CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 26: Slope Fields, 11/05/2021

Review

26.1. In the last lecture we have seen three type of differential equations
A) $\boxed{y'(t) = ry(t)}$ The general solution was $\boxed{y(t) = Ce^{rt}}$.
B) $\boxed{y'(t) = f(t)}$ The anti-derivative $\boxed{\int_0^t f(x)\,dx + C}$ was the general solution.
C) $\boxed{y'(t) = a + ry(t)}$ A banking problem with income a and interest rate r. The general solution is $\boxed{Ce^{rt} + a/r}$. (Notice that $y(t) = a/r$ is a solution and that we can add the solution in A).

26.2. A typical application to C) not related to banking are **in-and-out problems**. Assume a container of size c contains a mix of two fluids like water and syrup. Assume you fill in syrup at a rate of b gallons per minute and in the same time draw b gallons per minute from the mix. What differential equation models the amount $S(t)$ of syrup in the tank?
Solution: We have $S'(t) = b - (bS(t)/c)$ because $S(t)/c$ is the **syrup concentration** meaning that $bS(t)/c$ is taken out. We see again a differential equation of the form $\boxed{S'(t) = a + rS(t)}$ but now $r = -b/c$.

26.3. More complicated is the setup, where a different amount a gallons per minute from mix. In this case, we have the differential equation $\boxed{S'(t) = b - a(S(t)/(c - at))}$. The reason is that the container has fluid $c - at$, leading to the concentration $S(t)/(c - at)$.

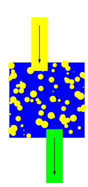

FIGURE 1. Don't worry, these **input/output problems** come back. Are also called "compartmental analysis".

Slope Fields

26.4. Given a differential equation $y' = F(t, y)$ where the right hand side can depend both on time and on y, we can draw the **slope field** in the $t - y$ plane. Draw at every point a short line with slope $f(t, y)$. Given an initial point (a, b) we can then draw the **curve** $y(t)$ which goes through the point (a, b) such that $y'(t)$ is the slope at the point (t, y).

26.5.

Example: a) Draw the slope field for $y'(t) = y(t)$.

26.6.

Example: b) Draw the slope field for $y'(t) = \sin(t)$.

26.7.

Example: c) Draw the slope field for $y'(t) = y(t)(2 - y(t))$ (rumors!)

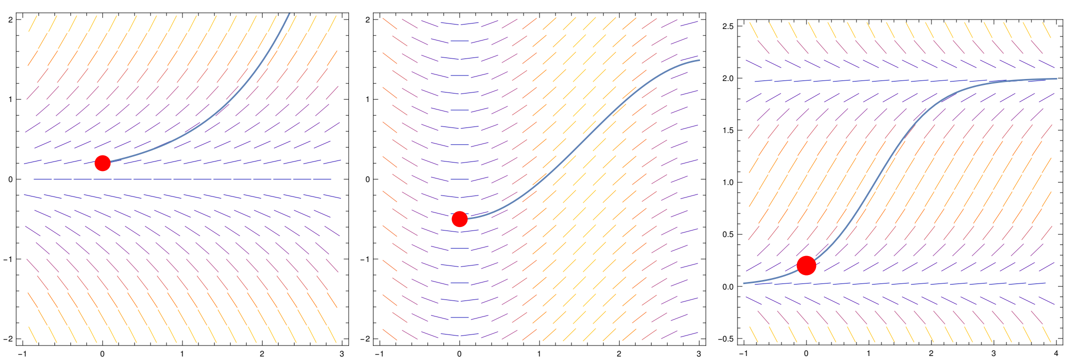

FIGURE 2. Slope fields to the three examples above. We see a solution.

Existence theorem

26.8. As mentioned in the first lecture, differential equations allow to compute the future from the present situation, if we have a model based on what has happened in the past. It is very important case that we have only one solution to the problem. Fortunately, this happens for reasonable models:

If $y'(t) = F(t, y)$ is a differential equation for which $F(t, y)$ is continuously differentiable both with respect to t and with respect to y, then given $y(0) = a$, there is exactly one solution with that initial condition.

Reminders

- Homework PS 24 is due next Monday.
- Partial point recovery problems is live since Thursday.

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