CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 28: Compartmental analysis, 11/10/2021

Input Output system

28.1. The autonomous input-output system $y' = a + ry$ has already appeared as a **banking problem** or a **eggnog** problem. The problem is also known as a **compartmental analysis** problem.

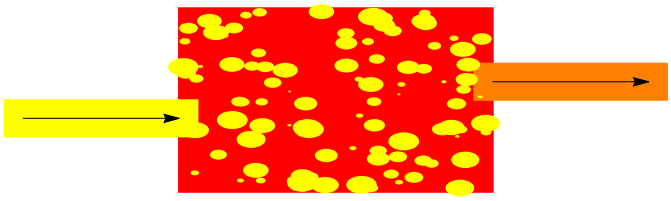


28.2. The worksheet starts with an other example. It deals with the amount $M(t)$ of **Mango juice** in an **Orange-Cranberry-Mango** blend.

> **Example:** *A 20-liter juice dispenser in a cafeteria is filled with juice mixture that is 10 percent mango juice and 90 percent cranberry juice. An orange-mango blend that is 50 percent orange and 50 percent mango is entering the dispenser at a rate of 4 liters per hour and the well-stirred mixture leaves at the same rate. Find the differential equation for the Mango concentration $M(t)$ and find $M(0)$.*

To solve this, first list the set-up in an organized way.

Input: 2 Liters per minute Mango, 2 Liters per minute Orange.
Dispenser: 20 Liter fill with 2 liters Mango and 18 liters of Cranberry.
Output: 4 Liters per minute is leaving the dispenser.



28.3. To set up a differential equation like that, we write down the rate of change of $M(t)$. It is a difference between input and output. The input is constant 2. The output is 4 times the current Mango concentration $M(t)/20$, which is $M(t)/5$. We have $M'(t) = 2 - 4M(t)/20 = 2 - M(t)/5$ and $M(0) = 2$. The answer is $\boxed{M' = 2 - M/5, M(0) = 2}$.

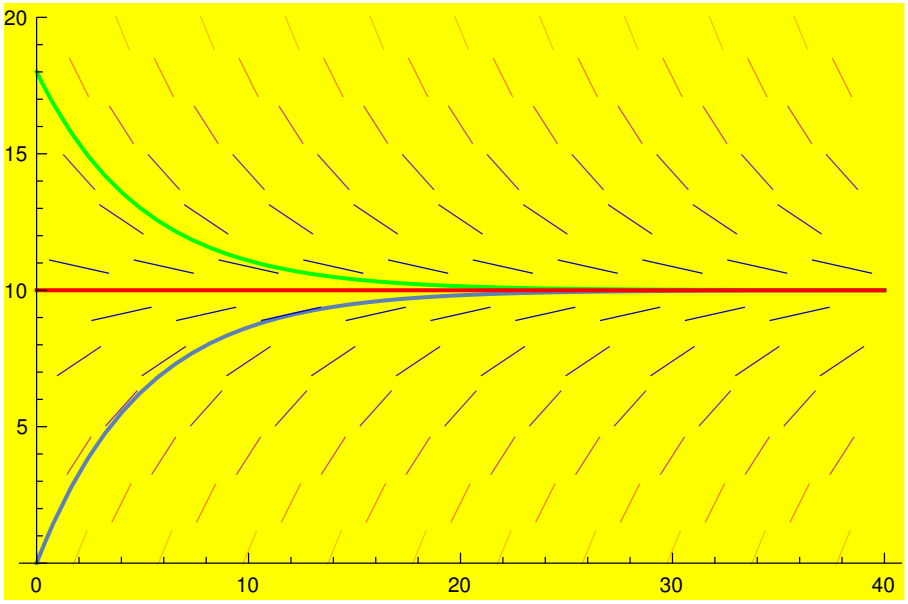


FIGURE 1. The slope field picture with trajectories of the Mango-Cranberry-Orange system. The value $M = 10$ is the only equilibrium.

EQUILIBRIA

28.4. Last time we have seen that it is important to look at the **equilibrium points** of a homogeneous differential equations.

28.5. In the juice problem case, the only equilibrium is $M(t) = 10$. We have also seen that for $M' = f(M)$, the derivative of $f'(10)$ matters. In our case it is $-1/5$. The negative sign shows that the equilibrium is stable. Indeed, we can even write down the general solution $M(t) = Ce^{-t/5} + 10$. We can fix C by plugging in $t = 0$. In the current case this gives $C = -8$.

REMINDERS

28.6. The general solution of the system $\boxed{y' = a + ry}$ is $\boxed{y(t) = -a/r + Ce^{rt}}$. It is the sum of the equilibrium solution $-q/r$ and the general solution of the exponential system $y' = ry$.

- Homework PS 26 is due next Friday.
- Partial point recovery problems is due tomorrow Thursday.
- The QRD on weather is due Friday.