Lecture 13: Taylor Error Term, 10/4/2021

THE ERROR TERM

13.1. The Taylor approximation of a function f at a point c is the polynomial

$$P_n(x) = \sum_{k=0}^{n} f^{(k)}(c) \frac{(x-c)^k}{k!} \ .$$

Definition: The **error term** is the function $R_n(x) = |f(x) - P_n(x)|$. Define M_n as the maximum of $|f^{(n)}(y)|$ for y between c and x.

13.2. The Lagrange form of the error tells:

$$|R_n(x)| \leq M_{n+1} \frac{(x-c)^{n+1}}{(n+1)!}.$$

13.3.

Example: For $f(x) = \exp(x)$ and $c = 0$ we have for $x > 0$ the error term $R_n(x) = e^x x^{n+1}/(n+1)!$. For $x = 1$, this gives the bound $R_n(1) = e/(n+1)!$. For $n = 5$ This is $R_5(1) = e/720 = 0.00377$. The actual error is $e - (1 + 1 + 1/2 + 1/6 + 1/24 + 1/120) = 0.00161$. The upper bound was about twice as good.

13.4.

Example: If $f(x) = \sqrt{x}$ and $c = 100$, the error term for $x = 110$ is $R_2(x) = M_3(x-100)^3/6$, where M_3 is the maximal third derivative on the interval $[100, 110]$ which is $\frac{3}{8}100^{-5/2}$. We can take $M_3 = 1/200000$. The error is bounded by

$$|R_3(x)| \leq \frac{1}{200000} 10^3/6 = 1/1200 = 0.00083333$$

The Taylor series of degree 2 at $x = 100$ is

$$10 + (x-100)/20 - (x-100)^2/8000 = 10.4875 \ .$$

The actual value is 10.4881 which is about -0.000588 off, smaller!

Calculus and Differential equations

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