CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 9: Improper Integrals I, 9/24/2021

Improper integrals

9.1. If we integrate over an infinite interval we deal with an **improper integral**. In this first lecture we look at integrals of the form $\int_a^\infty f(x)\,dx$ where f is continuous.

Definition: If the limit $\lim_{b\to\infty} \int_a^b f(x)\,dx$ exists, we say the integral $\int_a^\infty f(x)\,dx$ **converges**. Otherwise, it **diverges**.

9.2.

Example: To decide about $\int_1^\infty \frac{1}{x^4}\,dx$, integrate from $x=1$ to $x=b$

$$\frac{-1}{3x^3}|_1^b = (1/3 - 1/(3b^3)) \; .$$

and take the limit $b \to \infty$. We see that limit is $1/3$. The integral converges.

9.3.

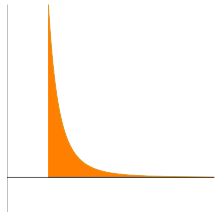
Example: What is $\int_1^\infty \frac{1}{x^{1/4}}\,dx$? Since the anti-derivative is $\frac{4}{3}x^{3/4}$, we have $\int_1^b \frac{1}{x^{1/4}}\,dx = \frac{4}{3}x^{3/4}|_1^b = \frac{4}{3}(b^{3/4}-1)$. The limit $b \to \infty$ does not exist. The integral diverges.

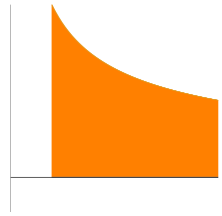
9.4.

Example: The integral $\int_1^\infty e^{-3x}\,dx$ converges.

9.5.

Example: The integral $\int_1^\infty \frac{1}{x}\,dx$ diverges because $\int_1^b \frac{1}{x}\,dx = \log(b)$ diverges for $b \to \infty$.





The figure illustrate the functions $1/x^4$ and $1/x^{1/4}$.

p-integrals

9.6. Bu explicit integration of the p-integral $\int_1^b \frac{1}{x^p}\,dx$, (do it yourself!), we see:

$\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$ and diverges for $p \leq 1$.

9.7. The value $p = 1$ is a threshold. It is important that on the threshold $p = 1$ we still have divergence.

Comparison test

9.8. The **comparison test** is

If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty f(x)\,dx$ converges then $\int_a^\infty g(x)\,dx$ converges.

9.9. By contraposition, we see

If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

For example, $\int_1^\infty (5 + \sin(x^5))e^{-x}\,dx$ converges because the integrand $g(x)$ is bounded above by $f(x) = 6e^{-x}$ which leads to a finite integral.

9.10. Note that in the result, it is assumed f, g are non-negative. Without that assumption, the result is wrong in general. Can you see why?

An application

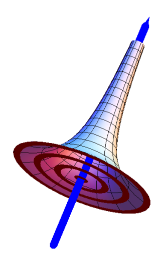
9.11. If we rotate $f(x) = 1/x$ around the x-axes for x between 1 and ∞ we get a trumpet. Its cross section area is π/x^2. The volume up to height b is

$$\int_1^b \frac{\pi}{x^2}\,dx = -\frac{\pi}{x}|_1^b = -\pi/b + \pi \to \pi\ .$$

The **surface area** A of the trumpet is

$$A = 2\pi \int_1^b \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx\ .$$

We can use the comparison test see that the integral diverges. We can fill the trumpet with paint but can not paint its surface!



On the radar and remarks

- HW 8 on is due Monday. Our first midterm is on October 4th.

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