
8/30/2021 near Mather house

Lecture 13

10/4/2021

Taylor Error Term

Table of Contents

1) Know the error of approximations!

2) $$|R_n(x)| \le M_{n+1} \frac{|x-c|^{n+1}}{(n+1)!}$$

3) Examples

4) Worksheet problems

5) HW 12 due Friday

A story

Integer factoring

Modern cryptology allowing secure transactions and communication is based on mathematics.

One of the hard problems which secures the vaults is factoring large integers.

Holy Grail

The holy grail of factoring large numbers n=p*q is to find a number x such that x^2 has a small reminder when dividing by n.

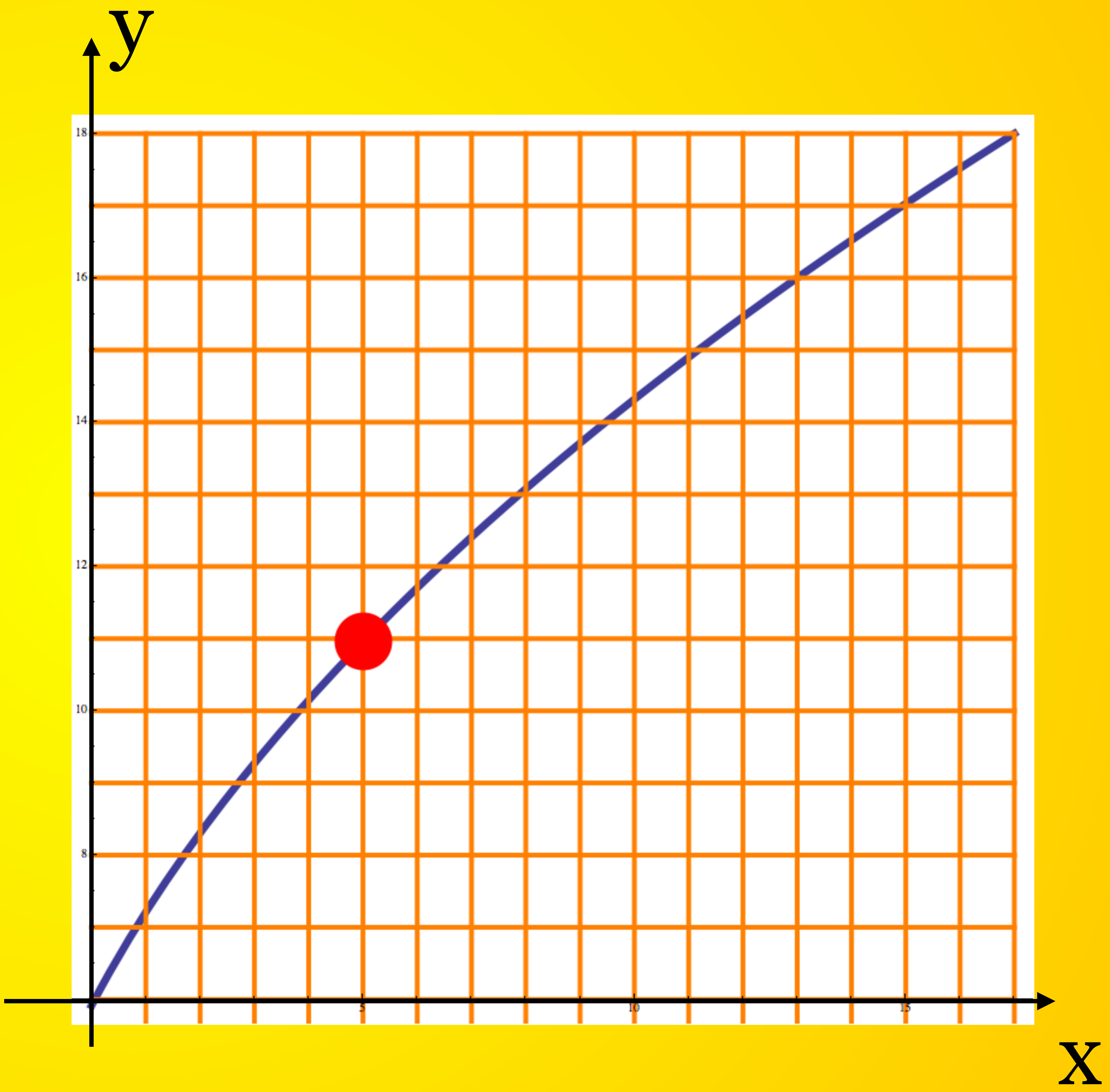
Fermat, Quadratic sieve, Morrison Brillard,...

Idea

Find x such that

$$y = \sqrt{2n + x\,n + 1}$$

is very close to an integer. Then y^2 has a small reminder.

