

CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 2: Density and definite integral

DENSITY AND DEFINITE INTEGRAL

2.1. The integral

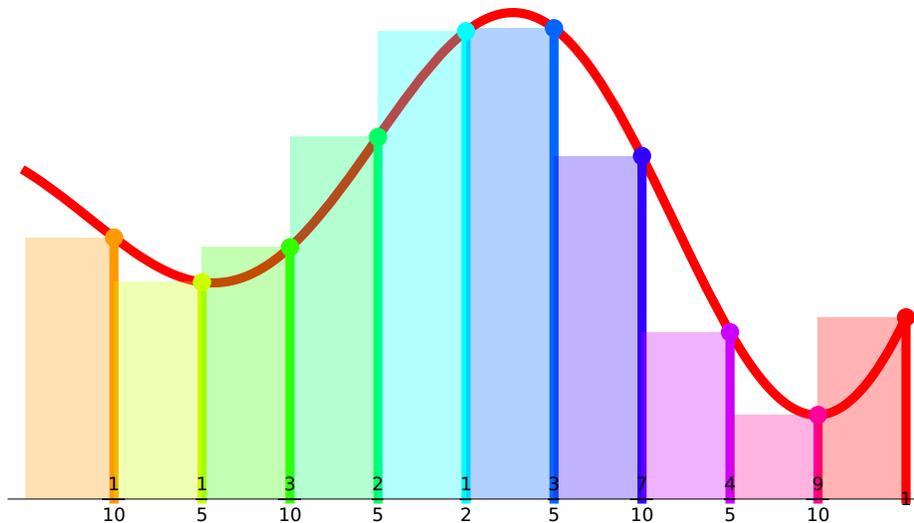
$$\int_a^b \rho(x) f(x) dx .$$

is the limit of the Riemann sum

$$\sum_{k=1}^n \rho(x_k) f(x_k) \Delta x$$

with $\Delta x = (b - a)/n$, $x_k = a + k\Delta x$.

2.2. The interpretation is that $f(x_k)\Delta x$ is the area of a **slice** and that $\rho(x_k)f(x_k)\Delta x$ is the content of the material included in that slice. We have seen examples, where $\rho(x)$ can be a color density, an ink density, a cheese density, a population density, a mass density.



2.3. When is it possible to take the limit $n \rightarrow \infty$? It turns out that we only need continuity for the limit to exist. If the function f is differentiable it is easier to see that the error of the difference between the Riemann sum and the integral goes to zero like $1/n$.

PHILOSOPHY OF THE INTEGRAL

2.4. It is a fascinating task to dig deeper what an integral actually is. Here are two things to think about:

2.5. A) Is space really a continuum? Is there between any two points an other point? Can we divide space up into smaller pieces? Or are there smallest pieces of space? Democritus, the Greek pre-socratic philosopher formulated an atomic theory of the universe. Everything is composed of atoms. So, what if also space is made of smallest atomic parts? This is perfectly possible. We do not know what happens on a very small scale like 10^{-35} . Atoms were smashed, maybe also such atomic smallest scale units can get smashed. Maybe on a scale of 10^{-100} the rules of quantum mechanics do no more hold and a continuum word again emerges. Maybe in each of the small Planck scale units there are entire universes hidden on very small scales. We have no idea because our current measurements can go maybe to the size of 10^{-20} .



2.6. B) Why does orientation matter? Here is a riddle: if we compute the mass density as just described, why does the result not depend on the orientation (from left to right or right to left) when we integrate? Remember that

$$\int_1^2 f'(x) dx$$

gives a different result than $\int_2^1 f'(x) dx$. By the **fundamental theorem of calculus** for example, we have in the first case $f(2) - f(1)$ and in the second case $f(1) - f(2)$. Obviously the integral comes with an orientation. It turns out that the **density integral** we have considered here is a fundamentally different kind of integral than the

integral we look at in the context of the fundamental theorem of calculus. Don't worry about it but keep it as a philosophical riddle.

OLIVER KNILL, KNILL@MATH.HARVARD.EDU, MATH 1B, HARVARD COLLEGE, 9/8/2023