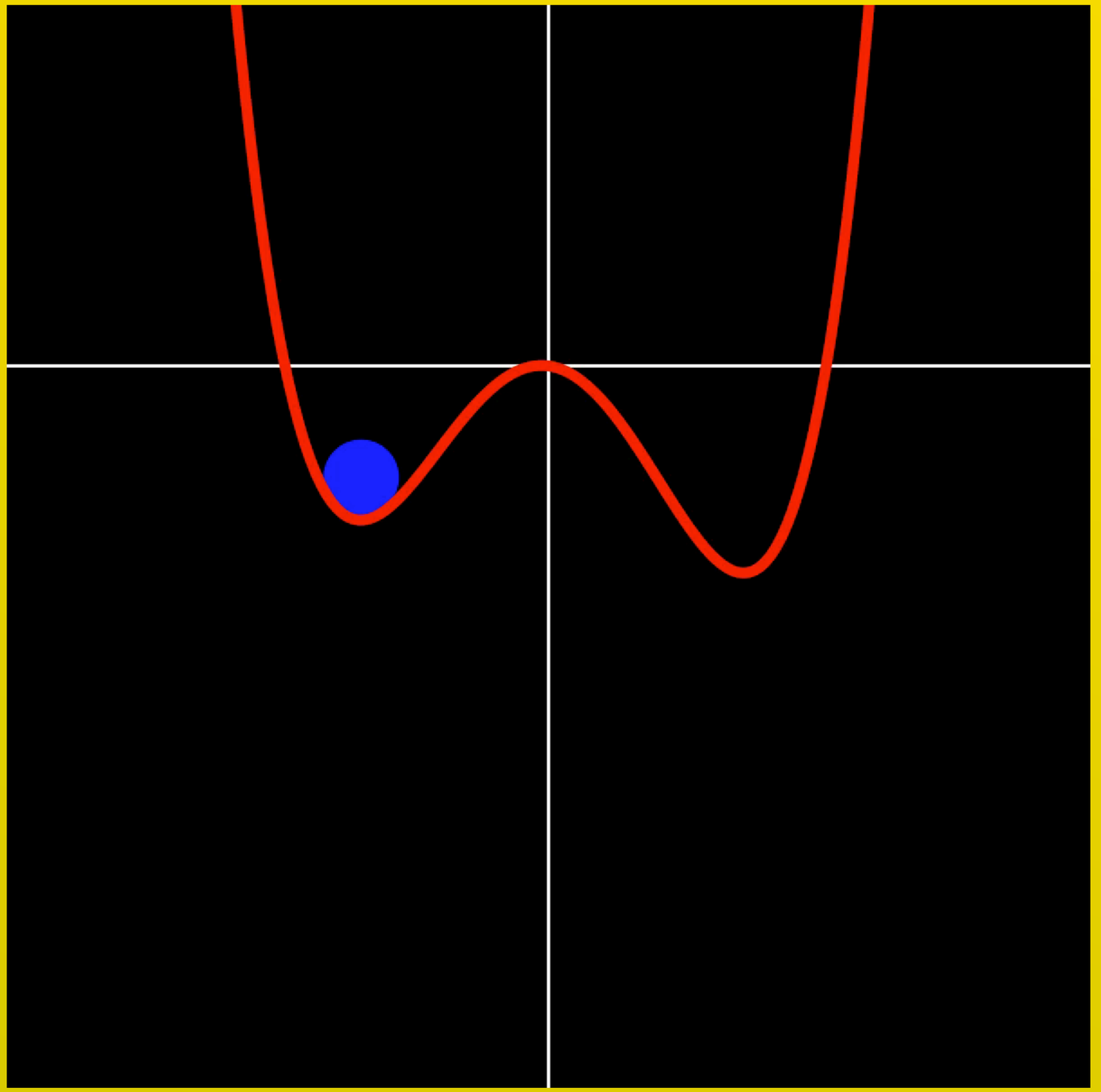
EQULIBRIUM CHANGE

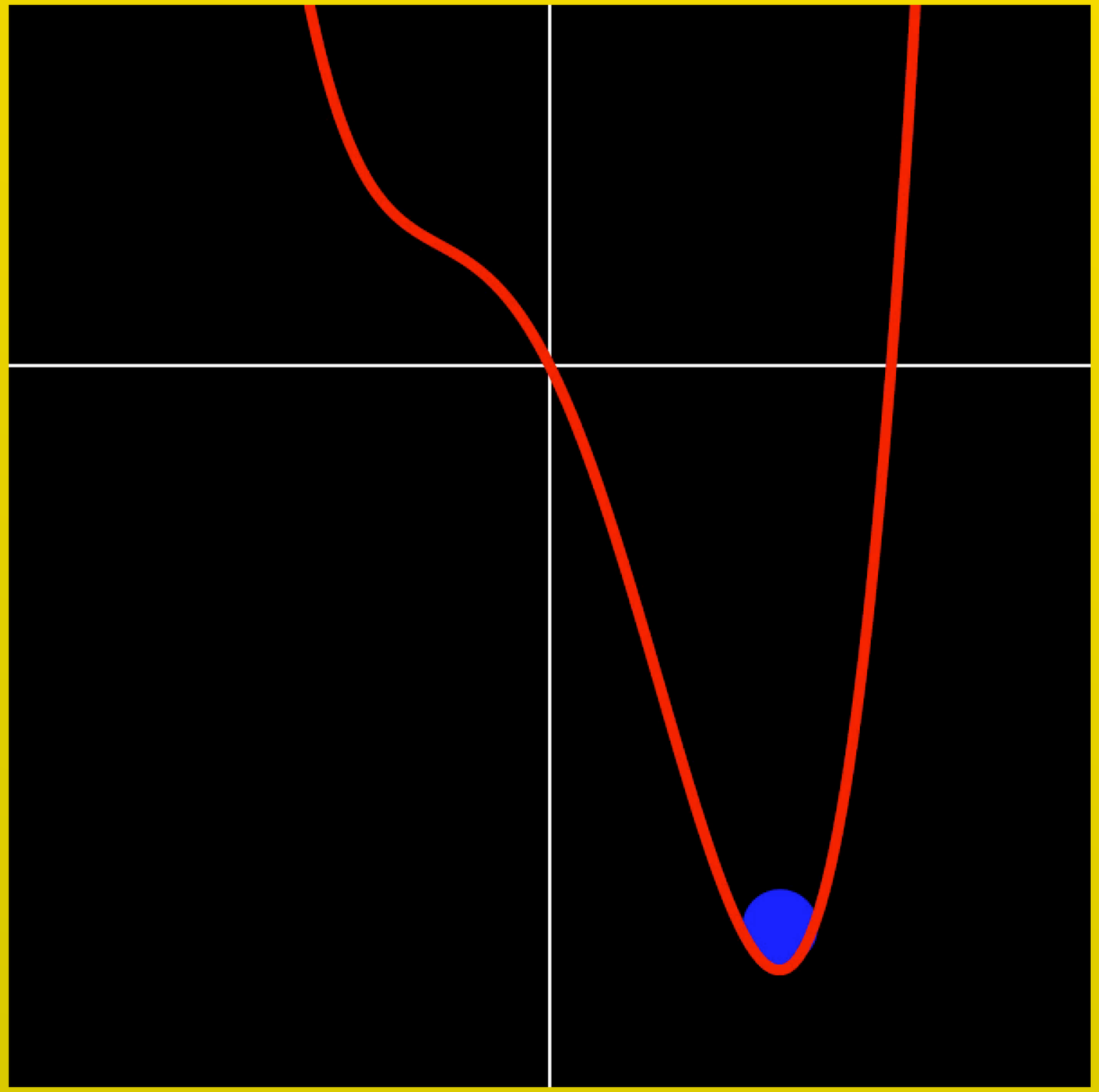




