
8/30/2021 near Mather house

Lecture 4

9/13/2021

Integration Techniques

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Part I

Information for Techniques of Integration test

9/15

- Wednesday, 9/15 from 6-7:30.
- Test will be remote. Download from Gradescope in a 15 minute window starting 6pm and uploading within 90 minutes of your download time. We will give more instructions closer to the test date.
- People with AEO letters should contact Cliff about getting extra time
- People with conflicts should contact Cliffabout starting earlier or later.
- There will be make-up Techniques test in mid-October where people can recoup any and all lost points.
- No aids of any kind (electronic, notes, books, people, ...) can be used during the test. Just you and your brain.
- Study material is on the Exam Information (Techniques test) page of the Math 1b website. This includes practice problems with solutions, practice exams with solutions and videos.
- The sections on Monday, 9/13 will be reviewing for the test.

Problem Sessions

The Math 1b Problem/review sessions will start this coming Thursday (Sept. 9) and run through the last week of the semester (except for holidays). Here is the schedule:

1. Tuesdays 9-10am in Science Center 309.
2. Thursdays 10:30-11:30am in Science Center 221.
3. Thursdays 12-1pm in Science Center 110
4. Thursdays 4:30-5:30pm in Science Center lecture hall E.

(The schedule and rooms are also on the Math 1b website; click the link on the left hand side-bar that says 'Problem/review session times and rooms'.)

The problem/review sessions for this coming Thursday and this coming Tuesday will focus in part on going over techniques of integration for the upcoming assessment test on September 15. In general, these sessions will review recent course material and homework problems. These are really good opportunities to ask questions about stuff you don't understand. (These are not mandatory; they are here for you to take advantage of. Go to more than one if you want.)

Keep in mind that you can also ask Math 1b questions (and get homework help) at the Math Question Center, room B10 of the Science Center in the evenings, from 7:30-10:30 on Sunday, Tuesday, Wednesday and Thursday; and on Monday from 8:30-10:30; and also go to the TF office hours.

Part 2

Overview

See Handout

CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 4: Integration techniques, 9/13/2021

Substitution

1.1. Identify part of the formula which you call u, then differentiate to get du in terms of dx, then replace dx with du. Example:

$$\int \frac{x}{1+x^4}\, dx \; .$$

Solution: Substitute $u = x^2, du = 2xdx$ gives $(1/2)\int du/(1+u^2)\, du = (1/2)\arctan(u) = (1/2)\arctan(x^2) + C$.

Remarks. Sometimes we have to try several times. In the example, we might first try $u = 1 + x^4$ but that does not give us a nice cancellation. If you should forget substitution, remember the chain rule. If $f(x) = g(u(x))$ then $f'(x) = g'(u(x))u'(x)$.

Integration by parts

1.2. Write the integrand as a product of two functions, differentiate one u and integrate the other dv. Then use $\int udv = uv - \int vdu$ from the product formula.

Example:

$$\int x\cos(x/3)\, dx$$

Solution: differentiate $u = x$ and integrate $dv = \cos(x/3)dx$. We have $3x\sin(x/3) - \int 1 \cdot 3\sin(x/3) = 3x\sin(x/3) + \cos(x/3)9 + C$.
Remarks. If you should forget the rule, remember the product rule $d(uv) = udv + vdu$ and integrate it, the solve for $\int u\, dv$.

Partial fractions

1.3. Use algebra to write a fraction as a sum of fractions we can integrate. Example:

$$\int \frac{1}{(x-3)(x-2)}\, dx$$

Solution: write $\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}$.

To get A, multiply with $x - 3$, cancel terms and put $x = 3$ which gives $A = 1$. To get B, multiply with $x - 2$, cancel terms and put $x = 2$ which gives $B = -1$.

Remarks. Most find the constants A, B by cross multiplication and comparing coefficients. The just explained method is **much faster**.

Trig substitution

1.4. Replace a term with $\sin(u)$ so that the formula simplifies.

Example: a prototype example is

$$\int \frac{1}{\sqrt{x^2-64}}\, dx \; .$$

Solution: $x = 8\sin(u)$ gives $\frac{1}{\sqrt{x^2-64}} = 1/(8\cos(u))$. As $dx = 8\cos(u)$. The integral is $\int 1/8du = u/8 + C = \arcsin(x/8) + C$.

Trig identities

1.5. The double angle formulas $\cos^2(x) = (1+\cos(2x))/2$ and $\sin^2(x) = (1-\cos(2x))/2$ are handy. Also consider using $\cos^2(x) = 1 - \sin^2(x)$ or $\sin^2(x) = 1 - \cos^2(x)$ or use the identity $2\sin(x)\cos(x) = \sin(2x)$.

Example:

$$\int \sin^4(x)\, dx = \int (1-\cos^2(x))\sin^2(x) = \int \sin^2(x) - \sin^2(2x)/4\, dx$$

we can now use the double angle formulas to write this as $\int (1-\cos(2x))/2 - (1 - \cos(4x))/8$ which now can be integrate $x/2 - \sin(2x)/4 - x/8 + \sin(4x)/32 + C$.

