

# CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

## Lecture 22: Power series, 10/27/2021

### POWER SERIES

**22.1.** A series

$$S(x) = \sum_{k=0}^{\infty} a_k x^k .$$

is a **power series**. More generally one can center them at a point  $c$

$$S(x) = \sum_{k=0}^{\infty} a_k (x - c)^k .$$

**22.2.** An important class of power series are **Taylor series**

$$S(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k .$$

**Example:**  $\sum_{k=0}^{\infty} x^k$  which is the Taylor series of  $1/(1-x)$ .

**Example:**  $\sum_{k=0}^{\infty} \frac{x^k}{k!}$  which is the Taylor series of  $\exp(x)$ .

### RADIUS OF CONVERGENCE

**22.3.** For any power series, there is an interval  $(c - R, c + R)$  centered at  $c$  on which the series converges. This is called the **interval of convergence**. The number  $R$  is called the **radius of convergence**.

**22.4.** If  $R$  is the radius of convergence then for  $|x - c| < R$ , the series converges for  $|x - c| > R$  the series is divergent.

**Example:** For the series  $\sum_{k=1}^{\infty} \frac{x^k}{k}$  for example, the series converges for  $|x| < 1$  by the ratio test we have seen before. For  $|x| > 1$  the series does not converge by the  $n$ 'th term test.

For  $x = 1$ , we have the **Harmonic series**, where the series diverges. For  $x = -1$  we have an **alternating series** which converges. The limit is  $\log(2)$ . We say then the "maximal interval of convergence" is  $[-1, 1)$  as the left point is included.

**22.5.** Here is a formula for the radius:

$$R = \lim_{k \rightarrow \infty} |a_k|/|a_{k+1}|, \text{ if the limit exists.}$$

**22.6.** Indeed, for  $|x - c| < R$ , the terms  $b_k = a_k(x - c)^k$  satisfy  $|b_{k+1}|/|b_k| = |x - c|/R < 1$  so that  $\sum_k b_k$  converges by the **ratio test**. For  $|x - c| > R$ , the terms satisfy  $|b_{k+1}|/|b_k| = |x - c|/R > 1$  so that  $\sum_k b_k$  diverges by the **n'th term test**.

WHY ARE THEY USEFUL?

**22.7.** Power series are mostly used for **Taylor series**. This allows us to work with functions by taking **polynomial approximations** with control about the error.

**22.8.** A power series not coming from a named function  $f$  is the **prime function**

$$S(x) = \sum_{p \text{ prime}} x^p = x^2 + x^3 + x^5 + x^7 + x^{11} + \dots$$

The coefficients of  $S(x)^2 = x^4 + 2x^5 + x^6 + 2x^7 + 2x^8 + 2x^9 + 3x^{10} + \dots$  tell in how many ways one can write a given number as a sum of two primes. The **Goldbach conjecture** states that every even number can be written as such. The **Goldbach comet** plotting the coefficients of the power series of  $S(x)^2$  illustrates that.

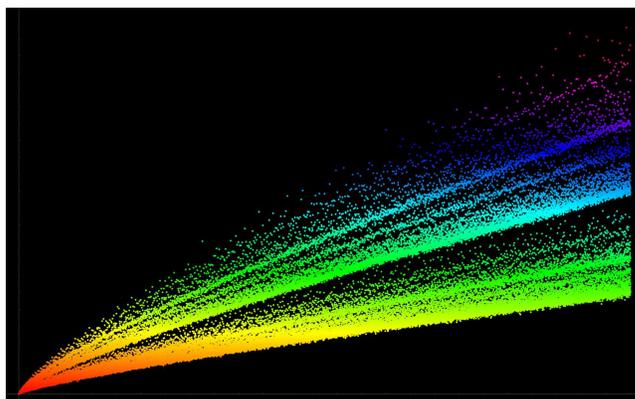


FIGURE 1. The Goldbach comet gives the coefficients of  $b_k$ .

**22.9.** An other application are **moment generating function** in statistics.

$$S(x) = \sum_{n=0}^{\infty} \frac{E[X^n]x^n}{n!}$$

where  $E[X]$  denotes the expectation or mean of a random variable  $X$ . Moments are important because for example the variance of a random variable is  $E[X^2] - E[X]^2$  and so expressible using moments.

EXAMPLES

- HW 21 is due Friday