CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 14: Taylor Series, 10/6/2021

Power series

14.1.

Definition: An function which is a sum $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is called a **power series**.

An example is the sum $f(x) = \sum_{n=0}^{\infty} x^n$. Obviously this does not always make sense. Take for example $x = 1$, then we get infinity. But for $x = 1/2$ this gives a finite sum 1 (we will talk about series later more in this course).

Taylor series

14.2.

Definition: The **Taylor series** of a function f at a point c is the series

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(c)\frac{(x-c)^k}{k!} = f(c) + f'(c)\frac{x-c}{1} + f''(c)\frac{(x-c)^2}{2!}... .$$

The infinite series converges if the **error term** $R_n(x) = |f(x) - P_n(x)|$ goes to zero.

14.3.

Example: For $f(x) = \exp(x)$ we have the series

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

14.4.

Example: For $f(x) = \exp(x^2)$, just substitute x^2 in the above to get

$$f(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + ...$$

14.5.

Example: For $f(x) = \sin(x)$ we have the series

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$$

14.6.

Example: For $f(x) = \sin(x)/x$ (just divide all terms by x, we have

$$f(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + ...$$

14.7.

Example: For $f(x) = \cos(x)$ we have the series

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$$

14.8.

Example: For $f(x) = 1/(1-x)$ we have at $c = 0$ the derivatives $f'(x) = -1/(1-x)^2, f''(x) = 2/(1-x)^3, f'''(x) = 6/(1-x)^4, ...$ We see that $f^{(k)}(0) = k!$ and so

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

This is an important series, called the **geometric series**.

14.9.

Example: For $f(x) = \log(1+x))$ we have the series

$$f(x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

14.10.

Example: For $f(x) = \arctan(x))$ we have the series

$$f(x) = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

Verify this by differentiating both sides using the geometric series

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6....$$

Cool application: $\pi/4 = \arctan(1) = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$

- HW 12 is due Wednesday together with HW 13