CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 30: Euler's formula, 11/15/2021

The most beautiful formula in math

30.1. Whenever one has made surveys what the most beautiful equation is in mathematics, there is always the same clear winner

$0 = 1 + e^{i\pi}$.

The formula combines **geometry** π, **analysis** e, and **algebra** i. Arithmetic comes in with **addition** with neutral 0 and **multiplication** with neutral 1 and **exponentiation**.

30.2. This formula is a special case of the **Euler formula**

$e^{it} = \cos(t) + i\sin(t)$.

Just plug in $t = \pi$ to get $e^{i\pi} = -1$. This is the winner of the beauty contest.

Figure 1. Leonhard Euler (1707-1783)

30.3. The Euler formula can be proven by **Taylor series**. Remember $e^x = 1 + x + x^2/2 + x^3/6 + ...$ and

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots$$
$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots \ .$$

Now plug in $x = it$ into the exponential function to get

$$e^{it} = 1 + it - \frac{t^2}{2!} - i\frac{t^3}{3!} + \frac{t^4}{4!} + i\frac{t^5}{5!} \cdots \ .$$

You see that the real part of the later is just $\cos(t)$ and that the imaginary part is $\sin(t)$.

The quadratic equation

30.4. The **quadratic equation**

$$r^2 + br + c = 0$$

can be solved by rewriting it as $r^2 + br + b^2/4 = b^2/4 - c$ so that the left is a square $(r + b/2)^2 = b^2/4 - c$. Now multiply both sides with 4 and take the square root. We get $2(r + b/2) = \sqrt{b^2 - 4c}$. Solving for r gives:

$r = \frac{-b \pm \sqrt{b^2-4c}}{2}$ solves $r^2 + br + c = 0$.

30.5. It follows

Every quadratic polynomial has exactly two roots. They are in general complex.

30.6. If $b^2 - 4c$ is negative, then we get complex solutions. Let us look at an example of a differential equation where this happens:

$$x''(t) + 4x'(t) + 5x(t) = 0$$

Plugging in $x(t) = e^{rt}$ leads to the equation $r^2 + 4r + 5 = 0$ which has the solutions $r = -2 + i$ and $r = -2 - i$. This means that

$$x(t) = C_1 e^{(-2+i)t} + C_2 e^{(-2-i)t} \ .$$

Now, since we can write $e^{it} = \cos(t) + i\sin(t)$ and adapt the constants, we can write the general solution also as

$$x(t) = Ae^{-2t}\cos(t) + Be^{-2t}\sin(t) \ .$$

Reminders

- Homework PS 28 is due next Wednesday

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