$$f(x) = x^4 - x^2$$

$$f(x) = x^4 - x^2 - 0.6x$$



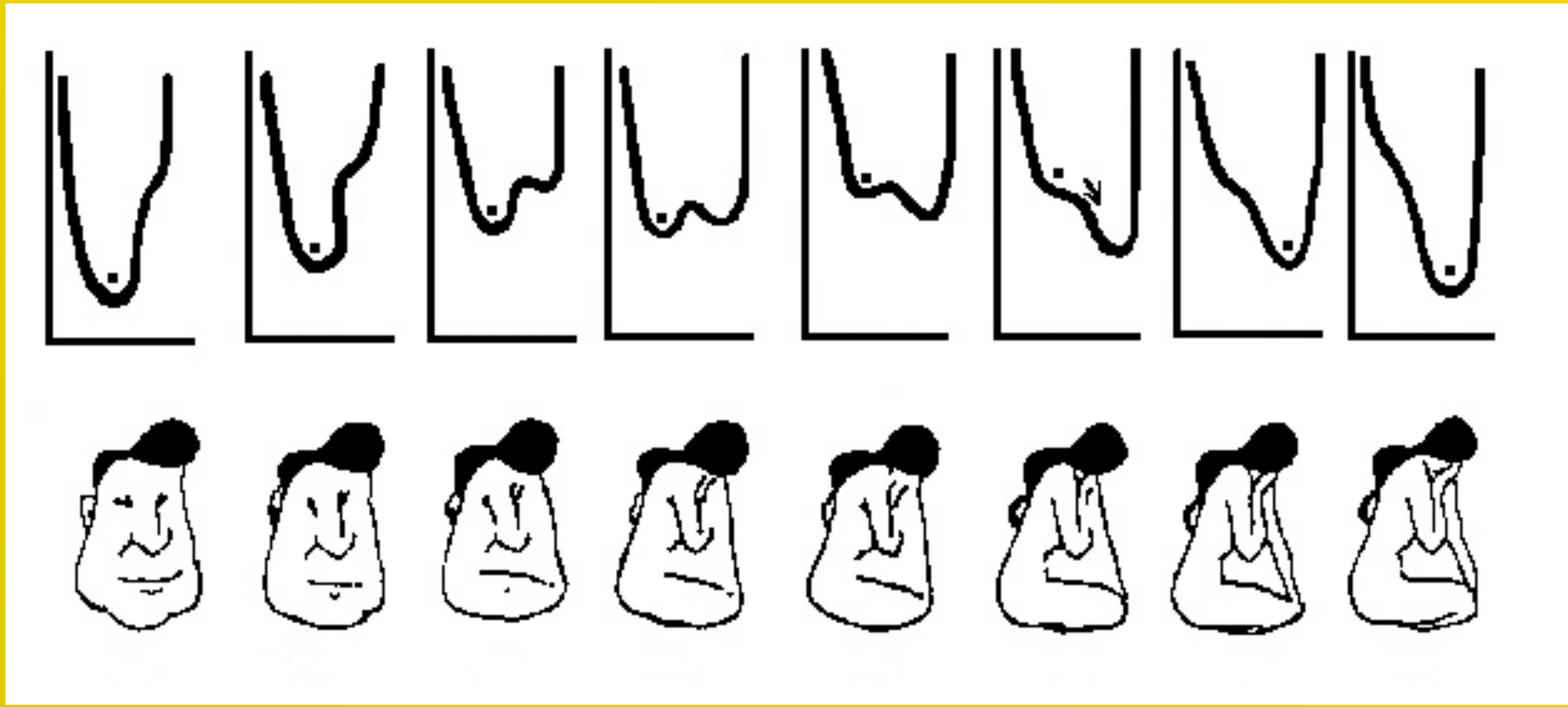


