



# *Lecture 16*

10/12/2021

*Divergence  
and comparison*

8/30/2021 near Mather house

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# *Review Series*

$$S = a_1 + a_2 + a_3 + \dots$$

is called a series written in ... notation.

$$S = \sum_{k=1}^{\infty} a_k$$

is the sum notation

# *Review Converge*

The series converges if the partial sums

$$S_n = a_1 + a_2 + \dots + a_n$$

converge to a number  $S$ .

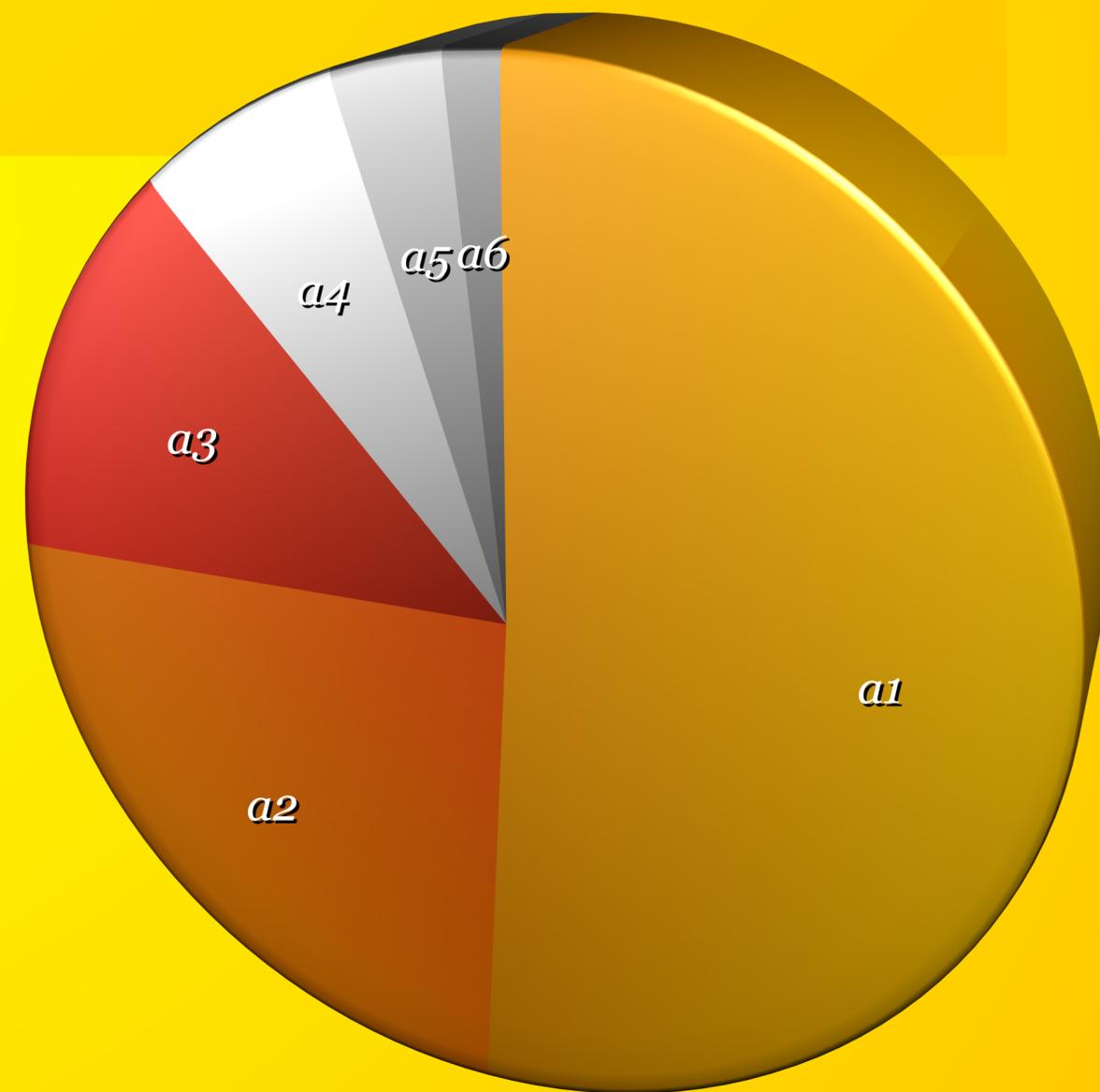
It converges absolutely if the sum of the absolute values converges.

# Example

$$S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

converges to 1.

Take half a pie, add a quarter (half of the rest) an eighth (half of the rest) etc



# *Grandi's Series*

$$S = 1 - 1 + 1 - 1 + 1 - 1 \dots$$

is called Grandi's series.

