CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 11: Taylor Approximation I, 9/29/2021

Finite degree approximations

11.1. Given a nice function $f(x)$ we can approximate it with a **quadratic Taylor approximation**

$$f(x) \sim P_2(x) = f(0) + f'(0)x + f''(0)x^2/2$$

This is a more refined approximation to the **linear approximation**

$$f(x) \sim P_1(x) = f(0) + f'(0)x \ .$$

11.2. We can justify this by just pretending that f is a quadratic function, $f(x) = a_0 + a_1 x + a_2 x^2$, then differentiate a few times and see what we get when we put $x = 0$. Here we go

$$f(0) = a_0$$

$$f'(0) = a_1$$

$$f''(0) = 2a_2$$

We can read off a_0, a_1, a_2 from this and see that the quadratic function is $f(0)+f'(0)x+f''(0)\frac{x^2}{2}$.

> **Example:** If $f(x) = \log(1 + x)$, then $f'(0) = 1, f''(0) = -1$. The quadratic approximation of $\log(1 + x)$ is $x - x^2/2$.

11.3. You can see from this how to continue. If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ then $f^{(k)}(0) = a_k \frac{d^k}{dx^j} x^k = a_k k!$. The expression $k! = k(k-1)(k-2)\cdots 1$ is called the **factorial**. Solving for a_k gives $a_k = f^{(k)}(0)/k!$. The **n'th degree Taylor series** of f is the **polynomial**

$$f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!} \ .$$

With the sum notation, we can write this as

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(0)\frac{x^k}{k!} \ .$$

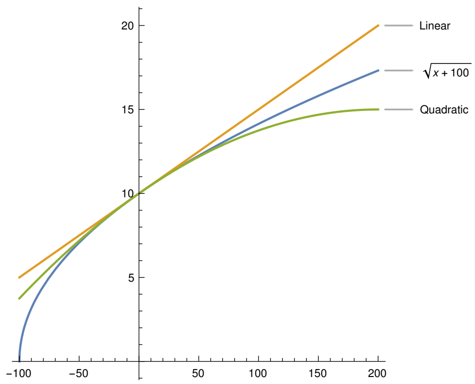
11.4.

Example: If $f(x) = \exp(x)$, then all derivatives are 1 at 0. The n'th degree approximation therefore is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} .$$

11.5.

Example: If $f(x) = \sqrt{100+x}$, then $f(0) = 10$, $f'(0) = 1/20$, $f''(0) = -1/8000$ and $f'''(0) = 1/1600000$. The cubic approximation is $10 + x/20 - x^2/8000 + x^3/1600000$. If we evaluate this at $x = 1$, we get a value $1607981/160000$ which is 5 million'th close to $\sqrt{101}$.



11.6.

Example: Going back to $f(x) = \log(1+x)$, we have the 5-th degree approximation

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} .$$

11.7. We do not really need to start the approximation at $x = 0$. By replacing 0 with c and x with $x - c$, we have the

The n'th degree approximation of a function f at a point c is the polynomial

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(c) \frac{(x-c)^k}{k!} .$$

On the radar and remarks

- HW 10 is due Friday
- Our first midterm is on October 4th, the upcoming Monday.

Lecture 11: Some worksheet problems

Remember that $P_2(x) = f(0) + f'(0)x + f''(0)x^2/2$ is the quadratic Taylor polynomial centered at $c = 0$.

PROBLEM 1

a) Write down the quadratic Taylor polynomial $P_2(x)$ for $f(x) = \sqrt{10000 + x}$.

b) Compare $\sqrt{10001}$ with $P_2(1)$.

PROBLEM 2

a) Write down the quadratic Taylor polynomial $P_2(x)$ for the function $f(x) = \sin(3x)$ centered at $c = 0$.

b) Compare $f(0.1) = \sin(0.3)$ with $P_2(0.1)$.

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