CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 6: Three dimensional density

The integral

6.1. Remember that if $\rho(x)$ was a density and $f(x)$ a length, then

$$\int_a^b \rho(x)f(x)\,dx$$

was the total amount of material in the solid. We had written this as a limit

$$\sum_{k=1}^{n} \rho(x_k)f(x_k)\Delta x\ ,$$

where $\Delta(x) = (b-a)/n$ and $x_k = a + k\Delta x$. Today, we look at **three dimensional versions** of that. It is very similar except that the length $f(x)$ is now an area $A(x)$ and the area slices $f(x_k)\Delta x$ are now **volume slices** $A(x_k)\Delta x$.

6.2. If $\rho(x)$ is the density and $A(x)$ is the cross section or shell of the solid at x which is given in the range $a \leq x \leq b$, then

$$\int_a^b \rho(x)A(x)\,dx$$

is the total amount of material we measure in the solid. This generalizes the volume

$$\int_a^b A(x)\,dx$$

the case when $\rho(x) = 1$ is constant. When looking at **3-dimensional density problems**, we do have less freedom to compute the total integral. The reason is that we want to slice in such a way that the density is constant on the slice.

6.3. Example:

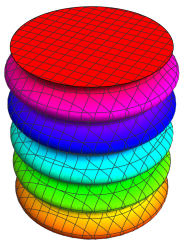
Assume that a wedding cake radius $10+\sin(x)$ where x ranges in $0 \leq x \leq 10\pi$. Assume the density of the cream at height h is $\rho(x) = 4 + \cos(x)$. Compute the total amount of cream.

Solution: If we slice the cake horizontally at height $x_k = k\Delta x$ with $\Delta x = 10\pi/n$, we get a washer of height Δx and area $A(x_k) = \pi(10 + \sin(h_k))^2$. The volume of that slice is $A(x_k)\Delta x$ and the cream content of that slice is $\cos(x_k)\pi(10+\sin(x_k))^2\Delta x$. The

total volume is $\sum_{k=1}^{n} \rho(x_k)\pi(10+\sin(x_k))^2\Delta x$. In the limit when n goes to infinity, this becomes the integral

$$\pi \int_0^{10\pi} (4+\cos(x))(10+\sin(x))^2 dx \ .$$

This integral is $4020\pi^2$.



6.4. In class, we will look at various objects. The heroes of the worksheet a **slushy**, where the slices are disks, a **planet** where the slices are spherical shells, a **candle** where the slices are cylindrical hells and a **muffin**, where the slices are forced to be disks. These examples show that the cross sections can be disks, washers, spherical shells, cylindrical shells or more general two dimensional shells which are rotated around an axis.