CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 4: Integration techniques, 9/13/2021

Substitution

4.1. Identify part of the formula which you call u, then differentiate to get du in terms of dx, then replace dx with du. Example:

$$\int \frac{x}{1+x^4}\, dx \; .$$

Solution: Substitute $u = x^2, du = 2xdx$ gives $(1/2)\int du/(1+u^2)\; du = (1/2)\arctan(u) = (1/2)\arctan(x^2) + C$.

Remarks. Sometimes we have to try several times. In the example, we might first try $u = 1 + x^4$ but that does not give us a nice cancellation. If you should forget substitution, remember the chain rule. If $f(x) = g(u(x))$ then $f'(x) = g'(u(x))u'(x)$.

Integration by parts

4.2. Write the integrand as a product of two functions, differentiate one u and integrate the other dv. Then use $\int udv = uv - \int vdu$ from the product formula.

Example:

$$\int x\cos(x/3)\; dx$$

Solution: differentiate $u = x$ and integrate $dv = \cos(x/3)dx$. We have $3x\sin(x/3) - \int 1 \cdot 3\sin(x/3) = 3x\sin(x/3) + \cos(x/3)9 + C$.
Remarks. If you should forget the rule, remember the product rule $d(uv) = udv+vdu$ and integrate it, the solve for $\int u\; dv$.

Partial fractions

4.3. Use algebra to write a fraction as a sum of fractions we can integrate. Example:

$$\int \frac{1}{(x-3)(x-2)}\; dx$$

Solution: write $\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$.

To get A, multiply with $x-3$, cancel terms and put $x=3$ which gives $A=1$. To get B, multiply with $x-2$, cancel terms and put $x=2$ which gives $B=-1$.

Remarks. Most find the constants A, B by cross multiplication and comparing coefficients. The just explained method is **much faster**.

Trig substitution

4.4. Replace a term with $\sin(u)$ so that the formula simplifies.

Example: a prototype example is

$$\int \frac{1}{\sqrt{x^2-64}}\, dx \; .$$

Solution: $x = 8\sin(u)$ gives $\frac{1}{\sqrt{x^2-64}} = 1/(8\cos(u))$. As $dx = 8\cos(u)$. The integral is $\int 1/8 du = u/8 + C = \arcsin(x/8) + C$.

Trig identities

4.5. The double angle formulas $\cos^2(x) = (1+\cos(2x))/2$ and $\sin^2(x) = (1-\cos(2x))/2$ are handy. Also consider using $\cos^2(x) = 1 - \sin^2(x)$ or $\sin^2(x) = 1 - \cos^2(x)$ or use the identity $2\sin(x)\cos(x) = \sin(2x)$.

Example:

$$\int \sin^4(x)\, dx = \int (1-\cos^2(x))\sin^2(x) = \int \sin^2(x) - \sin^2(2x)/4\, dx$$

we can now use the double angle formulas to write this as $\int (1-\cos(2x))/2 - (1-\cos(4x))/8$ which now can be integrate $x/2 - \sin(2x)/4 - x/8 + \sin(4x)/32 + C$.

Symmetries

4.6. Sometimes, the result of an integral can be seen geometrically.

Example:

$$\int_{-2}^{2} \sin^7(5x^3)\, dx$$

is an integral we can not compute so easily by finding the anti derivative. However we see that the function in the integrand is odd. If we integrate an odd function over a symmetric interval, we have a cancellation. The answer is 0.

Remember

- No Homework is due on Wednesday
- Techniques of integration test is on Wednesday 9/15.