EQUILIBRIA IN PSYCHOLOGY

FLIPPING PERCEPTION

PERCEPTION

MATHEMATICAL LINK



CONFLICTS

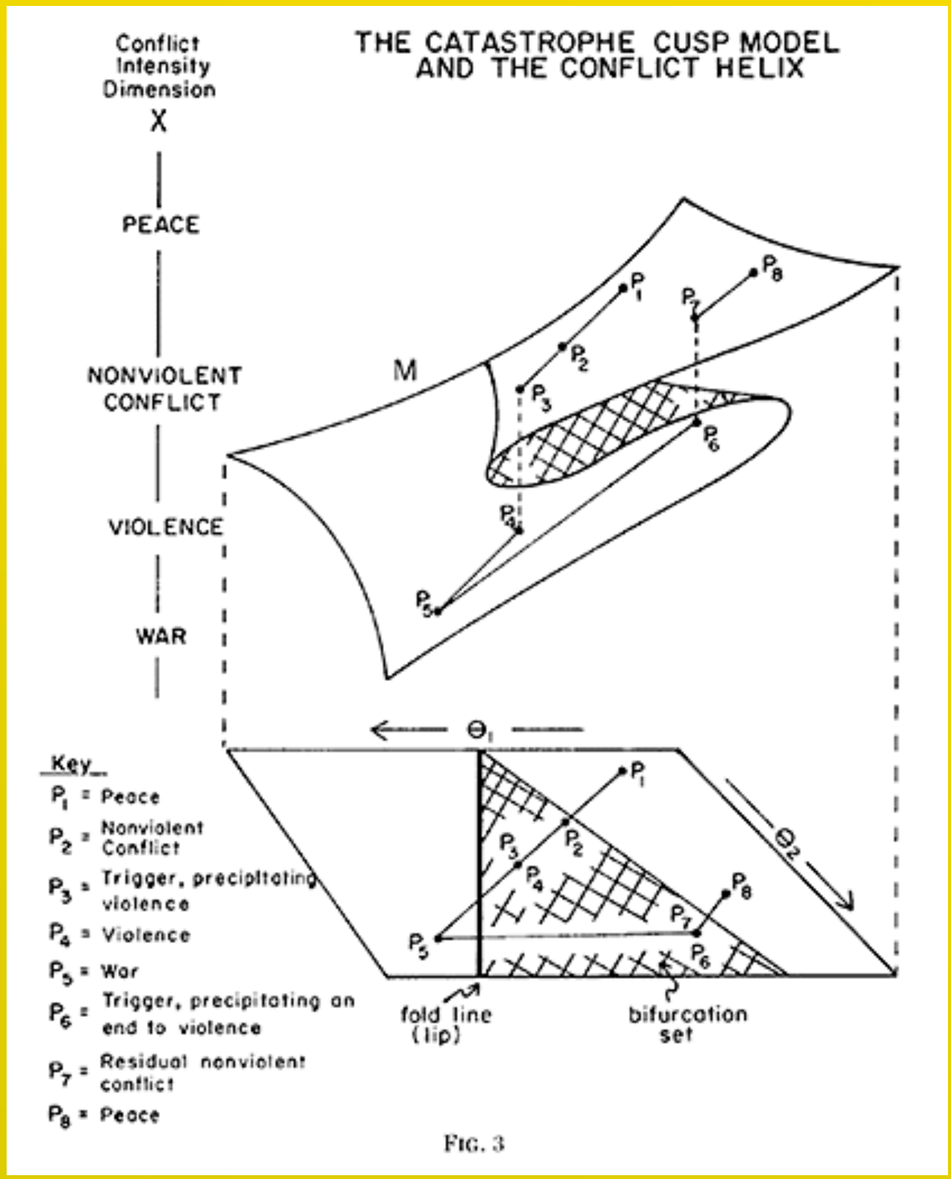
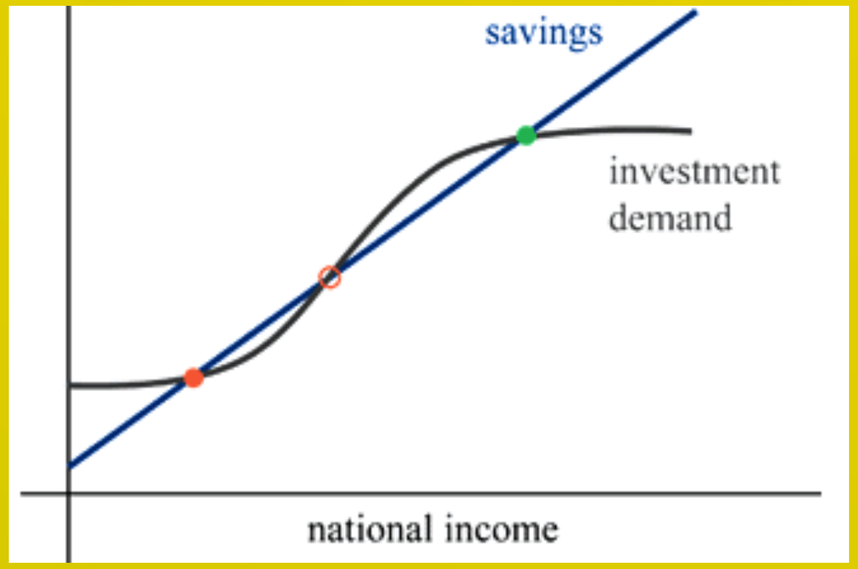


