CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 17: Geometric series, 10/15/2021

Geometric series

17.1. The **geometric series** $S = \sum_{n=0}^{\infty} x^n$ is no doubt the most important series in mathematics. Do not mix it up with $S = \sum_{n=1}^{\infty} n^x$ which is called the **zeta function** and which for $x = -1$ is the **harmonic series** we have seen last time. It is traditionally written as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, where the case $s = 1$ is the harmonic series.

17.2. Which of the following series are geometric series?

a) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + ...,$ b) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + ...$
c) $1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \frac{1}{243} - ....$ d) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + ...$
Can you write them down in sum notation? By the way, we often write the geometric series also as $\sum_{k=0}^{\infty} ar^k$, using the letter r instead of x and allowing for a more general constant a. Remember it as $\boxed{\frac{a}{1-r} = a + ar + ar^2 + ar^3 + ...}$.

17.3. a) A first justification for

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

is to multiply both sides with $1 - x$!
b) A second justification is to make the Taylor expansion of $f(x) = \frac{1}{1-x}$ at $x = 0$ and to see that $f^{(k)}(0) = k!$. c) A third justification is to look at the **finite sum** formula for the partial sum S_n below and which generalizes expressions like $1 + x = \frac{x^2-1}{x-1}$ or $1 + x + x^2 = \frac{x^3-1}{x-1}$ or $1 + x + x^2 + x^3 = \frac{x^4-1}{x-1}$.

Convergence

17.4.

The geometric series converges for $|x| < 1$.

The reason is the formula for the **partial sums**

$$S_n = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} .$$

If $|x| < 1$, then $x^{n+1} \to 0$ and the partial sums $S_n = \frac{1-x^{n+1}}{1-x}$ converge to $\frac{1}{1-x}$.

17.5. This matches all your experiments done so far. Remember the case $x = 1/10$ for example, where $1 + 1/10 + 1/100 + 1/1000 = 1.111111 = 10/9$ works.

17.6. The geometric series diverges for $x = 1$ and $x = -1$. In the case $x = 1$, the partial sums converge to infinity, in the case $x = -1$, we have the Grandi sum which oscillates and does not converge. The Harmonic series formula gives $1/(1 - x) = 1/(1 - (-1)) = 1/2$ and diplomatically chooses the middle of the oscillation but there is no identity.

Applications

17.7. Historically, the geometric series first appeared in **Zeno's paradox**. Assume Archilles races a turtle who is given an advance of 1 mile. Assume the turtle can run x times slower than Archilles. Both start running. Once Archilles reaches the start point of the turtle, the turtle is x miles ahead at $S_1 = 1 + x$. Once Archilles the point S_1 the turtle is at $S_2 = 1 + x + x^2$, if Archilles reaches S_2 the turtle is ahead again at $1 + x + x^2 + x^3$. Since the turtle is always ahead of Archilles, Artchilles will never be able to catch up. We see that the fallacy is that an infinite sum is not necessarily infinite. There is a definite point when Archilles will overtake the turtle and that is if Archilles has run $1/(1 - x)$ miles. This works. If both would run with the same speed $x = 1$, then the time to catch up would be infinite.

17.8. Next in history comes a computation of Archimedes about the **quadrature of the parabola**. Look at a line segment connecting $A = (a, a^2), B = (b, b^2)$ in a parabola. This defines a triangle with third point $C = ((a + b)/2, (a + b)^2/4)$. The two new segments AC and BC again define triangles as such of area $x = 1/4$ times of the original triangle. Repeat like this and add up all these triangle areas. They fill up the area bound by the parabola and the original segment. This area is therefore $1/(1 - x) = 4/3$ times the area of the original triangle.

17.9. The third application appears in fractal geometry and will be covered in the worksheet. We can compute the **area of the Koch snowflake**.

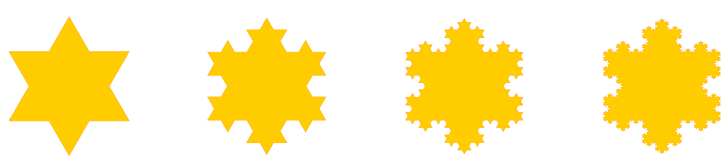

Figure 1. The first 4 approximations of the Koch snowflake.

