

CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 15: Divergence and Comparison

DIVERGENCE

15.1. We have seen that if $S = \sum_{k=1}^{\infty} a_k$ converges then $\lim_{n \rightarrow \infty} a_n = 0$. This is called the **n'th term test** and formulated as a **condition to assure that a series diverges**:

If $\lim_{n \rightarrow \infty} a_n$ does not exist or exists and is not zero, then $S = \sum_k a_k$ is divergent.

Note that $\lim_{n \rightarrow \infty} a_n \rightarrow 0$ does not suffice for convergence. The prototype is the **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$ which will be one of the main characters in this lecture, in homework, in this course and in your life!

15.2.

Example: The sum

$$S = \sum_{k=1}^{\infty} \left(1 + \frac{\sin(k)}{k}\right)$$

does not converge because the numbers a_k converge to 1.

Example: The sum

$$S = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

is called the **harmonic series**. We can see with the comparison test below why it diverges. Similarly as in the homework, you can group things

$$S = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots$$

to see that the sum diverges.

Example: The **Grandi's series** (named after Guido Grandi)

$$S = 1 - 1 + 1 - 1 + 1 \dots = \sum_{k=0}^{\infty} (-1)^k$$

is an example, where the limit $\lim_n a_n$ does not exist. This automatically implies that the sum $S = \sum_k a_k$ does not converge.

Example: The **Harmonic series**

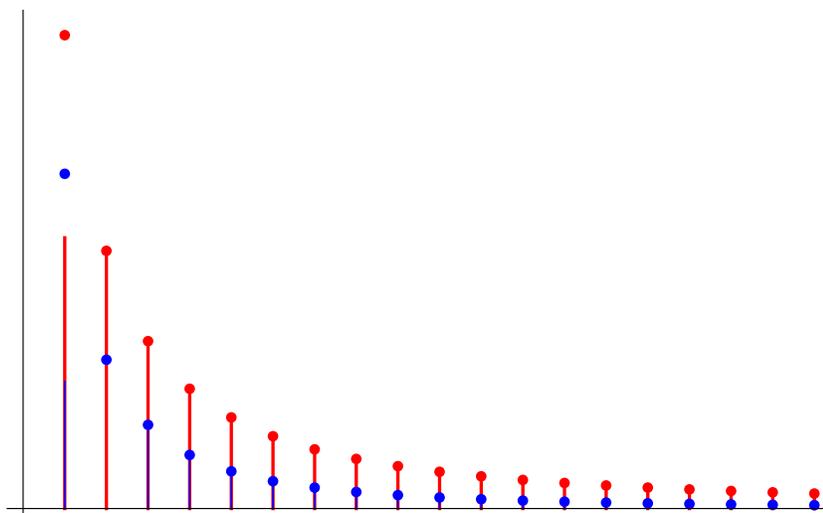
$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots$$

diverges also because we have seen in a previous homework that the Taylor expansion of $-\log(1-x)$ is $x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$. Evaluated at $x = -1$ this gives infinity and already is an indication why the harmonic series diverges. You will also see using comparison that the series diverges.

COMPARISON TEST

15.3. The **comparison test** has the same structure than the proper improper integrals. Indeed, it has the same structure just that the functions are now sequences.

If $0 \leq |a_k| \leq b_k$ and $\sum_k b_k$ converges, then $\sum_k a_k$ converges.



We see two sequences a_k and b_k plotted as a "graph".

Example: Verify that the sequence $\sum_k a_k = \sum_k \sin(2^k)/2^k$ converges.
 Solution: look at $\sum_k b_k = \sum_k 1/2^k$. We have $0 \leq |a_k| \leq b_k$ and $\sum_k b_k$ converges, therefore $\sum_k a_k$ converges.

15.4. One can also reverse this.

If $0 \leq b_k \leq a_k$ and $\sum_k b_k$ diverges then $\sum_k a_k$ diverges.

Example: Task: Show that the series $\sum_k a_k = \sum_k (1 + \sin^2(k))/k$ diverges.
 Solution: Look at the series $\sum_k b_k = \sum_k 1/k$. This is the harmonic series. It diverges. We also have $a_k \geq b_k$. Therefore $\sum_k a_k$ diverges.