FIG. 3

ECONOMICS

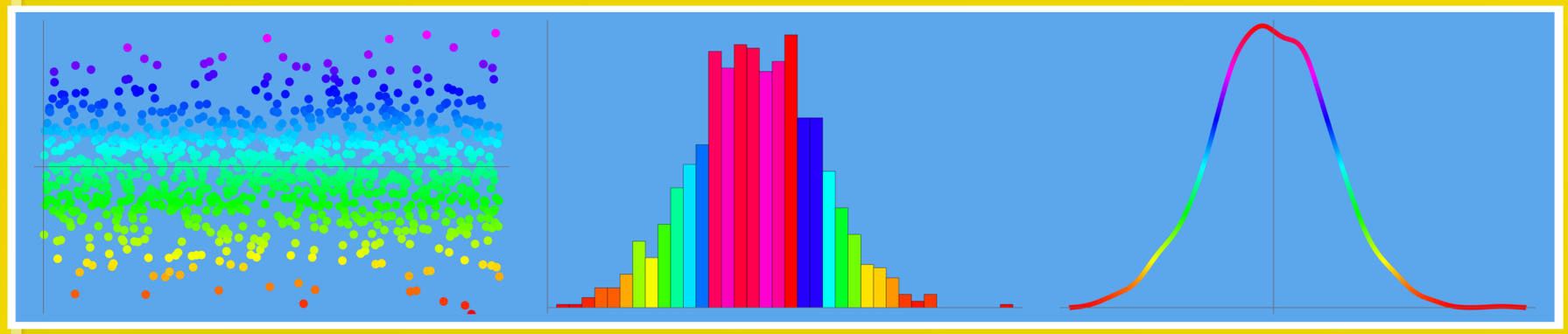


APROPOS PSYCHOLOGY

While sitting at your desk, lift your right foot off the floor and make clockwise circles. Now, while doing this draw the number "6" in the air with your right hand. Your foot will change direction and there's nothing you can do about it.

