CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 19: Asymptotics, 10/20/2021

Growth rates

19.1. We want to get a feel for which series converge and which diverge. In order to do that we need to see how fast something grows or decays asymptotically. Here, in the context of series, we are always interested in the limit $k \to \infty$. So, we say $3x^2/(4x^2+x) \to 3/4$ for example.

19.2.

Definition: We write $\boxed{a_k \sim b_k}$ and say "a_k **behaves like** b_k" if $a_k/b_k \to 1$.

Definition: We write $\boxed{a_k \ll b_k}$ and say "a_k **grows slower than** b_k" if $a_k/b_k \to 0$.

19.3.

Example: The numbers $a_k = k^5 + k^3$ behaves like $b_k = k^5$ because both grow asymptotically the same: we have $a_k/b_k = 1 + \frac{k^3}{k^5} \to 1$. We write therefore $a_k \sim b_k$.

19.4.

Example: The numbers $a_k = k^2$ grow slower than $b_k = 2^k$. Indeed $k^2/2^k \to 0$. How do we see this? The **l'Hopital rule** is a good way to see this. Since both k^2 and 2^k go to infinity, we can bring things to the Hospital. The limit is the same than $2k/(k\ln(2)2^k)$. An other Hospital visit gives $2/((\ln(2)+k^2\ln(2)^2)2^k$ which goes to zero.

19.5. It is good to have a feel for growth rates. If you need to compute something which depends on the size k, then it matters whether the task grows linearly, or logarithmically or quadratically or polynomially or exponentially or super exponentially:

Example	growth rate
$a_k = \sin(k)$	bounded
$a_k = \ln(k) + 11$	logarithmic
$a_k = \sqrt(k) + 4$	square root
$a_k = 3k + 4$	linear
$a_k = k^2 + 3k - 5$	quadratic
$a_k = k^{11} - \ln(k)$	polynomially
$a_k = e^{\sqrt{k}}$	sub-exponentially
$a_k = e^k + k^3$	exponentially
$a_k = k!$	factorial
$a_k = k^k$	super-exponentially

19.6. Asymptotically simplifying and expression means replacing terms with asymptotically equivalent terms, that means terms which behave in the same way.

Example:

$$\frac{5k^3 + k}{4k^2 + \ln(k)} \sim \frac{5k^3}{4k^2} \sim \frac{5k}{4} .$$

Example:

$$\frac{k! + k^k + k^5}{k^7 + 2^k} \sim \frac{k^k}{2^k} .$$

Example:

$$\frac{k^5 + 8}{\sqrt{k^11 + k^4 + 3}} \sim \frac{k^5}{k^{11/2}} .$$

Example:

$$\frac{\sqrt{k}}{\sqrt{k+1}} \sim \frac{\sqrt{k}}{\sqrt{k}} \sim 1 .$$

The **limit comparison test is**

Theorem: If $a_k \sim b_k$ then $\sum_k |a_k|$ converges if and only if $\sum_k |b_k|$ converges.

Example: Does

$$\sum_k a_k = \sum_k \frac{(4k^2 + 4k)(3k + 1)}{\sqrt{9k^9 + k^2 + 1}}$$

converge? Yes, because $a_k \sim 12k^3/3k^{9/2} \sim 4k^{-3/2} = b_k$. We know that $\sum_k b_k$ converges as it is 4 times the $p = 3/2$ series.

- HW 18 is due Friday

Oliver Knill, knill@math.harvard.edu, Math 1b, Harvard College, Fall 2021