CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 23: Series Overview, 10/29/2021

Series

Sum or series	$S = \sum_{k=0}^{\infty} a_k$
Partial sum	$S_n = \sum_{k=0}^{n} a_k$
Error term	$R_n = \sum_{k=n+1}^{\infty} a_k$
Taylor polynomial	partial sum to **Taylor series**
Polynomial	partial sum to **power series**

Convergence

Convergence	$R_n \to 0$
Divergence	$R_n \nrightarrow 0$
Absolute convergence	$\sum_{k=0}^{\infty} \lvert a_k \rvert$ converges
Conditional convergence	convergence but not absolute
Interval of convergence	maximal open interval $(c-R, c+R)$ of convergence
Radius of convergence	radius R of open interval

Convergence tests

N-term test	$a_k \nrightarrow 0$ implies divergence
Alternating series test	a_k alternating $\lvert a_k \rvert \to 0$ monotonically
Ratio test	$\lvert a_k \rvert / \lvert a_{k+1} \rvert \to r < 1$
Direct Comparison test	$0 \leq \lvert a_k \rvert \leq b_k$ and $\sum_k b_k$ converges
Asymptotic Comparison test	$a_k \sim b_k$ and $\sum_k b_k$ converges
Integral comparison test	$\lvert a_k \rvert \leq f(k)$ and $\int_n^{\infty} f(x)\,dx$ converges

Types of series

Power series	$\sum_k a_k (x-c)^k$
Taylor series	$\sum_k \frac{f^{(k)}(c)}{k!} (x-c)^k$
p-series	$\sum_k \frac{1}{k^p}$
Geometric series	$\sum_k ar^k$
Alternating series	$\sum_k (-1)^k b_k$ with b_k not changing sign.

Error bounds

Taylor series	$M_{n+1}\frac{\|x-c\|^{n+1}}{(n+1)!}$
Alternating series	a_{n+1}
Geometric series	$\frac{a_{n+1}}{1-r}$
Integral approximation	$\int_n^\infty f(x)\, dx$

Hall of fame

Cake series	$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$	2
Harmonic series	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+...$	∞
Basel series	$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+...$	$\pi^2/6$
Euler series	$1+\frac{1}{1}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...$	e
Log(2) series	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-...$	$\log(2)$
Leibniz series	$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...$	$\pi/4$
Grandi series	$1-1+1-1+1-...$	1/2 (*)
Youtube series	$1+2+3+4+5+...$	$-1/12$ (*)
Silly series	$1+1+1+1+1+...$	$-1/2$ (*)

Party knowledge

23.1. The p-series $\sum_{k=1}^\infty k^{-p}$ is also known as the **Riemann zeta function**

$$\zeta(s) = \sum_{k=1}^\infty k^{-s} .$$

$\zeta(1)$	Harmonic series
$\zeta(2)$	Basel series
$\zeta(-1)$	Youtube series
$\zeta(0)$	silly series
$\zeta(4) = \pi^4/90$	to really impress your friends

23.2. Of course, the series marked (*) are divergent by the n'th term test. Why $\zeta(-1) = -1/12$, $\zeta(0) = -1/2$, you have to look at youtube or read the book "Divergent series" by G.H. Hardy. To see why the Grandi series has a natural value 1/2, interpret it as a power series $\sum_k x^k = 1/(1-x)$ which for $x = -1$ which gives the value 1/2.

P.S. If you should solve the Riemann hypothesis settling, where the non-trivial roots of $s \to \zeta(s)$ are, tell Oliver. He will do the rest and arrange a trust fund with the million dollars for you. By the way, the trivial roots are $\zeta(-2) = \zeta(-4) = ... = 0$. For example, $\zeta(-2) = 1 + 4 + 9 + 16 + 25 + 36 + 49 + \cdots$ which is obviously zero. If you don't believe it, type $Zeta[-2]$ into Mathematica.

Reminders

The 2nd hourly on Monday 6-7:30 PM