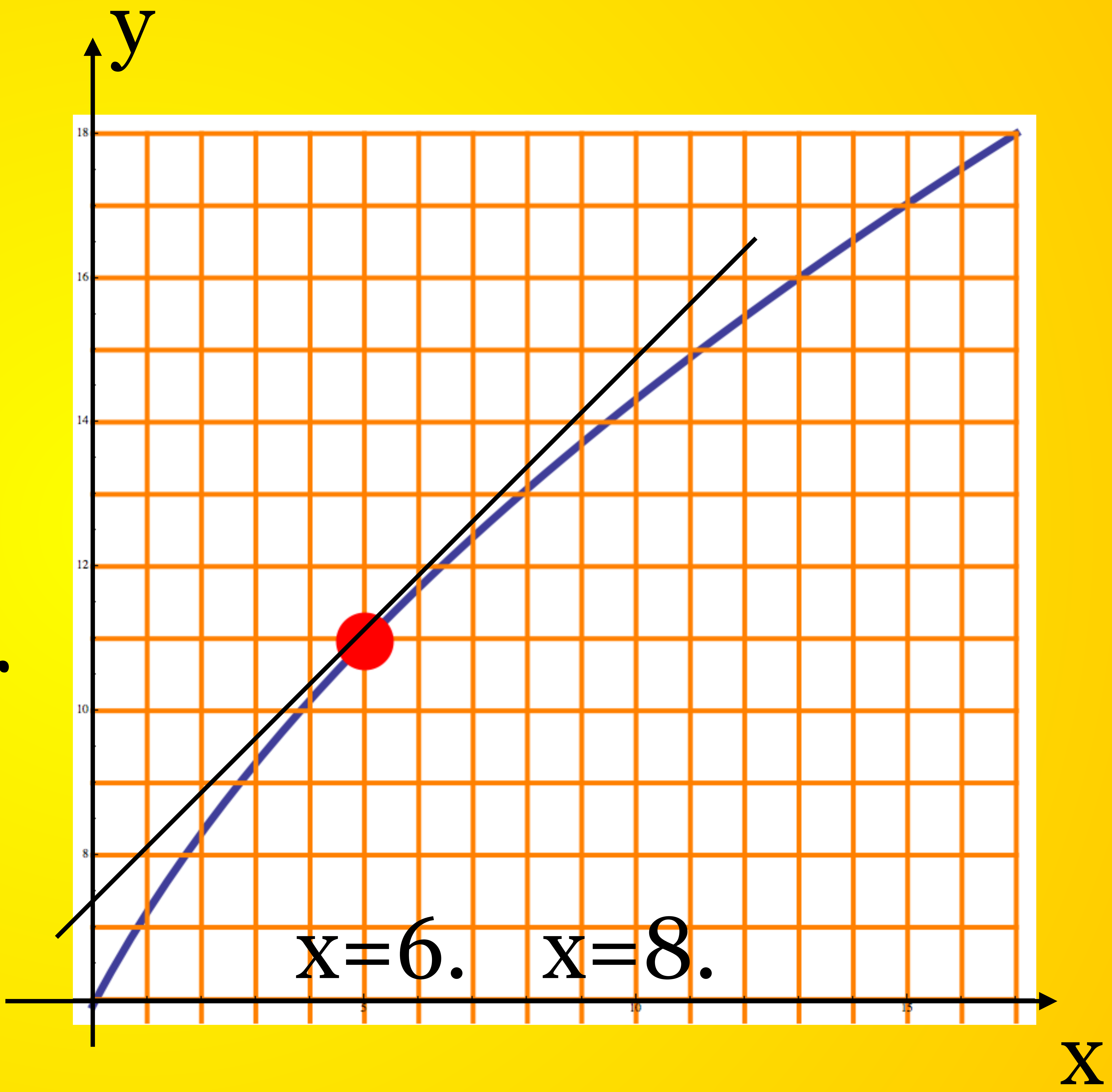


Taylor

assume n=10.

$$y = \sqrt{21 + 10x}$$

Make a degree 1 Taylor approximation at c=6 look at the error at x=8.



Error Bound

Key Formulas

$$P_n(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2} + \ldots + f^{(n)}(c)\frac{(x-c)^n}{n!}$$

$$|f(x) - P_n(x)| = |R_n(x)| \leq M_{n+1}\frac{|x-c|^{n+1}}{(n+1)!}$$

$M_{n+1} =$ maximum of n+1'th derivative of f on [x,c]

Proof

$$f(x) = P_n(x) + R_n(x) = P_n(x) + \int_c^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt$$

This is an exact formula which can be shown inductively using integration by parts.

The rest term can now be estimated by

$$\int_c^x M_{n+1}\frac{(x-t)^n}{n!}dt \leq M_{n+1}\frac{(x-c)^{n+1}}{(n+1)!}$$

$n=1$ example

$$R_n(x) = \int_c^x f^{(n+1)}(t)\frac{(x-t)^n}{n!}dt$$

Let us compute this for n=1:

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt$$

Now use integration by parts dv=f''(t) and u=(x-t)

to get $f(0) + f'(0)x + f'(t)(x-t)\,|_0^x + \int_0^x f'(t)dt = f(x)$

Estimations

Finding upper bounds

Estimate $|f'(x)|$ for $f(x)=\exp(5x)+\sin(x)$ on the interval [0,3].

Example 1

$$f(x) = \exp(x)$$

Take c=0 and x=1. How big is the error for a degree 4 Taylor series?

Worksheet

The End