Symmetries

1.6. Sometimes, the result of an integral can be seen geometrically.

Example:

$$\int_{-2}^{2} \sin^7(5x^3)\, dx$$

is an integral we can not compute so easily by finding the anti derivative. However we see that the function in the integrand is odd. If we integrate an odd function over a symmetric interval, we have a cancellation. The answer is 0.

Remember

- No Homework is due on Wednesday
- Techniques of integration test is on Wednesday 9/15.

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Part 3

Work sheet

Integration Techniques

1. (Weekly Problem TI 3. b).) Evaluate **three of the following four** integrals.

   (a) $\int \ln(3x+2)\, dx$.

   (b) $\int e^{-x^2}\, dx$.

   (c) $\int e^{\sqrt{x}}\, dx$.

   (d) $\int \ln \sqrt{x}\, dx$.

2. In each part, decide which method of integration you would use. If you would use substitution, what would u be? If you would use integration by parts, what would u and dv be? If you would use partial fractions, what would the partial fraction expansion look like? (Don't solve for the coefficients.)

   (a) $\int \frac{\cos x}{\sqrt{1+\sin x}}\, dx$.

   (b) $\int (\ln x)^2\, dx$.

   (c) $\int x^2 \sin x\, dx$.

   (d) $\int \frac{x}{x^2-1}\, dx$.

   (e) $\int x e^{x^2}\, dx$.

   (f) $\int \frac{x^2}{x^2+4x+3}\, dx$.

   (g) $\int \frac{e^t}{1+e^t}\, dt$.

   (h) $\int \arcsin x\, dx$.

3. In each part, find a substitution of the form x = *function of* u to change the left hand integral into the right hand integral.

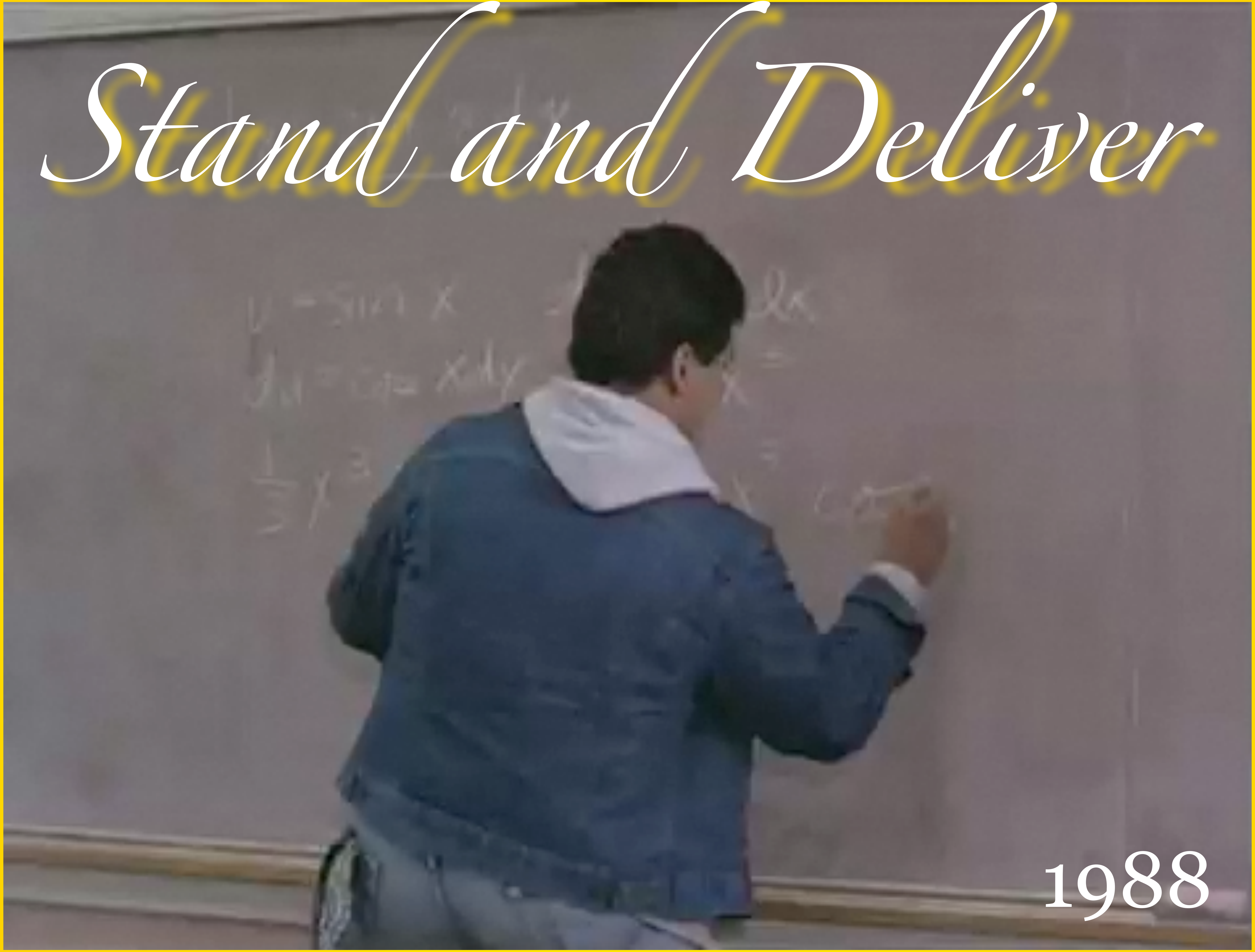
   (a) $\int \frac{x}{\sqrt{1+x}}\, dx \rightarrow \int \frac{\cos u \ \sin u}{\sqrt{1+\sin u}}\, du$. x = ____________________