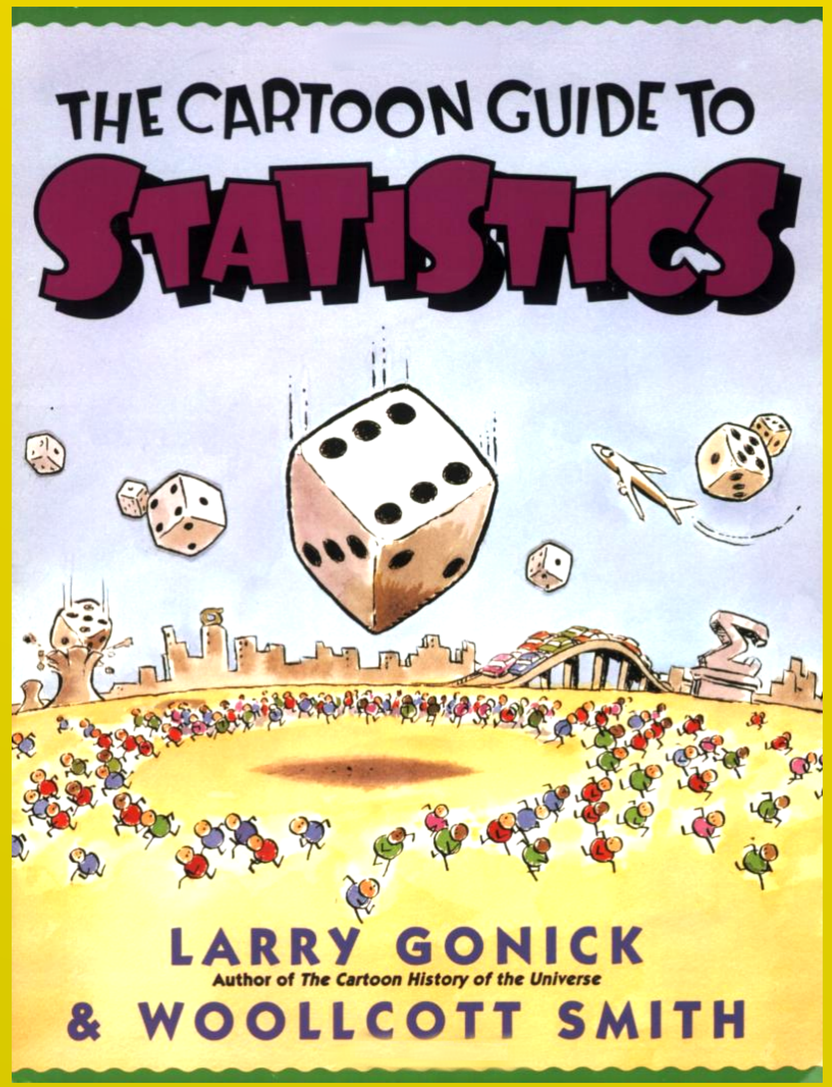
DATA TO FUNCTIONS

Data Histogram smoothed PDF

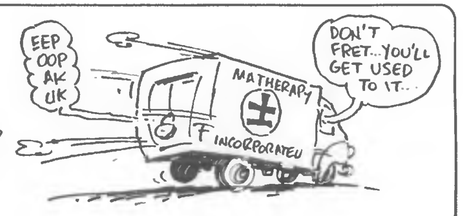


Data Histogram smoothed PDF

INFINITE INTEGRALS



ALTHOUGH THE NOTATION MAY BE UNFAMILIAR, ALL IT MEANS IS AN *AREA*.. THE INTEGRAL SIGN ITSELF IS A STRETCHED "S," FOR SUM, WHICH THE INTEGRAL, IN SOME SENSE, IS.



AS A SUMLIKE SOMETHING, THE INTEGRAL SERVES TO DEFINE THE

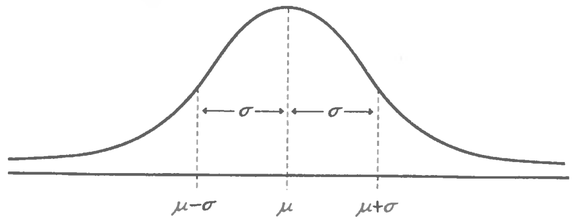
MEAN AND VARIANCE of a continuous random variable.

