

CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 10: Improper Integrals II, 9/27/2021

IMPROPER INTEGRALS FOR UNBOUNDED FUNCTIONS

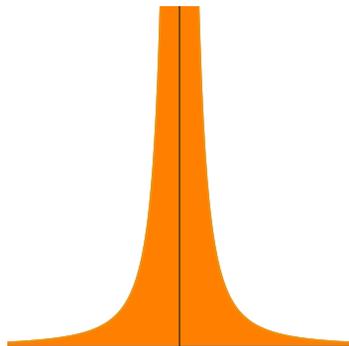
10.1. In this second lecture on improper integrals we look also at integrals of the form $\int_a^b f(x) dx$, where f can become unbounded at some point in the interval $a \leq x \leq b$. An other theme are integrals $\int_{-\infty}^{\infty} f(x) dx$.

Definition: If f is continuous except at $x = a$, then the limit $\lim_{a \rightarrow 0} \int_a^b f(x) dx$ exists we say the integral $\int_0^b f(x) dx$ **converges**. Otherwise, we say it **diverges**.

Definition: In the case when we look at an integral like $\int_{-1}^1 f(x) dx$, where f is unbounded at 0, we ask that both improper integral limits $\lim_{a \rightarrow 0} \int_a^1 f(x) dx$ and $\lim_{a \rightarrow 0} \int_{-1}^{-a} f(x) dx$ must exist.

10.2. In the following example, we see what can go wrong if one integrates blindly:

Example: What is $\int_{-1}^1 \frac{1}{x^2} dx$? We see the function $1/x^2$ on the interval $[-1, 1]$. Integrating blindly gives the value $-1/2$. This is obviously nonsense as the function is positive.



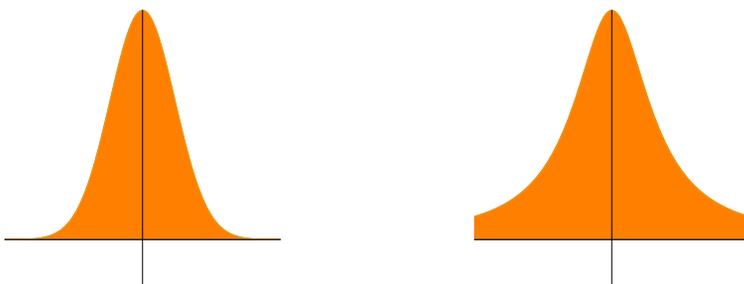
Definition: In the case of an integral like $\int_{-\infty}^{\infty} f(x) dx$, we need both sides $\int_0^{\infty} f(x) dx$ and $\int_{-\infty}^0 f(x) dx$ to exist.

10.3.

Example: What is $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$? In this case we can verify that the integral converges and is π . Do it!

10.4.

Example: What is $\int_{-\infty}^{\infty} e^{-x^2} dx$. In this case we can not integrate but see that the integral converges. See the worksheet. We know from a movie that the integral is $\sqrt{\pi}$.



The functions $\exp(-x^2)$ and $1/(1+x^2)$. Both are important in statistics. The first one leads to the **normal distribution**, the second leads to the **Cauchy distribution**.

P-INTEGRALS

10.5. By explicit integration of the p-integral $\int_a^1 \frac{1}{x^p} dx$, (do it yourself!), we see:

$\int_0^1 \frac{1}{x^p} dx$ converges for $p < 1$ and diverges for $p \geq 1$.

10.6. The value $p = 1$ is a threshold. As before, on the threshold $p = 1$ we still have divergence.

COMPARISON TEST

10.7. The same comparison test works as in the case of the last lecture. If $0 \leq g(x) \leq f(x)$ and the integral $\int_0^1 f(x) dx$ converges then the integral $\int_0^1 g(x) dx$ converges.

10.8.

Example: What is

$$\int_0^1 \frac{(\sin(4x) + 3x + \cos(x))}{\sqrt{x}} dx .$$

In this case we can not integrate but we see that the integral converges by applying the comparison test. Can you see the p-integral which is an upper bound?

ON THE RADAR AND REMARKS

- HW 9 is due Wednesday
- Our first midterm is on October 4th, today in a week.

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