17.10.

- HW 16 is due Monday
- The techniques test recovery on gradescope today 10/15 and due Thursday 10/21

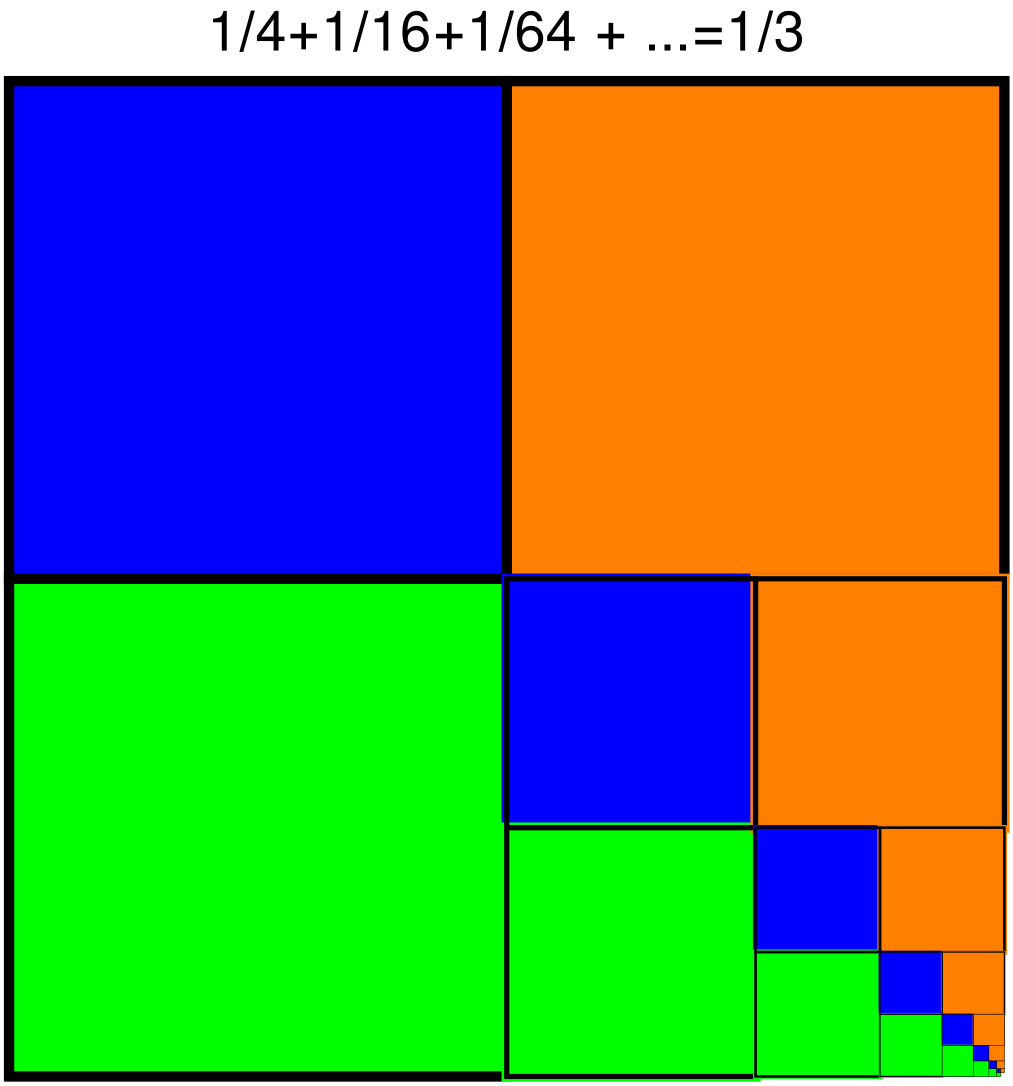

Figure 2. The identity $\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots=\frac{1}{3}$ proven with a picture.

Oliver Knill, knill@math.harvard.edu, Math 1b, Harvard College, Fall 2021