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

BY ANALOGY WITH THE DISCRETE FORMULAS:

$$\mu = \sum_{\text{all } x} x p(x) \qquad \sigma^2 = \sum_{\text{all } x} (x-\mu)^2 p(x)$$

ALTHOUGH IT MAY NOT BE OBVIOUS FROM THE FORMULAS, THESE DEFINITIONS OF MEAN AND VARIANCE ARE ENTIRELY CONSISTENT WITH THEIR ROLE AS CENTER AND AVERAGE SPREAD OF THE PROBABILITIES GIVEN BY THE DENSITY $f(x)$. THE PICTURE TO KEEP IN MIND IS THIS:



STATISTICS

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$ Assumption

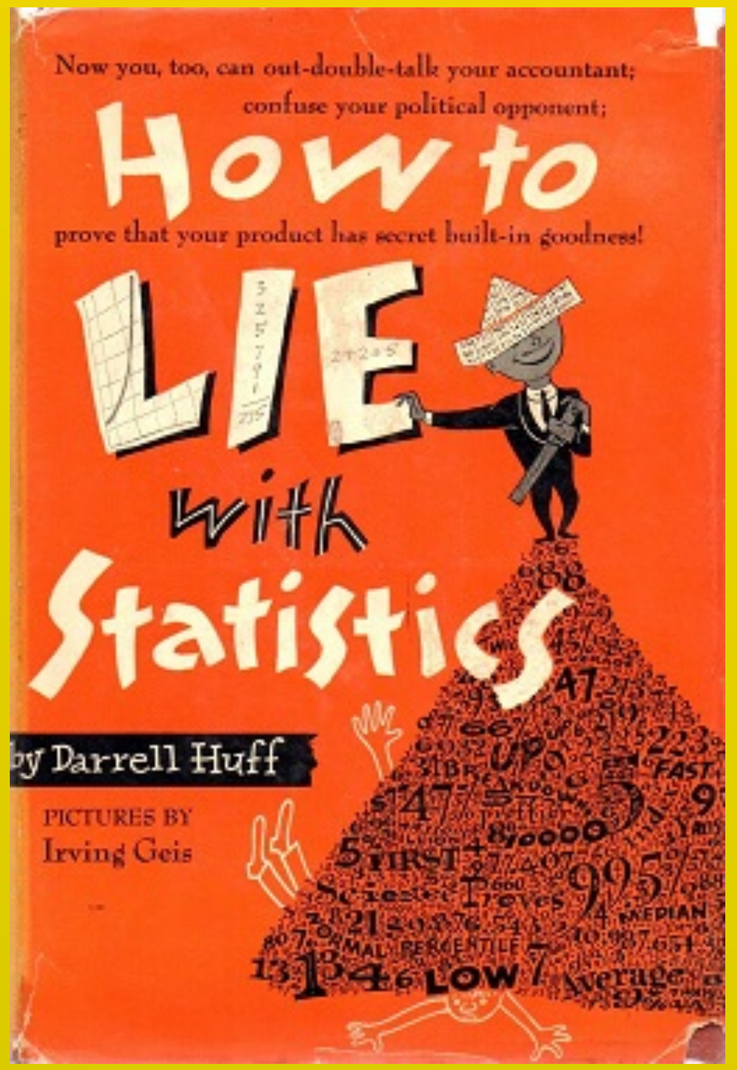
$$\int_{-\infty}^{\infty} xf(x)\,dx = m$$ Mean

$$\int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx$$ Variance

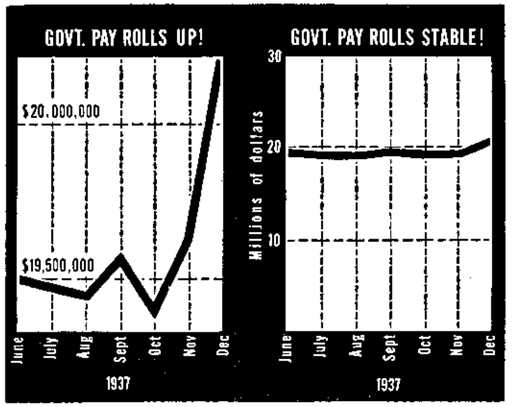
$$\sigma(f) = \sqrt{\operatorname{Var}(f)}$$ Standard Deviation

