

# CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

## Lecture 21: The Power of Taylor

### POWER SERIES

**21.1.** As a reminder, a series

$$S(x) = \sum_{k=0}^{\infty} a_k(x - c)^k .$$

is a **power series**. There exists  $R \geq 0$  possibly equal to  $\infty$  so that  $S(x)$  is convergent for  $x \in (-R, R)$ . This is the **maximal open interval of convergence**. The **interval of convergence** could additionally contain one or both of the boundary points.

**21.2.** To find the radius of convergence, we used the **ratio test**. By checking the boundary points we got the full interval of convergence. Here are examples of various cases:

- a)  $S(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$  converges in  $\mathbb{R} = (-\infty, \infty)$ .
- b)  $S(x) = \sum_{k=0}^{\infty} x^k$  converges in  $(-1, 1)$  and not on any boundary point.
- c)  $S(x) = \sum_{k=0}^{\infty} k!x^k$  converges only at 0.
- d)  $S(x) = \sum_{k=0}^{\infty} \frac{x^k}{k^2}$  converges at  $[-1, 1]$ .
- e)  $S(x) = \sum_{k=0}^{\infty} \frac{x^k}{k}$  converges at  $[-1, 1)$ .

**21.3.** An important subclass of power series are **Taylor series**

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

which by assumption is a power series with radius of convergence  $R > 0$ . [Some real analysis tests consider Taylor series with  $R = 0$ . A theorem of Borel then assures that every power series can come from a smooth function. We do not consider power series with  $R = 0$  as Taylor series. Except at halloween!] Unlike for power series which can be formal objects like the above example e), Taylor series always come for us with a **positive radius of convergence**.

**Theorem:** Every power series with positive radius of convergence is a Taylor series for the function  $f(x) = S(x)$ . For  $|x - c| < R$ , we can freely differentiate or integrate both sides of the equation  $f(x) = \sum_k a_k x^k$ . When integrating, we need to adapt the constant.

**21.4.** Here is the reason. More details in the next class: let us assume  $c = 0$ . Take the series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

which converges in  $(-R, R)$  and defines there a function  $f(x)$ . The Taylor series of  $f(x)$  produces a new power series

$$f(x) = \sum_{k=0}^{\infty} b_k x^k$$

with  $b_k = f^{(k)}(0)/k!$ . Comparing  $f(0)$  with  $S(0)$  shows  $a_0 = b_0$ . The ratio test shows that also the power series  $\sum_k a_k k x^{k-1}$  converges and that it agrees with  $f'(x)$ . Similarly,  $\sum_k b_k k x^{k-1}$  is the Taylor series of  $f'(x)$ . Comparing  $f'(0)$  with  $S'(0)$  shows  $a_1 = b_1$ . Continue to take derivatives to conclude  $a_k = b_k$ .

**21.5. Halloween note:** there is more to this but is beyond our world... One can ask for which power series  $\sum_k a_k x^k$ , there is a function  $f$  such that  $f^{(k)}(0)/k! = a_k$ . The answer is blood boiling and shocking: a **theorem of Borel** states that for every power series, there exists a smooth function  $f$  such that  $f^{(k)}(0)/k! = a_k$ , even if the radius of convergence is zero. Strange also that there are many functions which do the job for the same sequence. We can for example add the Taylor series of  $e^{-1/x}$ . This is a silly series because  $f^{(k)}(x) = 0$  for  $x \rightarrow 0$ . Its series does of course not represent the function  $e^{-1/x}$ ! **Trick or treat, smell my feet!**



FIGURE 1. A theorem of Borel assures that for any sequence of real numbers  $a_k$ , there is a smooth function  $f$  such that  $f^{(k)}(0)/k! = a_k$ .