CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 24: Representing functions, 11/01/2021

Power Series

24.1.

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

is called a **power series**. The **ratio test** allows to find the **radius of convergence** R as well as the **largest open interval of convergence** $(c-R, c+R)$. For example, the **geometric series** $\sum_{k=0}^{\infty} x^k$ has the interval of convergence $(-1, 1)$.

24.2. In the Taylor series case and x in the interval of convergence, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$$

The rest term $\sum_{k=n+1}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$ goes to zero for $n \to \infty$.

24.3.

For every x in the $(c-R, c+R)$, the Taylor series $S_n(x)$ converges to $f(x)$.

24.4. We can make use of this and compute Taylor series of functions we do not know before.

Example: Find the Taylor series of $f(x) = x^7/(1-x^9)$ at $c = 0$.

Solution: We know

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots .$$

Now replace x with x^9 to get

$$\frac{1}{1-x^9} = \sum_{k=0}^{\infty} x^{9k} = 1 + x^9 + x^{18} + x^{27} + \cdots .$$

Finally multiply with x^7 to get

$$\frac{x^7}{1-x^9} = \sum_{k=0}^{\infty} x^{7+9k} = x^7 + x^{16} + x^{25} + x^{35} + \cdots .$$

24.5. Having a Taylor series representation, we can use the series as the function. This looks like a crazy idea at first because the Taylor series appears to be more complicated. But say, you wanted to compute

$$\int_0^1 e^{-x^2}\, dx \ .$$

While no elementary function gives the anti-derivative of $f(x) = e^{-x^2}$ we can write

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...$$

and integrate term by term to get

$$\int_0^1 e^{-x^2}\, dx = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + ...|_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + ... \ .$$

The value $1 - 1/3 + 1/10 = 26/35 = 0.753...$ approximates the actual value 0.747.

Example: Find the Taylor series of $f(x) = \log(1-x)$ at $x = 0$.

Because

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \ .$$

we can integrate both sides with respect to x and get

$$-\log(1-x) = \sum_{k=0}^{\infty} x^{k+1} k = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

The radius of convergence is the same as before: $R = 1$. We have seen also that for $x = -1$, where we have $\log(2) = 1 - 1/2 + 1/3 - 1/4 + ...$ we have an alternating series which still converges. You had also explored that for $|x| > 1$, the Taylor series does not work any more.

Example: Find the Taylor series of the **sinc function** $\sin(x)/x$ at $c = 0$.

One can see with l'Hopital that this function is nice and smooth at $x = 0$. But we can also see this by looking at the Taylor series. We know the Taylor series

$$\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - ... \ .$$

Now we know also

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} = 1 - \frac{x^2}{3!} + \frac{x^5}{5!} - \frac{x^6}{6!} - ... \ .$$

We could use the ratio test to see that also this Taylor series converges everywhere.

Reminders

- The 2nd hourly is tonight from 6-7:30 PM