

CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 18: Asymptotics

GROWTH RATES

18.1. We want to get intuition and tools to decide whether a series converges or diverges. In order to do so, we need to see how fast something grows or decays asymptotically.

18.2.

Definition: We write $a_k \sim b_k$ and say " a_k is asymptotic to b_k " if $a_k/b_k \rightarrow 1$.

Definition: We write $a_k \ll b_k$ and say " a_k is asymptotically smaller than b_k " if $a_k/b_k \rightarrow 0$.

18.3.

Example: look at $a_k = k^5 + k^3$ and $b_k = k^5$. We have $a_k/b_k = 1 + \frac{k^3}{k^5} \rightarrow 1$, so that $a_k \sim b_k$.

18.4.

Example: $a_k = k^2 \ll b_k = 2^k$ because $k^2/2^k \rightarrow 0$. How do we see this? The **l'Hospital rule** is a good way to see this. First check that both are sick and go to infinity. We can then bring the **indefinite form** " ∞/∞ " to the Hospital. The limit is the same than $2k/(k \ln(2)2^k)$. An other Hospital visit gives $2/((\ln(2) + k^2 \ln(2)^2)2^k)$ which goes to zero.

18.5. It is good to have a feel for growth rates. If you need to compute something which depends on the size k , then it matters whether the task grows linearly, or logarithmically or quadratically or polynomially or exponentially or super exponentially:

Example	growth rate
$a_k = \sin(k)$	bounded
$a_k = \ln(k) + 11$	logarithmic
$a_k = \sqrt{k} + 4$	square root
$a_k = 3k + 4$	linear
$a_k = k^2 + 3k - 5$	quadratic
$a_k = k^{11} - \ln(k)$	polynomially
$a_k = e^{\sqrt{k}}$	sub-exponentially
$a_k = e^k + k^3$	exponentially
$a_k = k!$	factorial
$a_k = k^k$	super-exponentially
$a_k = k^{(k^k)}$	super-duper-exponentially

18.6. Asymptotically simplifying and expression means replacing terms with asymptotically equivalent terms, that means terms which behave in the same way.

Example:

$$\frac{5k^3 + k}{4k^2 + \ln(k)} \sim \frac{5k^3}{4k^2} \sim \frac{5k}{4}.$$

Example:

$$\frac{k! + k^k + k^5}{k^7 + 2^k} \sim \frac{k^k}{2^k}.$$

Example:

$$\frac{k^5 + 8}{\sqrt{k^11 + k^4 + 3}} \sim \frac{k^5}{k^{11/2}}.$$

Example:

$$\frac{\sqrt{k}}{\sqrt{k+1}} \sim \frac{\sqrt{k}}{\sqrt{k}} \sim 1.$$

The **limit comparison test** is for $a_k \geq 0, b_k \geq 0$.

Theorem: If $a_k \sim b_k$ then $\sum_k a_k$ converges if and only if $\sum_k b_k$ converges.

Proof for large k we have $a_k \leq 2b_k$ and $b_k \leq 2a_k$. Use comparison in both cases.

Example: Does

$$\sum_k a_k = \sum_k \frac{(4k^2 + 4k)(3k + 1)}{\sqrt{9k^9 + k^2 + 1}}$$

converge? Yes, because $a_k \sim 12k^3/3k^{9/2} \sim 4k^{-3/2} = b_k$. We know that $\sum_k b_k$ converges as it is 4 times the $p = 3/2$ series.