

CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 29: Second order systems, 11/12/2021

REAL ROOTS

29.1. Differential equations in which the second derivative of the unknown function appears are called **second order**. An example is the linear second order differential equation

$$x'' + 5x' + 6x = 0$$

If you plug in $x(t) = e^{rt}$ we get $(r^2 + 5r + 6)e^{rt} = 0$. This means $r^2 + 5r + 6 = 0$. For any root r of this equation, we have a solution! In this case it is $r = -2$ and $r = -3$. Because sums of solutions add up, the general solution of the system is

$$x(t) = Ae^{-2t} + Be^{-3t}$$

29.2. That was so much fun, lets try again, with the differential equation

$$x'' - x = 0 .$$

We look at the roots $r^2 - 1 = 0$ and see that $r = 1$ and $r = -1$ are solutions. The general solution of the differential equation is $x(t) = Ae^{-t} + Be^t$.

29.3. And here is a third example

$$x'' = 0$$

This is an example which we can solve directly by integration. Just integrate to get $x' = B$ and then integrate again to get $x(t) = A + Bt$. Note that we have again two parameters to fix. This is always the case

The general solution of a second order differential equation has two free parameters.

29.4. And here is a fourth example

$$x'' + 2x' + x = 0$$

This means we have to look at roots of $r^2 + 2r + 1 = 0$. We see now however that the root $r = -1$ appears twice. The general solution is now $x(t) = Ae^{-t} + Bte^{-t}$. Notice the additional t in the second part. It appears because we can not have twice e^{-t} . The reason why the additional t appears is preceded already in the previous example as $x(t) = Ae^{0t} + Bte^{0t}$.

COMPLEX ROOTS

29.5. This case will be discussed more on Monday. We write $i = \sqrt{-1}$. The imaginary numbers $i, -i$ are solutions to the equation $r^2 = -1$. A general **complex numbers** is written as $z = a + ib$. A nice property of the complex numbers is that every polynomial can be completely factored. Especially, we can factor any polynomial $r^2 + ar + b$ belonging to a differential equation $x'' + ax' + bx$. The roots are $(-a \pm \sqrt{a^2 - 4b})/2$. This two roots can become complex if $a^2 - 4b < 0$.

29.6. The mother of all second order differential equations is when we have only pure imaginary solutions like

$$x'' + x = 0.$$

This is an example of a **harmonic oscillator**. The functions $\cos(x)$ and $\sin(x)$ are both solutions. The general solution is $x(t) = A \cos(t) + B \sin(t)$.

29.7. The equation $r^2 + 1 = 0$ has only **imaginary solutions** $r = i$ and $r = -i$. If we proceed as above, we get $x(t) = Ae^{it} + Be^{-it}$. But we need some magic to relate this exponential function with the trig function. It is the **Euler formula** which does this connection

$$e^{it} = \cos(t) + i \sin(t).$$

29.8. Here is an example

$$x'' + x' + x = 0$$

where the roots of $r^2 + r + 1$ are $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Using constants C, D , the general solution is

$$x(t) = Ce^{(-\frac{1}{2} - i\frac{\sqrt{3}}{2})t} + De^{(-\frac{1}{2} + i\frac{\sqrt{3}}{2})t} \text{ which is } x(t) = Ae^{-\frac{t}{2}} \cos(\frac{\sqrt{3}}{2}t) + Be^{-\frac{t}{2}} \sin(\frac{\sqrt{3}}{2}t)$$

PHYSICAL SYSTEMS

29.9. $x''(t)$ is the acceleration. **Newton's law** assures that $mx''(t) = F(x(t), x'(t))$ is a force. The case $x''(t) + c^2x(t)$ is the **harmonic oscillator**, modelled by a friction-less system. Its solutions are $A \cos(ct) + B \sin(ct)$. It is a mass point hanging on a spring. If there is friction, the differential equation equation changes to $x''(t) + ax'(t) + c^2x(t)$. With $x(t) = e^{rt}$ Ansatz we get the polynomial $r^2 + ar + c^2 = 0$ which has the solutions $r = -\frac{a}{2} \pm \frac{\sqrt{a^2 - 4c^2}}{2}$. For $a > 2c$, we have real eigenvalues, an **over-damped system**. For $a < 2c$, the damping happens with oscillations.

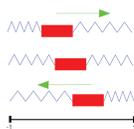


FIGURE 1. An oscillator $x''(t) + ax'(t) + c^2x(t) = 0$ with damping a and spring coefficient c^2 .

REMINDERS

- Homework PS 27 is due next Monday