CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 27: Autonomous systems, 11/08/2021

Autonomous systems

27.1. A differential equation of the form

$$y' = f(y)$$

is called an **autonomous differential equation**. An important example is the system $\boxed{y' = ry}$. It produces exponential growth for $r > 0$ and exponential decay for $r < 0$. An other example is the system $\boxed{y' = ry(1-y)}$ which is called the **logistic system**. A third example is $y' = a + ry$ which appeared last week in **banking problems** or in **input-output models** which are also called **compartmental problems**.

Equilibria

27.2. A root y of f is also called an **equilibrium**. Equilibria are important because if y is an equilibrium, then the quantity y does not move. It is stationary. Looking at equilibrium points vastly helps to understand an autonomous system.

27.3.

Example: The **exponential model** $y' = ry$ has only one equilibrium. It is $y = 0$.

Example: The **logistic model** $y' = ry(1-y)$ has two equilibria, $y = 0$ and $y = 1$.

Example: The **input-output** model $y' = a + ry$ has the equilibrium $y = -a/r$.

27.4.

Equilibria solutions are constant solutions to the differential equation!

You can remember that for an input-output model, the general solution is $y(t) = -a/r + Ce^{rt}$. It is the sum of the equilibrium solution $y = -a/r$ and the general solution Ce^{rt} which we know from the exponential model.

Stability

27.5. It is important to know whether an equilibrium is **stable** or **unstable**. The equilibrium $y = 0$ for example is stable for the system $y' = ry$ if r is negative and unstable if r is positive. Because near an equilibrium point c the function has a Taylor expansion $f(y) = r(y-c) + b(y-c)^2/2 + ...$ and higher order terms can be neglected, the stability is entirely determined by the value $r = f'(c)$.

If c is an equilibrium and $f'(c) < 0$, then c is stable. If $f'(c) > 0$ it is unstable.

Graphical analysis

27.6. In order to see this graphically, draw the graph $f(y)$. Note that this is a picture in the $y - y'$ **plane**. It has nothing to do with the $t - y$ **plane** we have looked at when we were discussing **slope fields**. Do not mix up these two pictures. There is no time axis in the $y - y'$ picture.

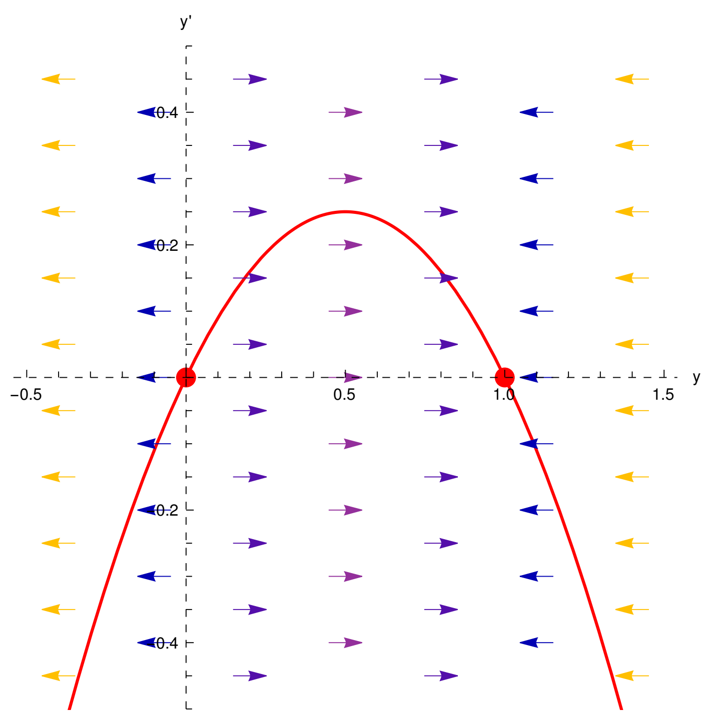

Figure 1. The **y-y' plane** in which one can plot the graph of the function $f(y) = y(1 - y)$ and the equilibrium points. If $f(y)$ is negative (like $y = 1$ here), we move towards the point near the equilibrium. The equilibrium is stable. On the other hand, the equilibrium point 0 is unstable. The solution moves away from it.

- Homework PS 25 is due next Wednesday
- Partial point recovery problems is due Thursday
- QRD on weather

Oliver Knill, knill@math.harvard.edu, Math 1b, Harvard College, Fall 2021