Luigi Grandi 1671 – 1742



# *Harmonic Series*

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

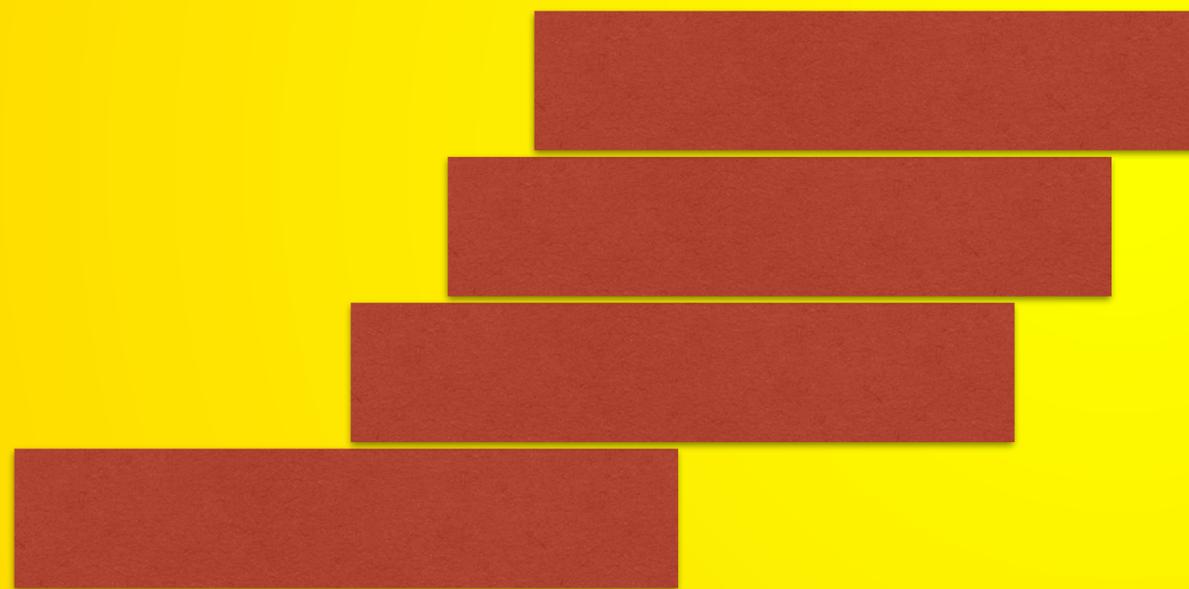
Is the Harmonic series. It diverges

$$S = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 1/2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 1/2} + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geq 1/2} + \dots$$

It is also the Taylor series of  $-\log(1-x)$  at  $x=1$

# *Block stacking problem*

How far can we go?



Center of mass  
needs to be above  
ground block in  
order to not to fall.

# *Tower of Lire*

It is a consequence of the divergence of the harmonic series that the tower can lean over arbitrary far.



image source:  
wikipedia

# Maximal Overhang

## Maximum Overhang

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### Abstract

How far can a stack of  $n$  identical blocks be made to hang over the edge of a table? The question dates back to at least the middle of the 19th century and the answer to it was widely believed to be of order  $\log n$ . Recently, Paterson and Zwick constructed  $n$ -block stacks with overhangs of order  $n^{1/3}$ , exponentially better than previously thought possible. We show here that order  $n^{1/3}$  is indeed best possible, resolving the long-standing overhang problem up to a constant factor.

## 1 Introduction

The problem of stacking  $n$  blocks on a table so as to achieve maximum overhang has a long history. It appears in physics and engineering textbooks from as early as the mid 19th century (see, e.g., [P1850], [W1855], [M1907]). The problem was apparently first brought to the attention of the mathematical community in 1923 when J.G. Coffin posed it in the “Problems and Solutions” section of the American Mathematical Monthly [C1923]; no solution was presented there.

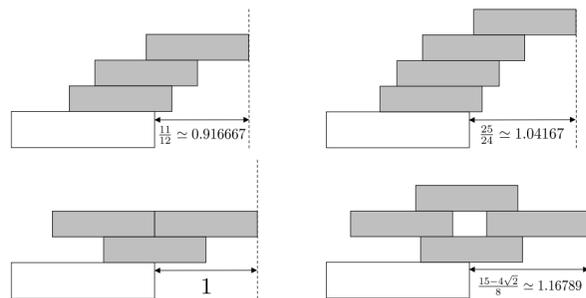


Figure 1: Optimal stacks with 3 and 4 blocks, compared to the corresponding harmonic stacks. The 4 block solution is from [A1979]. Like the harmonic stacks it can be made stable by minute displacements.

The problem recurred from time to time over subsequent years, e.g., [S1953, S1954, S1955, J1955, GS1958, E1959, G1964, G1971, A1979, D1981, GKP1988, H2005], achieving much added notoriety from its appearance in 1964 in Martin Gardner’s “Mathematical Games” column of *Scientific American* [G1964, G1971].

