CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 31: The general solution, 11/17/2021

Four Cases

31.1. The form of the solutions $r = \frac{b \pm \sqrt{b^2/4 - c}}{2}$ of the quadratic equation $r^2 + br + c = 0$ belonging to the differential equation $x'' + bx' + cx = 0$ leads to four possible cases:

Case	Example	Solution
Two different real eigenvalues	$x'' + 4x' - 21x = 0$	$x(t) = Ae^{3t} + Be^{-7t}$
Two equal real eigenvalues	$x'' + 12x' + 36x - 0$	$x(t) = Ae^{-6t} + Bte^{-6t}$
Pure complex eigenvalues	$x'' + 9x = 0$	$x(t) = A\cos(3t) + B\sin(3t)$
Mixed complex eigenvalues	$x'' + 2x' + 5x = 0$	$x(t) = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$

31.2. Match the differential equations with solutions to $x(0) = 0, x'(0) = 1$:

Diff equation	which graph A-F?
$x'' + 6x = 0$	
$x'' - x' + 2x = 0$	
$x'' = 0$	
$x'' + x' + 6x = 0$	
$x'' + x' + 2x = 0$	
$x'' + 2x = 0$	

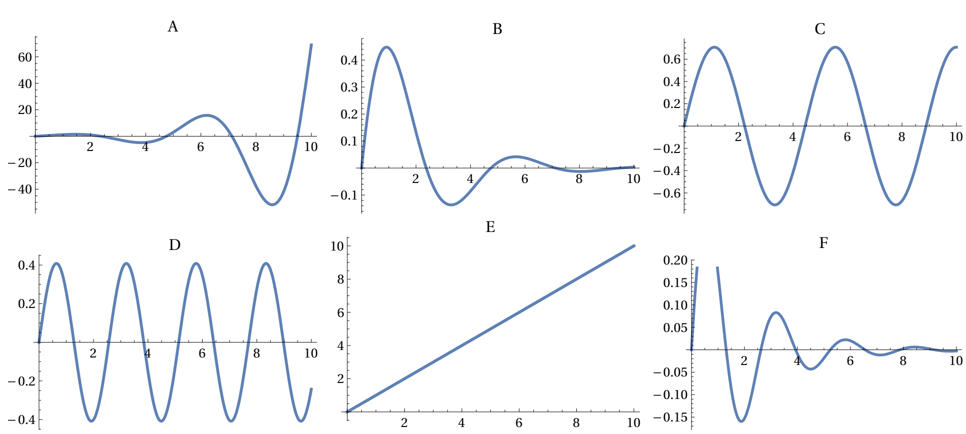

FIGURE 1. Solution curves $x(t)$ labeled A-F used for matching.

The ultimate solution

31.3.

A general linear second order differential equation $\boxed{x'' + bx' + cx = 0}$ is solved by plugging in $x(t) = e^{rt}$ to get the quadratic equation $\boxed{r^2 + br + c = 0}$ If r_1, r_2 are two different roots then $\boxed{x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$ is the general solution. If the two roots are the same, then $\boxed{x(t) = C_1 e^{r_1 t} + tC_2 e^{r_1 t}}$ is the general solution. With $x(0)$ and $x'(0)$ known, one can fix the constants C_1, C_2.

Looking beyond: Driven systems

31.4. Just to beyond this course a bit but related to the QRD project, we can look at a linear system of a Harmonic oscillator and drive it with a time-periodic function like in

$$x''(t) + 4x(t) = 2\sin(3t)$$

In this case we know how to get the general homogeneous solution as $x(t) = C_1\cos(2t) + C_2\sin(2t)$ solving $x''(t) + 4x(t) = 0$. An additional solution is obtained by trying $x(t) = C\sin(3t)$. Then $x'' + 4x = (9C + 4)\sin(3t) = 2\sin(3t)$ which tells us that $(9C + 4) = 2$ or $C = -2/9$. So, $\boxed{x(t) = C_1\cos(2t) + C_2\sin(2t) + (-2/9)\sin(3t)}$.

31.5. Now, lets look at the case

$$x''(t) + 4x(t) = 2\sin(2t)$$

The oscillator is hit with a force which has the same frequency than the oscillator. This phenomenon is linked to **resonances**. We still have the general solution $x(t) = C_1\cos(2t) + C_2\sin(2t)$ solving $x''(t) + 4x(t) = 0$. But to get a special solution we can not try $x(t) = C\sin(2t)$ because this gives us zero. Let us try $x(t) = Ct\cos(2t)$. Plugging this in gives $x'' + 4x = -4C\sin(2t)$. This shows that $C = -1/2$ works.

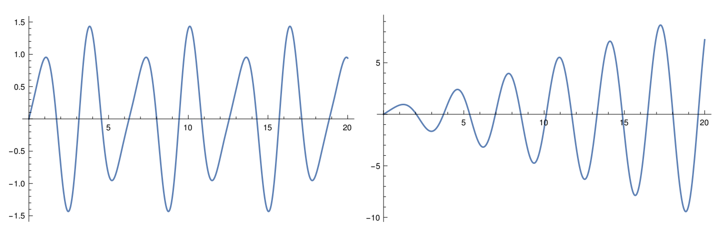

Figure 2. The solution of $x''(t) + 4x(t) = 2\sin(3t)$ and the solution of $x''(t) + 4x(t) = 2\sin(2t)$ are dramatically different. Both use the initial condition $x(0) = 0, x'(0) = 1$. In the second case, the oscillator is stimulated with a driving force $2\sin(2t)$ which is its own frequency. You see such a thing in the Tacoma QRD project.

Reminders

Homework PS 29 is due next Friday. This Thursday due: point recovery problem. Due 11/22: Tacoma Narrows bridge QRD.

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