LIE WITH STATS

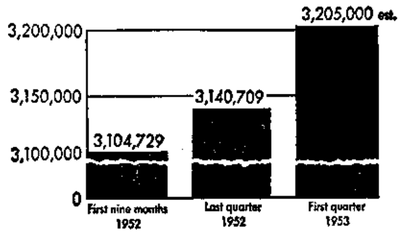
1954
Darrell Huff
1913-2001



honest red line that rose just four per cent, under this caption: GOVERNMENT PAY ROLLS STABLE.



Collier's has used this same treatment with a bar chart in newspaper advertisements. Note especially that the middle of the chart has been cut out:



From an April 24, 1953, newspaper advertisement for COLLIER'S

CALCULUS AND HUMOR

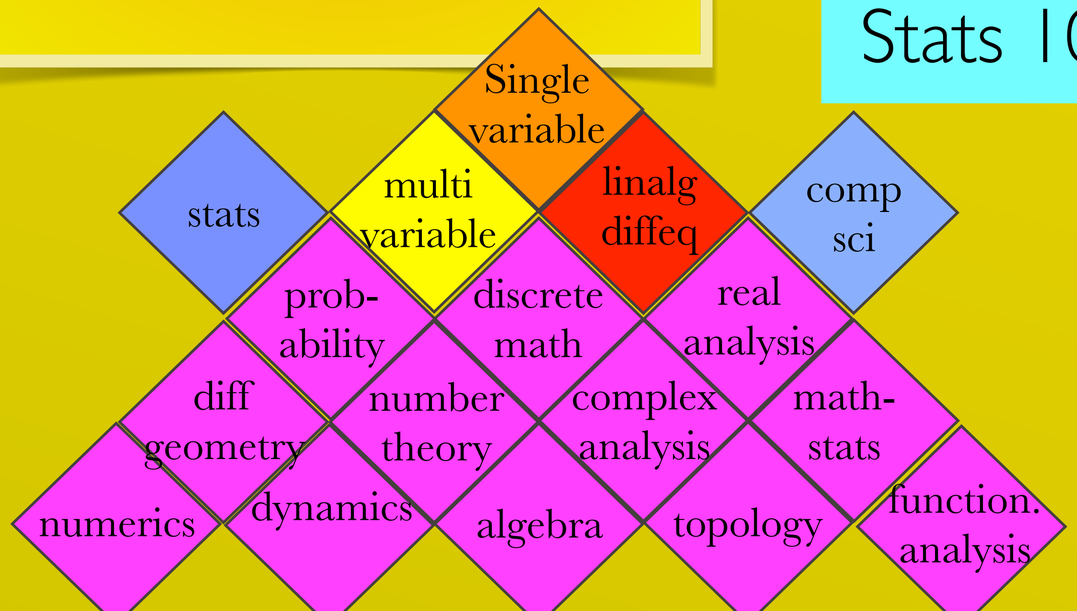
How do you make $\frac{1}{x}$ burn?

Differentiate a log

$$\frac{1}{x}$$

BEYOND 1A

Math 1B	Series and Diff equations
Math 21a	Multivariable Calculus
Math 21b	Linear Algebra and Diff Equ
Stats 101	Probability and Statistics



Computer Science

Physics Astronomy

Business Economics

BioChem LifeScience

Psychology Philosophy

THE END