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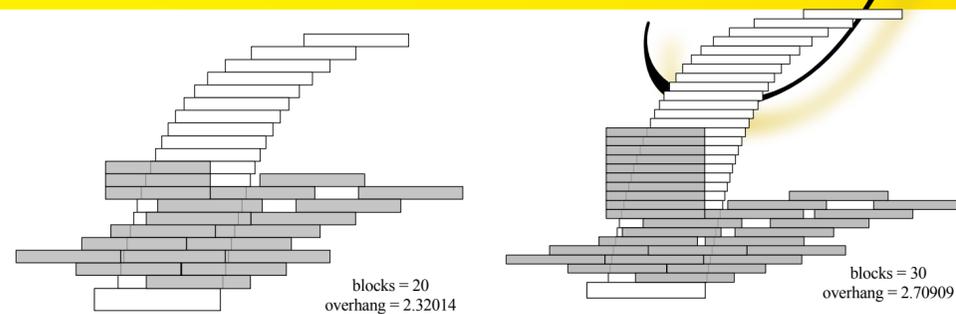


Figure 2: Optimal stacks with 20 and 30 blocks from [PZ2006] with corresponding harmonic stacks in the background.

Most of the references mentioned above describe the now classical *harmonic stacks* in which  $n$  unit-length blocks are placed one on top of the other, with the  $i^{\text{th}}$  block from the top extending by  $\frac{1}{2i}$  beyond the block below it. The overhang achieved by such stacks is  $\frac{1}{2}H_n = \frac{1}{2} \sum_{i=1}^n \frac{1}{i} \sim \frac{1}{2} \ln n$ . The cases  $n = 3$  and  $n = 4$  are illustrated at the top of Figure 1 above, and the cases  $n = 20$  and  $n = 30$  are shown in the background of Figure 2. Verifying that harmonic stacks are *balanced* and can be made *stable* (see definitions in the next section) by minute displacements is an easy exercise. (This is the form in which the problem appears in [P1850], [W1855], [M1907].) Harmonic stacks show that arbitrarily large overhangs can be achieved if sufficiently many blocks are available. They have been used by countless teachers as an introduction to recurrence relations, the harmonic series and simple optimization problems (see, e.g., [GKP1988]).

### 1.1 How far can you go?

Many readers of the above mentioned references were led to believe that  $\frac{1}{2}H_n (\sim \frac{1}{2} \ln n)$ , the overhang achieved by harmonic stacks, is the *maximum* overhang that can be achieved using  $n$  blocks. This is indeed the case under the restriction, explicit or implicit in some of these references, that the blocks should be stacked in a *one-on-one* fashion, with each block resting on at most one other block. It has been known for some time, however, that larger overhangs may be obtained if the one-on-one restriction is lifted. Three blocks, for example, can easily be used to obtain an overhang of 1. Ainley [A1979] found that four blocks can be used to obtain an overhang of about 1.16789, as shown at the bottom right of Figure 1, and this is more than 10% larger than the overhang of the corresponding harmonic stack. Using computers, Paterson and Zwick [PZ2006] found the optimal stacks with a given limited number of blocks. Their solutions with 20 and 30 blocks are shown in Figure 2.

Now what happens when  $n$  grows large? Can general stacks, not subject to the one-on-one restriction, improve upon the overhang achieved by the harmonic stacks by more than a constant factor, or is overhang of order  $\log n$  the best that can be achieved? In a recent cover article in the *American Journal of Physics*, Hall [H2005] observes that the addition of counterbalancing blocks to one-on-one stacks can double (asymptotically) the overhang obtainable by harmonic stacks. However, he then incorrectly concludes that no further improvement is possible, thus perpetuating the order  $\log n$  “mythology”.

Recently, however, Paterson and Zwick [PZ2006] discovered that the modest improvements gained for small values of  $n$  by using layers with multiple blocks mushroom into an exponential improvement for large values of  $n$ , yielding overhang of order  $n^{1/3}$  instead of just  $\log n$ .



