CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 16: Divergence and Comparison, 10/12/2021

Divergence

16.1. We have seen that $\lim_{n \to \infty} a_n = 0$ is necessary for convergence of $\sum_n a_n$. We can restate that as follows. It is called the **n'th term test**:

If $\lim_{n \to \infty} a_n$ does not exist or if the limit exists and is not 0, then the series $S = \sum_k a_k$ is divergent.

But $\lim_{n \to \infty} a_n \to 0$ does not guarantee convergence. The prototype is the **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$ which will be one of the main characters in this lecture and in our course.

16.2.

Example: The sum

$$S = k \sum_{k=1}^{\infty} (1 + \frac{\sin(k)}{k}$$

does not converge because the numbers a_k converge to 1.

Example: The sum

$$S = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ ...$$

is called the **harmonic series**. We can see with the comparison test below why it diverges. Similarly as in the homework, you can group things

$$S = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ ...$$

to see that the sum diverges.

Example: The **Grandi's series** (named after Guido Grandi)

$$S = 1 - 1 + 1 - 1 + 1 \ ... = \sum_{k=0}^{\infty} (-1)^k$$

is an example, where the limit $\lim_n a_n$ does not exist. This automatically implies that the sum $S = \sum_k a_k$ does not converge. You have been asked in a homework whether one can use $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ for the Grandi value $x = -1$.

Example: The **Harmonic series**

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \ ...$$

diverges also because we have seen in a previous homework that the Taylor expansion of $-\log(1-x)$ is $x + \frac{x^2}{2} + \frac{x^3}{3} + ...$ Evaluated at $x = -1$ this gives infinity and already is an indication why the harmonic series diverges. We will see also with a comparison test that the series diverges.

Comparison test

16.3. The comparison test looks similar as for improper integrals.

If $0 \leq |a_k| \leq b_k$ and $\sum_k b_k$ converges, then $\sum_k a_k$ converges.

16.4.

Example: Verify that the sequence $\sum_k a_k = \sum_k \sin(2^k)/2^k$ converges. Solution: look at $\sum_k b_k = \sum_k 1/2^k$. We have $0 \leq |a_k| \leq b_k$ and $\sum_k b_k$ converges, therefore $\sum_k a_k$ converges.

16.5. One can also reverse this.

If $0 \leq b_k \leq a_k$ and $\sum_k b_k$ diverges then $\sum_k a_k$ diverges.

Example: Show that the series $\sum_k a_k = \sum_k (1 + \sin(k))/k$ diverges. Solution: Look at the series $\sum_k b_k = \sum_k 1/k$. This is the harmonic series. It diverges. Therefore $\sum_k a_k$ diverges.

Reminders

16.6.

- HW 15 is due Friday
- The QRD product is due this Friday.
- The techniques test recovery on gradescope 10/15 and due Thursday 10/21