Lecture 18: p-series and p-integrals, 10/18/2021

p- series

18.1. The **p-series** $S = \sum_{n=1}^{\infty} \frac{1}{n^p}$ is an important benchmark series. It is also called the **zeta function** if one looks at it as a function of p.

18.2.

Example: For $p = 1$, we have the **Harmonic series**

$$S = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ...$$

which diverges.

Example: For $p = 0$, we have the series

$$S = 1 + 1 + 1 + ...$$

which diverges.

Example: The case $p = 2$ was solved by Euler first and called the Basel problem. It is

$$S = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$$

It turns out to be finite and have the value $\pi^2/6$.

Convergence

18.3. We can interpret S as an integral $\int_1^{\infty} f(x)\, dx$, where $f(x)$ is piecewise constant. Let us look at the case $p = 2$, where the function $f(x)$ is 1 for $0 \leq x \leq 1$ and $f(x) = 1/4$ for $1 \leq x \leq 2$ etc. Now $S = \int_1^{\infty} f(x)\, dx$. The sum is the **right Riemann sum** of the function f with spacing $\Delta x = 1$.

The p-series converges for $p > 1$ and diverges for $p \leq 1$.

The reason is that we can for any p define a piecewise constant function $f(x)$ such that $S = \int_1^{\infty} f(x)\, dx$ and such that $f(x) \leq 1/x^p$. Now remember what we knew about p-integrals. The integral converged for $p > 1$ and diverged for $p \leq 1$. We have been able to decide about convergence by comparing the sum with an integral.

Comparison

18.4. The comparison test from last time can be applied now: if $0 \leq |a_k| \leq b_k$ and $\sum_k b_k$ converges, then $\sum_k a_k$ converges.

Example:

$$S = 1 - 1/4 + 1/9 - 1/16 + 1/25 - ... = \sum_k a_k$$

converges because $|a_k| \leq b_k = |a_k| = 1/k^2$. The value by the way is $\pi^2/12$.

Example: The sum

$$S = \sum_{k=1}^{\infty} \frac{7}{k^5 + 3}$$

converges by comparison with a multiple of a p-series for $p = 5$.

Example: The sum

$$S = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

diverges because it is the p-series for $p = 1/2$ which according to the integral convergence test diverges.

Example: The sum

$$S = \sum_{k=2}^{\infty} \frac{1}{\ln(x)}$$

diverges because $1/\ln(x) > 1/x$ for $x \geq 2$.

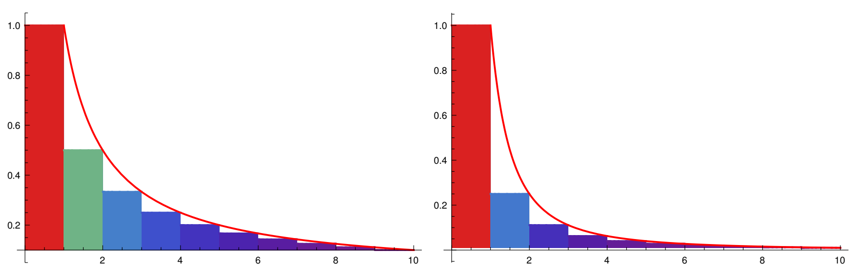

Figure 1. The function belonging to the Harmonic series (p=1) and the Basel problem series (p=2). The area under the curve is the sum in each case. $\int_1^\infty 1/x \, dx$ diverges and $\int_1^\infty 1/x^2 \, dx$ converges.

Reminders

- HW 17 is due Wednesday