The Leaning Tower of Lire, youtube Ding

*Worksheet*

*Reminders*

# QRD 3

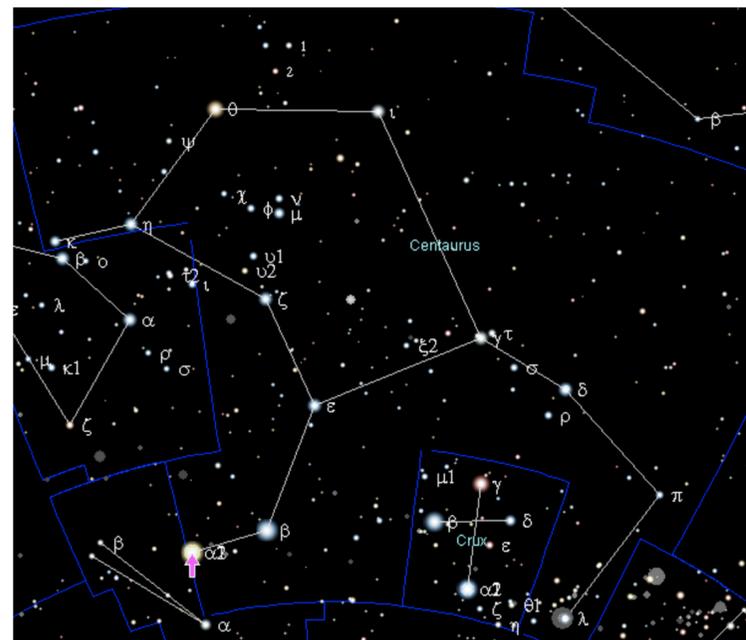
## The distance to the stars

Then nearest star to Earth is the sun. How far away is the next nearest star? And, how far away are the other stars? How can you possibly measure that? Probably you know that the next nearest star is named Proxima Centauri. It is too dim to be seen without a telescope. However, Proxima Centauri is part of a triple star system with the other two stars being so close to each other that we see them as the single bright star Alpha Centauri; a very bright star visible from near the equator or farther south (not from Boston anyway). What follows is an image from the Wikipedia article on Alpha Centauri. It is the brightest star, about center height on the left; Proxima Centauri is the reddish star at the center of the small red circle which is down and to the right of Alpha Centauri.



The two bright stars are (left) Alpha Centauri and (right) Beta Centauri, both binaries. The faint red star in the center of the red circle, at right angles to both and south-east of Alpha is Proxima Centauri, intensely red, smaller in size, weaker in brightness and a distant third element in a triple star system with the main close pair forming Alpha Centauri. Taken with Canon 85mm f/1.8 lens with 11 frames stacked, each frame exposed 30 seconds.

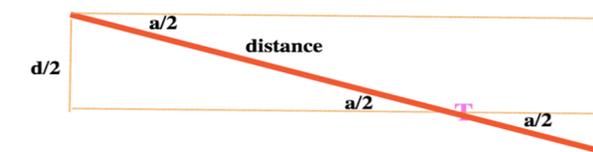
If you are near to or south of the equator, you can easily see Alpha Centauri. The small pink arrow points to Alpha Centauri on the following star chart



But how far away is Proxima Centauri? The current most accurate distance measurement comes from the European Space Agency's Gaia satellite (more about this momentarily) which (according to Wikipedia) finds Proxima Centauri to be  $4.2465 \pm 0.0003$  light years.

The unofficial definition of a 'light year' is the distance that light travels in a year. To determine that, you have to decide what precisely you mean by a 'year' which is a subtle issue if you are looking for precision. In any event, the International Astronomical Union (IAU) has an official definition of a light year which is 9,460,730,472,580.8 km, exactly. This is roughly 9.5 trillion kilometers.

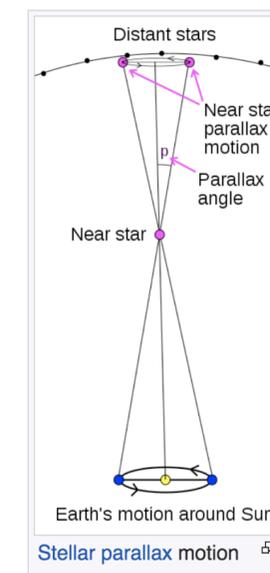
The Gaia space satellite has measured the distance to not just Proxima Centauri but to a lot of other stars. The ultimate goal of Gaia is to measure the distances to 20 million stars with accuracy within 1% and distances to 200 million stars with accuracy within 10%. When finished, it will have measured the distances of over 1 billion stars in our galaxy (and lots of other things besides). Take a look at the Gaia mission website if you want to know more <https://sci.esa.int/web/gaia/-/47354-fact-sheet>. (Gaia measures more than just distance. See these websites <https://www.cosmos.esa.int/web/gaia/early->



Because of this, the angle  $a/2$  can be determined by the viewer, which is to say you or me or Gaia depending on the context.

PROBLEM 2: Measure the length of your arm and the distance between your eyes. Use these numbers to determine the angle  $a/2$  in the case where you hold your finger at arms length and alternately view your finger using just your left eye and then just your right eye. (Give the arm and eye distance measurements to get full credit for this problem.)

In the astronomical context, the parallax picture is this one (also from Wikipedia):



What is denoted by  $p$  in this diagram is what was called  $a/2$  in the previous diagrams. The analog of  $d/2$  in this setting is the radius of the earth's orbit around the sun (thus,  $1/2$  of the diameter which is the analog of  $d$ ). With this geometry understood, then the task of measuring the distance to a star has the following components:

# due next week

# *Proxima Centauri*





Contact

*The End*