   (b) $\int (\ln x)^2\, x\, dx \rightarrow \int u^2 e^{2u}\, du$. x = ____________________

   (c) $\int \frac{e^x}{1+e^x}\, dx \rightarrow \int \frac{\cos(u)}{1+\sin(u)}\, du$. x = ____________________

   (d) $\int \frac{x}{x^2-1}\, dx \rightarrow \int \frac{u+3}{u^2+6u+8}\, du$. x = ____________________


RUSHMORE


Stand and Deliver
1988


Gifted 2017

Lecture 4 — Integration techniques

1) Substitution

$$\int \frac{x}{1+x^4}\,dx = \int \frac{1}{1+u^2}\,\frac{1}{2}\,du = \frac{1}{2}\arctan(x^2)+C$$

$u = x^2$, $du = 2x\,dx$, $dx = \frac{du}{2x}$

$$\int_{-2}^{3} \frac{x}{1+x^4}\,dx = \frac{1}{2}\arctan(3) - \frac{1}{2}\arctan(-2)$$

$$\int_{x=-1}^{x=3} \frac{x}{1+x^2}\,dx = \int_{u=1}^{u=9} \frac{1}{1+u^2}\,\frac{1}{2}\,du$$

no back subst. needed.

2) Parts

"Stand and deliver"

$$\int x^3 e^{2x}\,dx = x^3\frac{e^{2x}}{2} - \int 3x^2\,\frac{e^{2x}}{2}\,dx$$

($u = x^3$, $dv = e^{2x}\,dx$; uv)

Tic tac toe:

x^3	e^{2x}	
$3x^2$	$\frac{e^{2x}}{2}$	+
$6x$	$\frac{e^{2x}}{4}$	−
6	$\frac{e^{2x}}{8}$	+
0	$\frac{e^{2x}}{16}$	−

$$x^3\frac{e^{2x}}{2} - 3x^2\frac{e^{2x}}{4} + 6x\frac{e^{2x}}{8} - 6\frac{e^{2x}}{16} + C$$

3) Partial fractions

$$\int \frac{1}{(x-5)(x-6)}\,dx = \int \frac{A}{x-5} + \frac{B}{x-6}\,dx$$

$$\frac{1}{(x-5)(x-6)}(x-5) = \frac{A}{x-5}(x-5) + \frac{B}{x-6}(x-5)$$

$$\frac{1}{x-6} = A + \frac{B(x-5)}{x-6} \Rightarrow A = -1$$

Now put $x=5$

Cross multiply and compare coeff.

$$\frac{1}{(x-5)(x-6)}(x-6) = \frac{A}{x-5}(x-6) + \frac{B}{x-6}(x-6)$$

$$\frac{1}{x-5} = \frac{A(x-6)}{x-5} + B$$

Now put $x = 6$ $\Rightarrow B = 1$

$$\frac{1}{(x-5)(x-6)} = \frac{A(x-6)+B(x-5)}{(x-5)(x-6)}$$

$(A+B) = 0$

$A(-6) + B(-5) = 1$

system of eqn.

5) Symmetries

$$\int_{-1000}^{1000} \arctan(x^{777777})\,dx = 0$$

4) Trig identities

$$\cos^2(x) + \sin^2 x = 1$$

$$\cos^2(x) = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$\cos^2 2x = \frac{1+\cos 4x}{2}$

Differentiate the second identity: $2\sin x\cos x = \sin 2x$

$$\int \cos^4(x)\,dx = \int \cos^2(x)\cos^2(x)\,dx = \int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx$$

$$= \int \frac{1}{4} + \frac{1}{2}\cos 2x + \cos^2(2x)/4\,dx$$

6) Trig substit.

$$\int \frac{1}{\sqrt{100-x^2}}\,dx = \int \frac{10\cos u\,du}{10\cos u} = \int 1\,du = u + C = \arcsin\left(\frac{x}{10}\right) + C$$

$x = 10\sin u$, $dx = 10\cos u\,du$, $\sqrt{100-x^2} = 10\cos u$

$u = \arcsin\left(\frac{x}{10}\right)$

THE END